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A Semi-analytical Solution to Consumption and International Asset Allocation Problem

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Semi-Analytical Solution for Consumption and International Asset Allocation Problem*

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Abstract

We consider a continuous-time optimal consumption and international asset allocation problem for an agent with CRRA utility, under a quadratic factor international security market model, in which the latent factors are global and currency-specific ones. It is not typically straightforward to identify an analytical solution to the partial differential equation (PDE) for the agent's indirect utility function, since a non-homogeneous term appears in the PDE. Therefore, we apply the method of Liu [10] and Batbold *et al.* [2] to the PDE and derive a semi-analytical solution. In the optimal portfolio choice on domestic asset allocation, the global factor and the domestic market price of global risk exist. However, in the optimal portfolio choice on international asset allocation, there also exist the currency-specific factor, domestic market price of currency-specific risk, the difference between the domestic and foreign market prices of global risk, and the difference between the domestic and foreign market prices of currency-specific risk.

1 Introduction

The importance of asset formation for households has been emphasized in most developed countries against the deterioration of public pension financ-

*This is a revised version of Batbold *et al.* [3].

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ing due to low growth and population aging. International security investment has been recommended from the viewpoint of diversified investment, being essential for households in low growth countries. Considering that a household has limited investment knowledge, we should not promote risky active investments, but asset allocation towards domestic and foreign government bonds and main indices such as stock and REIT indices.

The purpose of this paper is to derive a semi-analytical solution to the optimal consumption and international asset allocation problem assuming a continuous-time international security market model; thus, it contributes to the discussions on exemplary international asset allocation for households.

Campbell and Viceira [5] considered a continuous-time optimal consumption and investment problem over an infinite time horizon under the assumption that an agent with CRRA utility invests in an instantaneously risk-free security and a zero-coupon bond with a constant time to maturity using the Vasicek one-factor term-structure model. A second-order partial differential equation (PDE) for the value function is deduced from the Hamilton-Yacobi-Bellman (HJB) equation, but it is not generally straightforward to identify an analytical solution to the PDE, due to the existence of a non-homogeneous term in this equation. As such, they derive an approximate analytical solution by applying the log-linear approximation proposed by Campbell [4] to the non-homogeneous term.

However, Liu [10] examined a continuous-time optimal consumption and investment problem over a finite time horizon under the assumption that an agent with CRRA utility invests in the risk-free security and risky securities under a highly general multi-factor security market model, in which the latent factors satisfy a diffusion process and both the drift and diffusion functions are quadratic functions of the factors, while both the market price of risk and the short-term interest rates are affine functions of the factors. He paid attention to the fact that a solution of the non-homogeneous PDE for the indirect utility function derived from the HJB equation is expressed as an integral of the solution for a homogeneous PDE, ignoring the non-homogeneous term of the non-homogeneous PDE, and derived a system of ordinary differential equations (ODEs) for the unknown parameters constituting the integrand.

More recently, Batbold, Kikuchi, and Kusuda [2] considered a continuous-time optimal consumption and asset allocation problem over a finite time horizon under the assumption that an agent with CRRA utility invests in an instantaneously risk-free asset, bonds, and indices under a multi-factor security market model, in which latent factors follow a multi-dimensional version of the Ornstein-Uhlenbeck process and both the market price of risk and the short-term interest rates are affine functions of the factors. They expressed the indirect utility function as an integral of the solution for the above homogeneous PDE by applying the method of Liu [10], and derived the system of ODEs for the unknown parameters constituting the integrand. They also solved the ODEs and derived a semi-analytical solution, which is a time-integrated analytic function.

The above cited studies assume a one-country security market model. Few studies deal with a continuous-time international security market model including both stock and bond markets. Recently, Kikuchi [8] unified the quadratic international bond market model of Leippold and Wu [9] with the affine one-country stock and bond market model of Mamaysky [11].

We assume a stationary latent factor international security continuous-time model that eliminates the non-stationary factor in Kikuchi's model and consider the same problem as Batbold *et al.* [2]. In the security market model, latent factors are constituted of global factor and currency-specific factor. These factors satisfy the multi-dimensional version of the Ornstein-Uhlenbeck process. In each country, the market price of global risk and of currency-specific risk is an affine function of the global factor and of the currency-specific factor, respectively, and the short-term interest rate, dividend-rate, and expected inflation-rate are quadratic functions of the global factors.

