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A Global Joint Pricing Model of Stocks and Bonds Based on the Quadratic Gaussian Approach *

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Abstract

This work presents a joint model for bond prices, stock prices, and exchange rates within multi-currency economies. The model includes three types of latent factors: systematic factors that determine the domestic and foreign interest rates, stock-specific factors, and currency-specific factors. By incorporating the stochastic discount factor reflecting these three risk factors, we derive an analytical formula for bond prices and stock prices, and exchange rates based on the quadratic Gaussian approach studied primarily in term structure modeling. Our model has the distinctive feature of capturing market rates in a low interest rate environment. Furthermore, the model not only enables a simultaneous estimation of bond, equity and currency risk premiums but also provides a foundation for solving an investment problem reflecting realistic market conditions.
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JEL Classification E43, F31, G10, G12

1 Introduction

Increased co-movements of financial asset prices are often seen across countries. To capture these co-movements, it is desirable to simultaneously model financial asset prices including currency rates in a unified framework. In this study, we construct a joint model bond prices, stock prices, and exchange rates within multi-currency economies.

A factor model can be useful in capturing co-movements in prices across various financial assets. Such a model is based on the premise that the origin of price movements is attributable to the dynamics of common factors across the assets. There is an extensive body of literature on factor-based asset pricing. For example, factor-based modeling has been frequently used in the context of modeling the term structure of interest rates such as Duffie and Kan[8] and Dai and Singleton[6].

In contrast to the literature on term structure modeling, there are fewer studies on a joint pricing model across different asset classes. In one study, Bakshi and Chen[3] derive an analytical formula for stock and bond prices in a single currency economy based on a market equilibrium approach. They introduce systematic and idiosyncratic factors that follow a mean-reverting square root process. The systematic factors are assumed to determine the stochastic discount factor. Thus, zero coupon bond prices are represented as the function of the systematic factors by taking the expectation of the stochastic discount factor. The stock

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dividend in the model is determined by both idiosyncratic factors as well as the systematic factors. Taking the expectation of the stochastic discount factor multiplied by dividend cash flows leads to the stock price representation as the function of the systematic and idiosyncratic factors.

As with Bakshi and Chen[3], Mamaysky[14] constructs a joint pricing model of stocks and bonds in a single currency economy under the no-arbitrage framework. In the model, the short rate is assumed to be an affine function of systematic factors following a mean-reverting square root process. This leads to analytical expressions for bond prices represented as the exponential of the affine function of the systematic factors. To model the stock price, Mamaysky[14] also models stock dividends as a function of systematic factors and idiosyncratic factors. The idiosyncratic factors in his model follow non-stationary processes whose drift terms depend on the systematic factors. Under these settings, he derives a stock price representation in the form of an exponential-affine function of systematic and idiosyncratic factors. Mamaysky[15] also constructs a joint pricing model of stocks and bonds in line with Mamaysky[14]. The difference is that Mamaysky[15] assumes the systematic factors follow Ornstein-Uhlenbeck processes rather than a square root process. Mamaysky[15] conducts an empirical analysis using monthly data for zero coupon interest rates of U.S. treasury bonds and the value-weighted equity market index.

Kikuchi[11] extends the model in Mamaysky[15] by incorporating the quadratic Gaussian term structure model (QGTM) used in Ahn et al.[1] and Leippold and Wu[12]. In Kikuchi's model, the short rate is defined as the quadratic function of systematic factors and the dividend is defined as a function that depends on both systematic and idiosyncratic factors. The assumptions imposed on the systematic and idiosyncratic factors in his model are the same as in Mamaysky[15]'s ones where systematic factors follow an Ornstein-Uhlenbeck process and idiosyncratic factors are non-stationary processes whose drift terms depend only on systematic factors. He derives the closed form formula for pricing zero coupon bonds and stocks under the no-arbitrage condition. In addition, he estimates bond and equity risk premia based on the proposed model using data from Japanese market.

Other than Kikuchi[11], there are studies on the nonlinear joint pricing model for stocks and bonds in a single currency economy. Bäuerle and Pfeiffer[5] construct a joint pricing model of stocks and bonds by introducing the hyperbolic Gaussian stochastic discount factor. Filipović and Willems[10] derive pricing formula not only for stocks and bonds but also for futures contracts whose payoff is determined by future dividends, by incorporating the polynomial jump-diffusion process studied in Filipović and Larsson[9].

There are some studies of factor-based asset pricing in multi-currency economies. One of them is Backus et al.[2]. Under the assumption of a complete market, they introduce stochastic discount factors for each economy and provide changes in exchange rates as the ratio of foreign and domestic stochastic discount factors. Leippold et al.[13] draw on the QGTM framework to construct the model dealing jointly with zero coupon interest rates and exchange rates. They add idiosyncratic factors to capture currency factors in addition to systematic factors that play a role in determining zero coupon bond prices for foreign and domestic economies. All factors follow Ornstein-Uhlenbeck processes. They estimate factors and parameters using U.S. and Japanese LIBOR and swap rates and the foreign exchange rates between the U.S. and Japan.

Bakshi et al.[4] derive a currency option pricing formula using the generalized Fourier transform in a setting where the stochastic discount factor is provided as a function of the short rate, global diffusion factors, and country-specific jump-diffusion factors. Using market data for exchange rates and currency options' implied volatilities among the U.S. dollar, Euro, and Japanese Yen, they estimate risk premia for interest rates, and for global and country-specific factors.

In terms of studies on factor based asset pricing in multi-currency economies, much less work has been on constructing a joint pricing model for assets belonging to different asset classes while including exchange rate dynamics. In this article, we propose a joint model for bond prices, stock prices, and exchange rates in multi-currency economies under a no-arbitrage condition. We extend Kikuchi[11] by incorporating domestic and foreign stochastic discount factors as shown in Leippold et al.[13]. Our model includes three types of factors: systematic factors, stock-specific factors and currency risk factors. The systematic factors are assumed to follow Ornstein-Uhlenbeck processes and their quadratic functions determine short rates and stock dividends in domestic and foreign countries. Stock-specific factors can be either stationary or non-stationary processes whose drift terms depend on them. This setting is different from Mamaysky[14][15] and Kikuchi[11] whose drift terms depend only on systematic factors, not on stock-specific factors. Although Mamaysky[14] and [15] insist that stock price dynamics follow a non-stationary process, this viewpoint is still controversial in finance. For this reason, we adopt a more general model for stock-specific factors. One of the features of our model is that it is based on the quadratic Gaussian approach. This allows interest rates and dividend yields the model implies to capture market rates more accurately in a low interest rate environment such as what has been observed in European countries and in Japan in the recent years.

The rest of this paper is organized as follows. In Section 2, we present the model setup. In Section 3, we derive an analytical bond pricing formula. In Section 4, we model cash flows paid by a stock and provide the formula for stock prices. In addition, we prove a theorem on sufficient conditions for obtaining a well-defined price. In Section 5, we formulate exchange rates using each country's stochastic discount factor. In Section 6, we indicate the invariant transformation to play a role in making the model simpler. Conclusions are presented in Section 7.

2 Setup

We fix l countries associated with a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leqq t}, \mathbb{P})$ satisfying the usual condition that $(\mathcal{F}_t)_{0 \leqq t}$ is complete and right continuous. $\mathbb{P}$ denotes the physical measure. We assume that each country issues its own currency and the financial markets in each country are complete. This assumption ensures the unique existence of the risk neutral measure equivalent to $\mathbb{P}$ for each country. We denote the risk neutral measure corresponding to the jth country as $\mathbb{Q}^j$. Suppose that $W_t^x \in \mathbb{R}^l$, $W_t^y \in \mathbb{R}^m$ and $W_t^z \in \mathbb{R}^n$ are $\mathcal{F}_t$-adapted standard Brownian motions under $\mathbb{P}$ and are independent of each other.

Our model incorporates three types of factors. First, we introduce a vector of systematic factors, X_t that takes values in a domain $\mathcal{D}_X \subseteq \mathbb{R}^l$ and follows the multivariate Ornstein-Uhlenbeck process:

$$dX_t = K_X(\theta_X - X_t)dt + \Sigma_X dW_t^x. \tag{2.1}$$

where all eigenvalues of K_X are assumed to take positive values. This assumption implies that X_t follows a stationary process. X_t determines the risk-free short rate in every country; consequently, X_t becomes the determinant of all financial asset prices. We define the risk-free short rate r_t^j in the jth country's currency as

$$r_t^j = X_t' \Psi^j X_t + {\varphi^j}' X_t + \eta^j, \tag{2.2}$$

where the superscript of X_t and φ^j represents their transpositions. We assume that Ψ^j is a positive definite matrix. Hereafter, we sometimes denote r_t^j by $r_t^j(X_t)$ to emphasize that r_t^j depends on X_t.

As previous studies (Backus et al.[2], Leippold and Wu[13]) have already pointed out, currency dynamics are determined not only by interest rate-related factors but also by currency-specific factors related to currency risk premiums. We denote a vector of currency-specific factors as Y_t. This vector takes values in a domain $\mathcal{D}_Y \subseteq \mathbb{R}^m$ and follows the multivariate Ornstein-Uhlenbick process:

$$dY_t = K_Y(\theta_Y - Y_t)dt + \Sigma_Y dW_t^y. \tag{2.3}$$

where all eigenvalues of K_Y are assumed to take positive values. This assumption is essentially the same as in Leippold and Wu[13].

In this study, we derive an analytical representation for stock prices. Z_t denotes a vector of stock-specific factors that affect stock prices aside from the systematic and currency-specific factors. Stock-specific factors are assumed to have the following dynamics

$$dZ_t = \mu_Z(X_t, Y_t, Z_t)dt + \Sigma_{Z,1}dW_t^x + \Sigma_{Z,2}dW_t^y + \Sigma_{Z,3}dW_t^z, \tag{2.4}$$

where $\Sigma_{Z,1} \in \mathbb{R}^{n\times l}$, $\Sigma_{Z,2} \in \mathbb{R}^{n\times m}$ and $\Sigma_{Z,3} \in \mathbb{R}^{n\times n}$. Later, we will provide the specification of the drift term $\mu_Z(X_t, Y_t, Z_t)$ and the volatilities of Z_t.

If we denote the Radon-Nikodym derivative of the jth country's risk neutral measure with respect to the physical measure by

$$\mathcal{R}^j = \frac{d\mathbb{Q}^j}{d\mathbb{P}},$$

then Girsanov's theorem tells us that there exist the adapted processes $\Lambda_t^{j,x}$, $\Lambda_t^{j,y}$, and $\Lambda_t^{j,z}$ such that the Radon-Nikodym derivative process $\mathcal{R}_t^j = E[\mathcal{R}^j|\mathcal{F}_t]$ is provided as

$$\begin{aligned}\mathcal{R}_t^j &= \exp\left\{-\int_0^t \Lambda_u^{j,x}\cdot dW_u^x - \frac{1}{2}\int_0^t ||\Lambda_u^{j,x}||^2 du\right\}\\ &\cdot \exp\left\{-\int_0^t \Lambda_u^{j,y}\cdot dW_u^y - \frac{1}{2}\int_0^t ||\Lambda_u^{j,y}||^2 du\right\}\\ &\cdot \exp\left\{-\int_0^t \Lambda_u^{j,z}\cdot dW_u^z - \frac{1}{2}\int_0^t ||\Lambda_u^{j,z}||^2 du\right\}\\ &\equiv \xi_t^{j,x}\cdot\xi_t^{j,y}\cdot\xi_t^{j,z}.\end{aligned} \tag{2.5}$$

From equation (2.5), the stochastic discount factor of the jth country, $\mathcal{M}_t^j$ is provided as

$$\mathcal{M}_t^j = \exp\left(-\int_0^t r_s^j ds\right)\mathcal{R}_t = \exp\left(-\int_0^t r_s^j ds\right)\xi_t^{j,x}\xi_t^{j,y}\xi_t^{j,z}. \tag{2.6}$$

Girsanov's theorem leads to the fact that $\tilde{W}_t^{j,x}$, $\tilde{W}_t^{j,y}$, and $\tilde{W}_t^{j,z}$ defined in equation (2.7) as indicated below follow the standard Brownian motions under $\mathbb{Q}^j$.

$$\tilde{W}_t^{j,x} := W_t^x + \int_0^t \Lambda_u^{j,x}du,\ \tilde{W}_t^{j,y} := W_t^y + \int_0^t \Lambda_u^{j,y}du,\ \tilde{W}_t^{j,z} := W_t^z + \int_0^t \Lambda_u^{j,z}du. \tag{2.7}$$

Furthermore, we assume the essentially affine setting introduced by Duffee[7] for $\Lambda_t^{j,x}$, $\Lambda_{t,}^{j,y}$, and $\Lambda_t^{j,z}$:

$$\Lambda_t^{j,x} = \lambda_x^j + \Lambda_x^j X_t,\quad \Lambda_t^{j,y} = \lambda_y^j + \Lambda_y^j Y_t,\quad \Lambda_t^{j,z} = \lambda_z^j + \Lambda_z^j Z_t. \tag{2.8}$$

3 Bond Price Representation

In this section, we derive the zero coupon bond pricing formula based on the setup indicated in the previous section. $P_t^{T-t,j}$ denotes the price at time t of the jth country's zero coupon bond with maturity date T. With T-$t = \tau$, we denote the bond price by $P_t^{\tau,j}$. $P_t^{\tau,j}$ is given by

$$P_t^{\tau,j} = E_t\left[\frac{\mathcal{M}_T^j}{\mathcal{M}_t^j}\right] = E_t^{\mathbb{Q},j}\left[\exp\left(-\int_t^T r_u^j(X_u)du\right)\right], \tag{3.1}$$

where $E_t^{\mathbb{Q},j}[\cdot]$ is the conditional expectation operator under $\mathbb{Q}^j$ with respect to the filtration $\mathcal{F}_t$.