The main results of this paper are summarized as follows. We apply the method of Liu [10] and Batbold *et al.* [2] to our problem, and derive a semi-analytical solution. In the optimal portfolio choice on domestic asset allocation, the global factor and the domestic market price of global risk exist. However, in the optimal portfolio choice on international asset allocation, there also exist the currency-specific factor, domestic market price of currency-specific risk, the difference between the domestic and foreign market prices of global risk, and the difference between the domestic and foreign market prices of currency-specific risk. This indicates that, in international security investment, an investor should always estimate the global and currency-specific factors, the difference between the domestic and foreign market prices of global risk, and the difference between the domestic and foreign market prices of currency-specific risk.

The rest of this paper is organized as follows. In Section 2, we explain the stationary latent factor international security market model and the agent's optimal consumption and security investment problem. In Section 3, we derive a semi-analytical solution to this problem and present the optimal consumption and portfolio choice.

2 Stationary Quadratic International Security Market Model and Consumer's Problem

Here, we first introduce the stationary quadratic international security market model and present the stochastic differential equations (SDEs) that domestic and foreign security's return rate processes satisfy under a no arbitrage condition. Then, we present the consumer international asset allocation problem.

2.1 Market Environment

We consider a frictionless international security market economy that consists of USA and N different currency areas over time span $[0, \infty)$. Agents' common subjective probability and information structure are modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathrm{P})$ where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$ is the natural filtration generated by a $\mathbf{N}$-dimensional standard Brownian motion B_t. We indicate the expectation operator under P with E, and the conditional expectation operator with E_t.

In the US, there are markets for the consumption commodity and securities at every date $t \in [0, \infty)$. The traded securities are the nominal-risk-free security called the *money market account*, a continuum of zero-coupon bonds whose maturity dates are $(t, t + \bar{\tau}]$, each of which has a one US dollar payoff at maturity, and J types of main indices (stock indices, REIT indices, *etc.*).

In the n-th currency area ($n \in \{1, \cdots, N\}$), there are security markets at every date $t \in [0, \infty)$. The traded securities are a continuum of zero-coupon bonds whose maturity dates are $(t, t + \tau_n]$, each of which has a payoff of one unit of the n-th currency at maturity, and J_n types of main indices. There are foreign exchange markets between any two currency areas at $t \in [0, \infty)$.

At every date t, let P_t, P_t^T, and S_t^j denote US dollar prices of the money market account, the zero-coupon bond with maturity date T, and the j-th index, respectively, in the US. Similarly, at every date, let P_{nt}^T and S_{nt}^j denote prices in the n-th currency of the zero-coupon bond with maturity date T, and the j-th index, respectively, in the n-th currency area.

2.2 Stationary Quadratic International Security Market Model

Recently, Kikuchi [8] unified the quadratic international bond market model of Leippold and Wu [9] with the affine domestic market model of stocks and bonds, presented by Mamaysky [11]. We eliminate a non-stationary factor on stock prices in this model. Following Kikuchi [8], we illustrate our quadratic international security market model.

Let $\mathbf{N} = M + N$ and

$$B_t = \begin{pmatrix} B_t^X \\ B_t^Y \end{pmatrix},$$

where B_t^X and B_t^Y are M-dimensional and N-dimensional Brownian motions, respectively.

Assumption 1. *State vector processes X_t and Y_t are controlled by the following SDEs:*

$$dX_t = -K_X X_t \, dt + dB_t^X, \tag{2.1}$$

$$dY_t = -K_Y Y_t \, dt + dB_t^Y, \tag{2.2}$$

where K_X is an $M \times M$ constant matrix, K_Y is an $N \times N$ constant matrix, and each matrix is a positive lower triangular matrix.

State vector processes X_t and Y_t follow multivariate Ornstein-Uhlenbeck processes with mean reversion. For identification, the two processes are normalized based on the regular affine transformation of Dai and Singleton [6] to have zero long-run means and identity diffusion matrices.[1]

Each country's state price deflator is assumed to be orthogonally decomposed into a deflator related to state process X_t and one related to Y_t.

Assumption 2. *The domestic and the n-th foreign state-price deflators π_t and π_{nt} are expressed as:*

$$\pi_t = \pi_t^X \, \pi_t^Y, \qquad \pi_{nt} = \pi_{nt}^X \, \pi_{nt}^Y, \tag{2.3}$$

where π_t^X and π_{nt}^X are diffusion processes that only depend on B_t^X, and π_t^Y and π_{nt}^Y are expressed as

$$\frac{d\pi_t^Y}{\pi_t^Y} = -\Lambda_t^Y \, dB_t^Y, \qquad \frac{d\pi_{nt}^Y}{\pi_{nt}^Y} = -\Lambda_{nt}^Y \, dB_t^Y. \tag{2.4}$$

Furthermore, any security price process follows a diffusion process that only depends on B_t^X.