Equations (2.1), (2.7), and (2.8) provide the dynamics of X_t under $\mathbb{Q}$:

$$dX_t = \tilde{K}_X^j(\tilde{\theta}^j - X_t)dt + \Sigma_X d\tilde{W}_t^{j,x}, \tag{3.2}$$

where the coefficients of the above equation satisfy the following relationship:

$$\begin{aligned}\tilde{K}_X^j &= K_X + \Sigma_X \Lambda_x^j,\\ \tilde{K}_X^j\tilde{\theta}_X^j &= K_X\theta_X - \Sigma_X\lambda_x^j.\end{aligned}$$

Applying the Feynman-Kac theorem to equation (3.1), we obtain the following partial differential equation for $P_t^{\tau,j}$:

$$\begin{aligned}&\frac{\partial P_t^{\tau,j}}{\partial t} + \tilde{\kappa}^j(X_t)'\frac{\partial P_t^{\tau,j}}{\partial X_t} - r_t^j(X_t)P_t^{\tau,j} + \frac{1}{2}\mathrm{Tr}\left(\Sigma_X\Sigma_X'\frac{\partial^2 P_t^{\tau,j}}{\partial X_t'\partial X_t}\right) = 0,\\ &P_T^{0,j} = 1,\end{aligned} \tag{3.3}$$

where $\tilde{\kappa}^j(X_t) = \tilde{K}_X^j(\tilde{\theta}^j - X_t)$.

To find a solution to equation (3.3), we make a guess at the following form:

$$P_t^{\tau,j} = \exp\left(X_t'A_\tau^jX_t + b_\tau^{j\prime}X_t + c_\tau^j\right). \tag{3.4}$$

Without loss of generality, A_τ^j is assumed to be symmetric.

Substituting equation (3.4) into equation (3.3) and applying coefficient comparison, we obtain the system of ordinary differential equations:

$$\begin{aligned}\frac{dA_\tau^j}{d\tau} &= -2\tilde{K}_X^{j\prime}A_\tau^j + 2A_\tau^j\Sigma_X\Sigma_X'A_\tau^j - \Psi^j,\ \ A_0^j = 0_{l\times l},\\ \frac{db_\tau^j}{d\tau} &= 2A_\tau^j\tilde{K}_X^j\tilde{\theta}_X^j - \tilde{K}_X^{j\prime}b_\tau^j + 2A_\tau^j\Sigma_X\Sigma_X'b_\tau^j - \varphi^j,\ \ b_0^j = 0_{l\times 1},\\ \frac{dc_\tau^j}{d\tau} &= (\tilde{K}_X^j\tilde{\theta}_X^j)'b_\tau^j + \frac{1}{2}\mathrm{Tr}(2\Sigma_X\Sigma_X'A_\tau^j + b_\tau^jb_\tau^{j\prime}) - \eta^j,\ \ c_0^j = 0.\end{aligned} \tag{3.5}$$

We can solve equation (3.5) by a numerical method such as the Runge-Kutta method.

Bond price dynamics under the physical measure $\mathbb{P}$ are given by

$$\begin{aligned}\frac{dP_t^{\tau,j}}{P_t^{\tau,j}} &= \left\{X_t'\left(\Psi^j + 2A_\tau^j\Sigma_X\Lambda_x^j\right)X_t + \left(\varphi^{j\prime} + 2\lambda_x^{j\prime}\Sigma_X'A_\tau^j + b_\tau^{j\prime}\Sigma_X\Lambda_X^j\right)X_t + b_\tau^{j\prime}\Sigma_X\lambda_X^j + \eta^j\right\}dt\\ &\quad + (2X_t'A_\tau^j + b_\tau^{j\prime})\Sigma_X dW_t^x.\end{aligned} \tag{3.6}$$

4 Stock Price Representation

In this section, we derive an analytical representation for the stock price denominated in the jth county's currency. No-arbitrage asset pricing theory tells us that the stock price is obtained as the expectation of the integration of the discounted cash flows the stock pays under the risk neutral measure corresponding to the jth country. We apply the Feynman-Kac theorem to obtain the partial differential equation followed by the stock price. We solve the resulting partial differential equation to derive the analytical representation for stock prices.

4.1 Specification of Stock-Specific Factors

Before proceeding, we specify the drift term and the volatility term of Z_t in equation (2.4). To retain computational tractability, one of the candidates for the specification is provided as follows:

$$dZ_t = (\mu_Z + K_Z Z_t)dt + \Sigma_{Z,z} dW_t^z, \tag{4.1}$$

where we make no assumptions for K_Z other than that it is invertible. To allow the formulation of a more general model, we formulate Z_t in the way that it can be either stationary or non-stationary.

The formulation in equation (4.1) is different from the one provided in Mamaysky[14][15] and Kikuchi[11]. Mamaysky[14] emphasized the importance of considering the non-stationarity of stock price dynamics and Mamaysky[15] showed an empirical analysis using U.S. market data with his proposed model. However, it cannot be stated that the result has the robust validity of a non-stationary model for stock dynamics. Therefore, whether or not stock prices follow a non-stationary process is still controversial. Thus, in this paper, we formulate Z_t such that it can be stationary or non-stationary.

The process for Z_t in equation (4.1) provided under $\mathbb{P}$ is represented under the measure $\mathbb{Q}^j$ using equation (2.7) and (2.8) as follows:

$$\begin{aligned} dZ_t &= \left(\mu_Z + K_Z Z_t - \Sigma_{Z,z}(\Lambda_z^j Z_t + \lambda_z^j)\right) dt + \Sigma_{Z,z} d\tilde{W}_t^{j,z} \\ &= (\tilde{\mu}_Z^j + \tilde{K}_Z^j Z_t)dt + \Sigma_{Z,z} d\tilde{W}_t^{j,z}, \end{aligned} \tag{4.2}$$

where $\tilde{K}_Z^j = K_Z - \Sigma_{Z,z}\Lambda_z^j$ and $\tilde{\mu}_Z^j = \mu_Z - \Sigma_{Z,z}\lambda_z^j$.

4.2 Pricing a Dividend-Paying Security with a Finite Maturity

Here, we assume that the stock denominated by the jth country's currency is non-defaultable and pays a dividend $D_t^j(X_t, Z_t)$ per unit of time.

In this subsection, we suppose a security pays $D_t^j(X_t, Z_t)$ per unit of time continuously until maturity date T, and $\overline{D}_T^j$ at T. We model the dividend $D_t^j(X_t, Z_t)$ as

$$\begin{aligned} D_t^j(X_t, Z_t) =& (X_t' \Phi_X^j X_t + \delta_X^{j\prime} X_t + Z_t' \Phi_Z^j Z_t + \delta_Z^{j\prime} Z_t + \delta_0^j) \\ & \exp\left(X_t' E_X^j X_t + f_X^{j\prime} X_t + Z_t' E_Z^j Z_t + f_Z^{j\prime} Z_t + g^j t + h^j\right), \end{aligned} \tag{4.3}$$

where Φ_X^j, Φ_Z^j, E_X^j and E_Z^j are assumed to be symmetric positive definite. At maturity date T,

$$\overline{D}_T^j = \exp\left(X_T' E_X^j X_T + f_X^{j\prime} X_T + Z_T' E_Z^j Z_T + f_Z^{j\prime} Z_T + g^j T + h^j\right) \tag{4.4}$$

is paid to the security holder.

$S_t^{T,j}$ denotes the price of this security at t.

The cumulative discounted gain from time 0 to time t denotes G_t^j, formulated as

$$G_t^j = \int_0^t \exp\left(-\int_0^s r_u^j(X_u)du\right) D_s^j(X_s,Y_s)ds + \exp\left(-\int_0^t r_u^j(X_u)du\right) S_t^{T,j}. \tag{4.5}$$

Under $\mathbb{Q}^j$, G_t^j must be a martingale process, so that $G_t^j{=}E_t^{\mathbb{Q},j}[G_T^j]$. This equation and equation (4.5) lead to the following equation for $S_t^{T,j}$:

$$S_t^{T,j} = E_t^{\mathbb{Q},j}\left[\int_t^T \exp\left(-\int_t^s r_u^j(X_u)du\right) D_s^j(X_s,Z_s)ds + \exp\left(-\int_t^T r_u^j(X_u)du\right)\overline{D}_T^j\right]. \tag{4.6}$$

Applying the Feynman-Kac theorem to equations (4.3), (4.4), and (4.6), we obtain the following PDE:

$$\begin{aligned}&\frac{\partial S_t^{T,j}}{\partial t} + \tilde{\kappa}_X^j(X_t)'\frac{\partial S_t^{T,j}}{\partial X_t} + \tilde{\kappa}_Z^j(Z_t)'\frac{\partial S_t^{T,j}}{\partial Z_t} + \frac{1}{2}\mathrm{Tr}\left(\Sigma_X\Sigma_X'\frac{\partial^2 S_t^{T,j}}{\partial X_t'\partial X_t}\right)\\ &+\frac{1}{2}\mathrm{Tr}\left(\Sigma_{Z,z}\Sigma_{Z,z}'\frac{\partial^2 S_t^{T,j}}{\partial {Z_t}'\partial Z_t}\right) - r_t^j(X_t)S_t^{T,j} + D_t^j = 0,\quad S_T^{T,j} = \overline{D}_T^j,\end{aligned} \tag{4.7}$$

where $\tilde{\kappa}_X^j(X) = \tilde{K}_X^j(\tilde{\theta}_X^j - X)$ and $\tilde{\kappa}_Z^j(Z) = \tilde{\mu}_Z^j + \tilde{K}_Z^j Z$.

We guess at a solution to equation (4.7) in the following form:

$$S_t^{T,j} = \exp\left(X_t'E_X^jX_t + f_X^{j\prime}X_t + Z_t'E_Z^jZ_t + f_Z^{j\prime}Z_t + g^jt + h^j\right). \tag{4.8}$$

Substituting equation (4.8) into equation (4.7) and applying the coefficient comparison, we obtain the following system of equations:

$$\begin{aligned}&2E_X^j\Sigma_X\Sigma_X'E_X^j - 2\tilde{K}_X^{j\prime}E_X^j + \Phi_X^j - \Psi^j = 0,\\ &2E_Z^j\Sigma_{Z,z}\Sigma_{Z,z}'E_Z^j + 2\tilde{K}_Z^{j\prime}E_Z^j + \Phi_Z^j = 0,\\ &-f_X^{j\prime}(\tilde{K}_X^j - 2\Sigma_X\Sigma_X'E_X^j) + 2\tilde{\theta}_X^{j\prime}\tilde{K}_X^{j\prime}E_X^j + \delta_X^j - \varphi^j = 0,\\ &f_Z^{j\prime}(\tilde{K}_Z^j + 2\Sigma_{Z,z}\Sigma_{Z,z}'E_Z^j) + 2\tilde{\mu}_Z^{j\prime}E_Z^j + \delta_Z^j = 0,\\ &g^j + \frac{1}{2}\mathrm{Tr}(\Sigma_X\Sigma_X'(2E_X^j + f_X^jf_X^{j\prime})) + \frac{1}{2}\mathrm{Tr}(\Sigma_{Z,z}\Sigma_{Z,z}'(2E_Z^j + f_Z^jf_Z^{j\prime}))\\ &+ f_X^{j\prime}\tilde{K}_X^j\tilde{\theta}_X^j + f_Z^{j\prime}\tilde{\mu}_Z + \delta_0^j - \eta^j = 0.\end{aligned} \tag{4.9}$$

Note that the variables in equation (4.9) are E_X^j, E_Z^j, f_X^j, f_Z^j, and g^j, and h_j is a normalized constant.

We argue a sufficient condition for the existence of a solution to the equations shown in (4.9) later. If a solution exists, then we find that $S_t^{T,j}{=}S_t^{T',j}$ for $T{\neq}T'$ because E_X^j, E_Z^j, f_X^j, f_Z^j, and g^j are computed from equation (4.9) independently of the maturity of the security. Hence, we abbreviate this security's price $S_t^{T,j}$ as S_t^j by eliminating T.