We call X_t the global factor and Y_t the currency-specific factor. This naming is justified by the following lemma.

[1] See Kikuchi [8] for a detailed discussion of the identification issue.

Lemma 1. *Under Assumptions 1 and 2, there is no arbitrage iff 1 and 2 below hold.*

1. π_t^X *and* π_{nt}^X *satisfy*

$$\frac{d\pi_t^X}{\pi_t^X} = -r_t\,dt - \Lambda_t^X\,dB_t^X, \qquad \frac{d\pi_{nt}^X}{\pi_{nt}^X} = -r_{nt}\,dt - \Lambda_{nt}^X\,dB_t^X, \quad (2.5)$$

   where r_t *and* r_{nt} *are the domestic and the* n*-th foreign instantaneous interest rates and* Λ_t^X *and* Λ_{nt}^X *are the domestic and the* n*-th foreign market prices of global risk, respectively.*

2. *The process of the exchange rate for the US dollar against the* n*-th foreign currency satisfies*

$$\frac{d\varepsilon_{nt}}{\varepsilon_{nt}} = \left(r_t - r_{nt} + \begin{pmatrix} \Lambda_t^X - \Lambda_{nt}^X \\ \Lambda_t^Y - \Lambda_{nt}^Y \end{pmatrix}' \begin{pmatrix} \Lambda_t^X \\ \Lambda_t^Y \end{pmatrix} \right) dt + \begin{pmatrix} \Lambda_t^X - \Lambda_{nt}^X \\ \Lambda_t^Y - \Lambda_{nt}^Y \end{pmatrix}' dB_t. \quad (2.6)$$

Proof. See Appendix A.1. □

Remark 1. *Leippold and Wu [9] estimate their international bond market model using US and Japanese LIBOR and swap rates and the exchange rate between the two economies. They conclude that independent currency factors are essential to capture the portion of the exchange rate movement that is independent of the term structure movement.*

Assumption 3. 1. *The domestic and the* n*-th foreign market prices of global risk are affine functions of the global factor.*

$$\Lambda_t^X = \lambda_X + \Lambda_X X_t, \qquad \Lambda_{nt}^X = \lambda_X^n + \Lambda_X^n X_t, \quad (2.7)$$

   where $K_X + \Lambda_X$ *is regular.*

2. *The domestic and the* n*-th foreign market prices of currency-specific risk are affine functions of the currency-specific factor.*

$$\Lambda_t^Y = \lambda_Y + \Lambda_Y Y_t, \qquad \Lambda_{nt}^Y = \lambda_Y^n + \Lambda_Y^n Y_t. \quad (2.8)$$

3. *The domestic and the* n*-th foreign instantaneous interest rates are quadratic functions of the global factor.*

$$r_t = \bar{\rho} + \rho' X_t + \frac{1}{2} X_t' \mathcal{R} X_t, \qquad r_{nt} = \bar{\rho}_n + \rho_n' X_t + \frac{1}{2} X_t' \mathcal{R}_n X_t, \quad (2.9)$$

   where $\mathcal{R}$ *and* $\mathcal{R}_n$ *are positive-definite symmetric matrices.*

4. *The domestic and the n-th foreign dividends are given by:*

$$
\begin{aligned}
D_t^j &= \left(\bar{\delta}_j + \delta_j' X_t + \frac{1}{2} X_t' \Delta_j X_t\right) \exp\left(\bar{\sigma}_j t + \sigma_j' X_t + \frac{1}{2} X_t' \Sigma_j X_t\right), \\
D_{nt}^j &= \left(\bar{\delta}_{nj} + \delta_{nj}' X_t + \frac{1}{2} X_t' \Delta_{nj} X_t\right) \exp\left(\bar{\sigma}_{nj} t + \sigma_{nj}' X_t + \frac{1}{2} X_t' \Sigma_{nj} X_t\right),
\end{aligned} \tag{2.10}
$$

where $(\bar{\delta}_j, \delta_j, \Delta_j)$ *and* $(\bar{\delta}_{nj}, \delta_{nj}, \Delta_{nj})$ *are such that* Δ_j *and* Δ_{nj} *are positive definite symmetric matrices, and*