Since $\overline{D}_T^j{=}S_T^{T,j}{=}S_T^j$ by equations (4.4) and (4.8), we can rewrite equation (4.6) as follows:

$$S_t^j = E_t^{\mathbb{Q},j}\left[\int_t^T \exp\left(-\int_t^s r_u^j(X_u)du\right) D_s^j(X_s,Z_s)ds + \exp\left(-\int_t^T r_u^j(X_u)du\right) S_T^j\right]. \tag{4.10}$$

Equation (4.10) implies that if the following equation called the transversality condition:

$$\lim_{T\to\infty} E_t^{\mathbb{Q},j}\left[\exp\left(-\int_t^T r_u^j(X_u)du\right) S_T^j\right] = 0 \tag{4.11}$$

holds, then

$$S_t^j = \lim_{T\to\infty} E_t^{\mathbb{Q},j}\left[\int_t^T \exp\left(-\int_t^s r_u^j(X_u)du\right) D_s^j(X_s,Z_s)ds\right]. \tag{4.12}$$

Equation (4.12) shows us that S_t^j uniquely exists in the case where S_t^j exists and the transversality condition holds.

4.3 Stock Prices

In this subsection, we derive the price representation for a non-defaultable stock paying the dividend $D_t^j(X_t, Z_t)$ defined in equation (4.3) continuously per unit of time. We denote the stock price by $S_t^{\infty,j}$.

The cumulative discounted gain G_t^j for the stock from time 0 to time t is provided as

$$G_t^j = \int_0^t \exp\left(-\int_0^s r_u^j(X_u)du\right) D_s^j(X_s,Z_s)ds + \exp\left(-\int_0^t r_u^j(X_u)du\right) S_t^{\infty,j}. \tag{4.13}$$

Under $\mathbb{Q}^j$, G_t^j must be a martingale so that $G_t^j = E_t^{\mathbb{Q},j}[G_T^j]$ for $T > t$. Applying this equation to equation (4.13) leads to the following equation:

$$S_t^{\infty,j} = E_t^{\mathbb{Q},j}\left[\int_t^T \exp\left(-\int_t^s r_u^j(X_u)du\right) D_s^j(X_s,Z_s)ds + \exp\left(-\int_t^T r_u^j(X_u)du\right) S_T^{\infty,j}\right]. \tag{4.14}$$

Here, we prove a theorem on $S_t^{\infty,j}$.

Theorem 1. Suppose that S_t^j exists and the transversality condition shown in equation (4.11) holds. Then, $S_t^{\infty,j}{=}S_t^j$.

Proof. As shown above, when S_t^j exists and the transversality condition holds, S_t^j uniquely exists. On the other hand, equation (4.6) holds for $S_t^{\infty,j}$ according to equation (4.14). If $S_t^j \neq S_t^{\infty,j}$, then this contradicts the fact that S_t^j uniquely exists. Hence, $S_t^{\infty,j}{=}S_t^j$. □

Theorem 1 and equation (4.14) provide the following corollary.

Corollary 1. When S_t^j exists and equation (4.11) holds,

$$\lim_{T\to\infty} E_t^{\mathbb{Q},j}\left[\exp\left(-\int_t^T r_u^j(X_u)du\right) S_T^{\infty,j}\right] = 0. \tag{4.15}$$

4.4 Sufficient Condition for a Well-defined Stock Price

In order for a stock price to be well-defined, the tansversality condition (equation (4.11)) must hold and S_t^j must exist. In this subsection, we first prove the sufficient condition of the transversality condition for the stock price. The existence of S_t^j depends on whether a solution to equation (4.9) exists or not. We present the conditions of the model parameters to be satisfied for a solution to equation (4.9) to exist.

Regarding the transversality condition for the well-defined stock price, Mamaysky[14] provides a sufficient condition in the affine framework. Even in the quadratic framework used in this study, the sufficient condition is consistent with that of Mamaysky[14].

Theorem 2. If there exists $\epsilon{>}0$ such that $D_t^j(X_t, Z_t){\geq}\ \epsilon S_t^j{>}0$ for any t, then the transversality condition for the stock price denominated in the jth county's currency, shown in equation (4.11), holds.

Proof. We denote $X_t'\Phi_X^j X_t + \delta_X^{j\prime} X_t + Z_t'\Phi_Z^j Z_t + \delta_Z^{j\prime} Z_t + \delta_0^j$ as $\delta^j(X_t, Z_t)$ and provide ζ_t^j as follows:

$$\zeta_t^j = \exp\left(\int_0^t \delta^j(X_u, Z_u)du\right)\exp\left(-\int_0^t r^j(X_u)du\right)S_t^j.$$

Ito's lemma leads to the following equation:

$$(\mathcal{D}_X^j + \mathcal{D}_Z^j)S^j + \frac{\partial S^j}{\partial t} = (r^j - \delta^j)S^j,$$

where $\mathcal{D}_X^j$ and $\mathcal{D}_Z^j$ are defined as follows:

$$\begin{aligned}
\mathcal{D}_X^j &= (\tilde{K}_X^j(\tilde{\theta}^j - X_t))'\frac{\partial}{\partial X_t} + \frac{1}{2}\mathrm{Tr}\left(\Sigma_X\Sigma_X'\frac{\partial^2}{\partial X_t \partial X_t'}\right),\\
\mathcal{D}_Z^j &= (\tilde{\mu}_Z^j + \tilde{K}_Z^j Z_t)'\frac{\partial}{\partial Z_t} + \frac{1}{2}\mathrm{Tr}\left(\Sigma_{Z,z}\Sigma_{Z,z}'\frac{\partial^2}{\partial Z_t \partial Z_t'}\right).
\end{aligned}$$

Applying Ito's lemma to ζ_t^j,

$$\begin{aligned}
d\zeta_t^j =& \exp\left(\int_0^t (\delta^j(X_u, Z_u) - r^j(X_u))du\right)\cdot\\
&\left\{\left(\mathcal{D}_X^j + \mathcal{D}_Z^j + \frac{\partial}{\partial t}\right)S_t^j dt + (\delta^j(X_t, Z_t) - r^j(X_t))dt + \frac{\partial S_t^j}{\partial X_t'}\Sigma_X d\tilde{W}_t^j + \frac{\partial S_t^j}{\partial Z_t'}\Sigma_{Z,z} d\tilde{W}_t^j\right\} dt\\
=& \exp\left(\int_0^t (\delta^j(X_u, Z_u) - r^j(X_u))du\right)\cdot\left(\frac{\partial S_t^j}{\partial X_t'}\Sigma_X d\tilde{W}_t^j + \frac{\partial S_t^j}{\partial Z_t'}\Sigma_{Z,z} d\tilde{W}_t^j\right).
\end{aligned}$$

This leads to the following equation:

$$\zeta_t^j = \zeta_s^j + \int_s^t \exp\left(\int_s^h (\delta^j(X_h, Z_u) - r^j(X_u))du\right)\cdot\left(\frac{\partial S_h^j}{\partial X_h'}\Sigma_X d\tilde{W}_h^j + \frac{\partial S_h^j}{\partial Z_h'}\Sigma_{Z,z} d\tilde{W}_h^j\right) \equiv \zeta_s^j + \mathcal{I}_s^j(t). \tag{4.16}$$

Since $\zeta_t^j > 0$, we find that $\mathcal{I}_s^j(t) > -\zeta_s^j$, that is, $\mathcal{I}_s^j(t)$ has the lower bound $-\zeta_s^j$. In addition, $\mathcal{I}_s^j(t)$ is a local martingale; hence, $\mathcal{I}_s^j(t)$ becomes a supermartingale.

Taking the expectation of equation (4.16) conditional on time $\mathcal{F}_s$ under the risk neutral measure of the jth currency's economy, we obtain the following inequality:

$$E_s^{\mathbb{Q},j}[\zeta_t^j] = \zeta_s^j + E_s^{\mathbb{Q},j}[\mathcal{I}_s^j(t)] < \zeta_s^j + \mathcal{I}_s^j(s) < e^{\epsilon s}\exp\left(\int_0^s -r^j(X_u)du\right)S_s^j,$$

where we use $\delta^j(X_t, Z_t) \geq \epsilon$ for any t due to the relationship $D_t^j(X_t, Z_t) \geq \epsilon S_t^j$ seen in the above inequality.

This inequality leads to the following relationship:

$$0 < E_s^{\mathbb{Q},j}\left[\exp\left(\int_0^t -r^j(X_u)du\right)S_t^j\right] < e^{-\epsilon(t-s)}\exp\left(\int_0^s -r^j(X_u)du\right)S_s^j. \tag{4.17}$$

Using equation (4.17),

$$\lim_{t\to\infty} E_s^{\mathbb{Q},j}\left[\exp\left(\int_0^t -r^j(X_u)du\right)S_t^j\right] = 0. \tag{4.18}$$

Equation (4.18) indicates that the transversality condition for the stock price holds. $\square$

The existence of S_t^j is guaranteed by the existence of the solution to equation (4.9). First, we prove the following proposition regarding the first equations in (4.9).

Proposition 1. When $\Psi^j - \Phi_X^j$ are positive definite, a matrix pair $(\tilde{K}_X^j, \Sigma_X)$ is observable and a matrix pair $(\tilde{K}_X^j, I)$ is controllable where I is the identity matrix, the first equation in (4.9) has a unique symmetric positive definite matrix solution for E_X^j.

Proof. By adding the transposition of the first equation in (4.9) to itself and multiplying it by $\frac{1}{2}$, we obtain the algebraic Riccati equation as follows:

$$-2E_X^j \Sigma_X \Sigma'_X E_X^j + \tilde{K}_X^{j\prime} E_X^j + E_X^j \tilde{K}_X^j + \Psi^j - \Phi_X^j = 0.$$

When $\Psi^j - \Phi_X^j$ are positive definite, a matrix pair $(\tilde{K}_X^j, \Sigma_X)$ is observable and a matrix pair $(\tilde{K}_X^j, I)$ is controllable, the above algebraic Riccati equation has a unique symmetric positive definite matrix solution for E_X^j (refer to Wonham[16]). □

The following proposition guarantees the existence of solution f_X^j for the third equation in (4.9).

Proposition 2. When $\Psi^j - \Phi_X^j$ are positive definite, a matrix pair $(\tilde{K}_X^j, \Sigma_X)$ is observable and a matrix pair $(\tilde{K}_X^j, I)$ is controllable where I is the identity matrix, the third equation in (4.9) has a solution for f_X^j.

Proof. When $\Psi^j - \Phi_X^j$ are positive definite and a matrix pair $(\tilde{K}_X^j, \Sigma_X)$ is observable and a matrix pair $(\tilde{K}_X^j, I)$ is controllable, the solution E_X^j of the equation exists from Proposition 1. We must confirm that the coefficient matrix $\tilde{K}_X^j - 2\Sigma_X \Sigma'_X E_X^j$ of f_X^j in the third equation is invertible. Here we set $\tilde{K}_X^j - 2\Sigma_X \Sigma'_X E_X^j$ by A. Then,

$$E_X^j A = E_X^j \tilde{K}_X^j - 2E_X^j \Sigma_X \Sigma'_X E_X^j = -\tilde{K}_X^{j\prime} E_X^j + \Phi_X^j - \Psi^j.$$

Since E_X^j is invertible by Proposition 1, we only have to confirm that $-\tilde{K}_X^{j\prime} E_X^j + \Phi_X^j - \Psi^j$ is invertible to confirm that A is invertible. We denote $-\tilde{K}_X^{j\prime} E_X^j + \Phi_X^j - \Psi^j$ by B. By adding B to its transposition, B', we obtain the following:

$$B + B' = -\tilde{K}_X^{j\prime} E_X^j - E_X^j \tilde{K}_X^j + 2(\Phi_X^j - \Psi^j) = -2E_X^j \Sigma_X \Sigma'_X E_X^j = -2E_X^j \Sigma_X \Sigma'_X E_X^j + \Phi_X^j - \Psi^j.$$

This indicates that $B + B'$ is a negative definite matrix; therefore, B is invertible. □

Unfortunately, we have not yet found proper sufficient conditions to guarantee the existence of solution E_Z^j in the second equation of (4.9). Moreover, even if the second equation has a solution, that does not guarantee the existence of the solution f_Z^j in the fourth equation. In other words, the matrix $K_Z^j + 2\Sigma_{Z,z} \Sigma'_{Z,z} E_Z^j$ is not always invertible.

We suppose a class of model parameters satisfying the case where the second equation has a solution and $K_Z^j + 2\Sigma_{Z,z} \Sigma'_{Z,z} E_Z^j$ is invertible. Then, from Proposition 1 and Proposition 2, we can prove the following theorem on the existence of the well-defined $S_t^{\infty,j}$ and its analytical representation.

Theorem 3. We suppose a class of model parameters satisfying the case where the second equation of (4.9) has a solution E_Z^j and $K_Z^j + 2\Sigma_{Z,z} \Sigma'_{Z,z} E_Z^j$ is invertible. Furthermore, we assume the following:

- Φ_X^j, Φ_Z^j and $\Psi^j - \Phi_X^j$ are positive definite,

- a matrix pair $(\tilde{K}_X^j, \Sigma_X)$ is observable,
- a matrix pair $(\tilde{K}_X^j, I)$ is controllable where I is the identity matrix,
- $\delta_0^j > \frac{1}{4}\delta'^j_X \Phi'^{-1}_X \delta_X^j + \frac{1}{4}\delta'^j_Z \Phi'^{-1}_Z \delta_Z^j$.

Then, the non-defaultable stock price is well-defined and has the following representation:

$$S_t^{\infty,j} = \exp\left(X_t' E_X^j X_t + f_X^{j\prime} X_t + Z_t' E_Z^j Z_t + f_Z^{j\prime} Z_t + g^j t + h^j\right), \tag{4.19}$$

where E_X^j, E_Z^j, f_X^j, f_Z^j and g^j are solutions of equation (4.9) and E_X^j becomes symmetric positive definite.

Proof. Φ_X^j, Φ_Z^j, and $\delta_0^j>\frac{1}{4}\delta'^j_X \Phi'^{-1}_X \delta_X^j+\frac{1}{4}\delta'^j_Z \Phi'^{-1}_Z \delta_Z^j$ imply $0<D_t^j(X_t, Z_t)/S_t^j$. Therefore, the transversality condition for S_t^j holds from Theorem 2. Thus, $S_t^{\infty,j}=S_t^j$ according to Theorem 1. Consequently,

$$S_t^{\infty,j} = \exp\left(X_t' E_X^j X_t + f_X^{j\prime} X_t + Z_t' E_Z^j Z_t + f_Z^{j\prime} Z_t + g^j t + h^j\right).$$

In the above equation, Proposition 1 guarantees the existence of the symmetric positive definite solution E_X^j. Furthermore, at this time, the solution f_X^j exists. Once we have solutions E_X^j, E_Z^j, f_X^j, and f_Z^j, g^j in the fifth equation is easily computed. □

4.5 Stock Price Dynamics

We denote the well-defined stock price as defined in Theorem 3 by S_t. Stock price dynamics under the physical measure $\mathbb{P}$ are given by

$$\begin{aligned} \frac{dS_t}{S_t} &= \left\{X_t'\left(\Psi^j - \Phi_X^j + 2E_X^j \Sigma_X \Lambda_x^j\right) X_t + Z_t'\left(2E_Z^j \Sigma_Z \Lambda_z^j - \Phi_Z^j\right) Z_t\right\} dt \\ &+ \left\{\left(\varphi^{j\prime} + 2\lambda_x^{j\prime} \Sigma'_X E_X^j + f_X^{j\prime} \Sigma_X \Lambda_X^j - \delta_X^{j\prime}\right) X_t + \left(2\lambda_z^{j\prime} \Sigma'_Z E_Z^j + f_Z^{j\prime} \Sigma_Z \Lambda_Z^j - \delta_Z^{j\prime}\right) Z_t\right\} dt \\ &+ (f_Z^{j\prime} \Sigma_Z \lambda_Z^j + f_X^{j\prime} \Sigma_X \lambda_X^j + \eta^j - \delta_0^j) dt + (2X_t' E_X^j + f_X^{j\prime}) \Sigma_X dW_t^x + (2Z_t' E_Z^j + f_Z^{j\prime}) \Sigma_Z dW_t^z. \end{aligned} \tag{4.20}$$

5 Exchange Rate Modeling

In this section, we model the dynamics of exchange rates.

When j=1, the jst country (the first and only country) is, by definition, its own home country. When $j\geq 2$, the jth country is a foreign country.

By equation (2.5), stochastic discount factor of the jth country, $\mathcal{M}_t^j$ is provided as

$$\mathcal{M}_t^j = \exp\left(-\int_0^t r_s^j ds\right) \mathcal{R}_t = \exp\left(-\int_0^t r_s^j ds\right) \xi_t^{j,1} \xi_t^{j,2} \xi_t^{j,3}. \tag{5.1}$$

Here suppose that $j \geq 2$. We denote the value of the jth country's currency per the value of one unit of the home country's currency by F_t^j. Backus et al.[2] prove that in a complete market, F_t^j is provided as the ratio of the stochastic discount factors corresponding to the foreign and home countries as follows:

$$F_t^j = \frac{\mathcal{M}_t^j}{\mathcal{M}_t^1}. \tag{5.2}$$

To model exchange rate dynamics, we derive the stochastic discount factor dynamics by applying Ito's lemma to equation (5.1) and using equation (2.5). The process is given as follows:

$$\frac{d\mathcal{M}_t^j}{\mathcal{M}_t^j} = -r_t^j - \Lambda_t^{j,x} dW_t^x - \Lambda_t^{j,y} dW_t^y - \Lambda_t^{j,z} dW_t^z. \tag{5.3}$$

By applying Ito's lemma to equation (5.2) and using equation (5.3), we obtain the exchange rate dynamics as follows:

$$\begin{aligned} \frac{dF_t^j}{F_t^j} &= \left\{ r_t^1 - r_t^j + (\Lambda_t^{1,x\prime}, \Lambda_t^{1,y\prime}, \Lambda_t^{1,z\prime})(\Lambda_t^{1,x\prime} - \Lambda_t^{j,x\prime}, \Lambda_t^{1,y\prime} - \Lambda_t^{j,y\prime}, \Lambda_t^{1,z\prime} - \Lambda_t^{j,z\prime})' \right\} dt \\ &\quad + (\Lambda_t^{1,x\prime} - \Lambda_t^{j,x\prime}, \Lambda_t^{1,y\prime} - \Lambda_t^{j,y\prime}, \Lambda_t^{1,z\prime} - \Lambda_t^{j,z\prime})(dW_t^{x\prime}, dW_t^{y\prime}, dW_t^{z\prime})'. \end{aligned} \tag{5.4}$$

The representation given in equation (5.4) clarifies that the volatility term of exchange rate dynamics depends on the difference between the domestic and foreign currency's market prices of risks corresponding to X_t, Y_t, and Z_t respectively. Moreover, we find that the drift term consists of the gap between domestic and foreign interest rates and the term that depends on the difference between the domestic and foreign currencies' market prices of risks.

6 Invariant Transformation

When factors are unobservable, the invariant transformation gives the model a simpler form. In this section, we demonstrate the invariant transformation to our model where it does not change the bond and stock prices; moreover, the model becomes simpler. The transformation also allows us to avoid over-fitting or under-fitting in implementing the estimation of parameters and latent factors. Dai and Singleton[6] derive the canonical form of the affine term structure model using the invariant transformation. Regarding the QGTM, Ahn et al.[1] and Leippold and Wu[12] provide the canonical form of their models.

First, we deal with the parallel shift transformation for X_t. We consider the transformation: $\hat{X} = X + a$ for any constant vector a. This new vector of factors $\hat{X}$ has the following dynamics under $\mathbb{P}$:

$$d\hat{X}_t = dX_t = K_X(\theta_X - X_t) + \Sigma_X dW_t^x = K_X((\theta_X + a) - \hat{X}_t) + \Sigma_X dW_t^x.$$

As a result, θ_X is transformed into $\hat{\theta}_X{=}\theta_X{+}a$ and K_X and Σ_X remain unchanged. From equation (2.2), this parallel shift transforms φ^j into $\hat{\varphi}^j = \varphi^j - 2\Psi^j a$, η^j to $\hat{\eta}^j{=}\eta^j{+}\, a'\Psi^j a{-}\varphi^{j\prime} a$ and makes Ψ^j invariant. From equation (2.8), the shift transforms λ_X^j into $\hat{\lambda}_x^j = \lambda_X^j{-}\Lambda_x^j a$ with Λ_x^j unchanged. As for the dividend yield of stock in equation (4.3), the shift transforms δ_X^j into $\hat{\delta}_X^j = \delta_X^j{-}2\Phi_X^j a$, δ_0^j into $\hat{\delta}_0^j{=}\delta_0^j{+}a'\Phi_X^j a{-}\delta_X^{j\prime} a$ and Φ_X^j is invariant.

We must first confirm that the value of equation (3.4) does not change based on the above parameter changes. In other words, we have to confirm the following equation:

$$X_t' A_\tau^j X_t + b_\tau^{j\prime} X_t + c_\tau^j = \hat{X}_t' \hat{A}_\tau^j \hat{X}_t + \hat{b}_\tau^{j\prime} \hat{X}_t + \hat{c}_\tau^j, \tag{6.1}$$

where $\hat{A}_\tau^j$, $\hat{b}_\tau^j$, and $\hat{c}_\tau^j$ satisfy the system of ordinary differential equations (3.5), likewise A_τ^j, b_τ^j, and c_τ^j satisfy the same system of ordinary differential equations. Using the relationship $\hat{X}_t = X_t + a$ and the parameter changes as indicated above, we can prove that equation (6.1) always holds.

We must also confirm the value of equation (4.19); specifically, we confirm whether or not the following equation holds:

$$\exp\left(X_t' E_X^j X_t + f_X^{j\prime} X_t + g^j t \right) = \exp\left(\hat{X}_t' \hat{E}_X^j \hat{X}_t + \hat{f}_X^{j\prime} \hat{X}_t + \hat{g}^j t \right). \tag{6.2}$$

where $\hat{E}_X^j$, $\hat{f}_X^j$, and $\hat{g}^j$ satisfy the system of algebraic equations (4.9), likewise E_X^j, f_X^j, and g^j satisfy the same system of equations. We can prove that equation (6.2) holds based on the relationship $\hat{X}_t = X_t{+}a$ and the parameter changes indicated above.

In this way, the parallel shift $\hat{X}{=}X{+}a$ does not change the model. The choice of $a = -\theta_X$ leads to the dynamics of X given by

$$dX_t = -K_X X_t + \Sigma_X dW_t^x. \tag{6.3}$$

Next, we consider a linear transformation for X_t, or $\hat{X} = LX$ for any invertible square matrix L. Here, we assume equation (6.3) as X's dynamics. This linear transformation changes model parameters as follows: $\hat{\Psi}^j = L'^{-1}\Psi^j L^{-1}$, $\hat{\varphi}^j = L'^{-1}\varphi^j$, $\hat{K}_X = LK_X L^{-1}$, $\hat{\Sigma}_X = L\Sigma_X$, $\hat{\Phi}_X^j = L'^{-1}\Phi_X^j L^{-1}$, $\hat{\delta}_X^j = L'^{-1}\delta_X^j$, and $\hat{\Lambda}_x^j = \Lambda_x^j L^{-1}$. Other parameters remain unchanged by the linear transformation of X. We also note that $\hat{\Psi}^j$ and $\hat{\Phi}_X^j$ become positive definite and all of the eigenvalues of $\hat{K}_X$ are positive.

In the same way as in the case of a parallel shift, we can prove that equations (6.1) and (6.2) hold under $\hat{X} = LX$ and the parameter changes indicated above according to this transformation. Consequently, a linear transformation $\hat{X} = LX$ does not change the model. Thus, the choice of L as Σ_X^{-1} leads to the dynamics of X being given by

$$dX_t = -K_X X_t + dW_t^x, \tag{6.4}$$

where all of the eigenvalues of K_X are positive.

For X's dynamics provided in equation (6.4), we demonstrate a transformation $\hat{X} = UX$ where U is an orthogonal matrix. Since this transformation is a linear transformation, the model remains unchanged. $\hat{X} = UX$ transforms equation (6.4) as

$$d\hat{X}_t = -UK_X U^{-1}\hat{X}_t + U dW_t^x. \tag{6.5}$$

In equation (6.5), we note that $UK_X U^{-1}$'s eigenvalues are always positive and UW_t^x is a multivariate standard Brownian motion. By applying Schur's decomposition to K_X and choosing a proper orthogonal matrix U, we obtain the lower triangular matrix $UK_X U^{-1}$ with all diagonal elements positive.

Summarizing the above, by taking a proper affine transformation for X_t, we obtain the following X_t without changing bond and stock prices:

$$dX_t = -K_X X_t + dW_t^x, \tag{6.6}$$

where K_X is the lower triangular matrix with all diagonal elements positive.

We can also take the similar affine transformation for Y_t and Z_t without changing bond and stock prices or exchange rate dynamics. As a result, we obtain simpler models for Y_t and Z_t given by

$$\begin{aligned} dY_t &= -K_Y Y_t + dW_t^y, \\ dZ_t &= K_Z Z_t + dW_t^z, \end{aligned} \tag{6.7}$$

where K_Y is the lower triangular matrix with all diagonal elements positive.

7 Conclusion

This work presents a joint model for bond prices, stock prices and exchange rates in multi-currency economies by introducing a stochastic discount factor based on three types of risk factors: systematic factors that determine the interest rates, stock-specific factors, and currency-specific factors. Using the quadratic Gaussian approach, we derive analytical representations for bond and stock prices and for exchange rates. Our model has a distinctive feature in that interest rates and dividend yields it implies capture market rates more accurately under a low interest rate environment, as has been seen in many countries in recent years.

Although this study focuses on constructing a model, it would be interesting to conduct a simultaneous estimation of bond, equity and currency risk premiums based on our model, using actual market data, and to disentangle the dependence between them. By including these concepts, we will conduct an empirical study using our model in the near future.

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