$$
\bar{\delta}_j \geq \frac{1}{2} \delta_j' \Delta_j^{-1} \delta_j, \quad \bar{\delta}_{nj} \geq \frac{1}{2} \delta_{nj}' \Delta_{nj}^{-1} \delta_{nj},
$$

and Σ_j *and* Σ_{nj} *are symmetric matrices.*

5. *The domestic price index satisfies*

$$
\frac{dp_t}{p_t} = i_t \, dt, \qquad p_0 = 1, \tag{2.11}
$$

where i_t *is the expected instantaneous inflation rate, and it is a quadratic function of* X_t*.*

$$
i_t = \bar{\iota} + \iota' X_t + \frac{1}{2} X_t' \mathcal{I} X_t, \tag{2.12}
$$

where $\mathcal{I}$ *is a positive-definite symmetric matrix such that* $\mathcal{R} - \mathcal{I}$ *is positive-definite.*

2.3 Security Return Rate Processes

Let O and $\tau = T - t$ denote a zero matrix or vector and the time to maturity of bond P_t^T, respectively. We use the following notations:

$$
\Lambda_t = \begin{pmatrix} \Lambda_t^X \\ \Lambda_t^Y \end{pmatrix}, \qquad \Lambda_{nt} = \begin{pmatrix} \Lambda_{nt}^X \\ \Lambda_{nt}^Y \end{pmatrix}.
$$

Kikuchi [8] presents the following lemma.

Lemma 2. *Under Assumptions 1–3, there is no arbitrage iff 1–3 below hold:*

1. *Domestic security return rate processes satisfy the following:*

   (i) The short-term bond:

$$
\frac{dP_t}{P_t} = r_t \, dt, \qquad P_0 = 1. \tag{2.13}
$$

(ii) The default-free bond with time τ to maturity:

$$\frac{dP_t^T}{P_t^T} = \left(r_t + (\sigma(\tau) + \Sigma(\tau)X_t)'\Lambda_t^X\right) dt + (\sigma(\tau)+\Sigma(\tau)X_t)' dB_t^X, \quad P_T^T = 1, \tag{2.14}$$

where

$$\frac{d\Sigma(\tau)}{d\tau} = \Sigma^2(\tau) - 2\Sigma(\tau)(K_X + \Lambda_X) - \mathcal{R}, \quad \Sigma(0) = 0, \tag{2.15}$$

$$\frac{d\sigma(\tau)}{d\tau} = \left(\Sigma(\tau) - (K_X + \Lambda_X)'\right)\sigma(\tau) - \Sigma(\tau)\lambda_X - \rho, \quad \sigma(0) = 0, \tag{2.16}$$

(iii) The j-th index:

$$\frac{dS_t^j + D_t^j dt}{S_t^j} = \left(r_t + (\sigma_j + \Sigma_j X_t)'\Lambda_t^X\right) dt + (\sigma_j + \Sigma_j X_t)' dB_t^X, \tag{2.17}$$

where

$$\Sigma_j^2 - (K_X + \Lambda_X)'\Sigma_j + \frac{1}{2}(\Delta_j - \mathcal{R}_j) = 0, \tag{2.18}$$

$$\sigma_j = (K_X + \Lambda_X - \Sigma_j)'^{-1}(\delta_j - \rho - \Sigma_j \lambda_X). \tag{2.19}$$

2. *n-th foreign security return rate processes denominated in domestic currency satisfy the following:*

(i) The default-free bond with time τ to maturity:

$$\begin{aligned}\frac{d(P_{nt}^T \varepsilon_t)}{P_{nt}^T \varepsilon_{nt}} = &\left\{ r_t + \left(\begin{pmatrix} \sigma_n(\tau) + \Sigma_n(\tau)X_t \\ O \end{pmatrix} + \begin{pmatrix} \Lambda_t^X - \Lambda_{nt}^X \\ \Lambda_t^Y - \Lambda_{nt}^Y \end{pmatrix} \right)' \begin{pmatrix} \Lambda_t^X \\ \Lambda_t^Y \end{pmatrix} \right\} dt \\ &+ \left(\begin{pmatrix} \sigma_n(\tau) + \Sigma_n(\tau)X_t \\ O \end{pmatrix} + \begin{pmatrix} \Lambda_t^X - \Lambda_{nt}^X \\ \Lambda_t^Y - \Lambda_{nt}^Y \end{pmatrix} \right)' dB_t, \end{aligned} \tag{2.20}$$

where

$$\frac{d\Sigma_n(\tau)}{d\tau} = \Sigma_n^2(\tau) - 2\Sigma_n(\tau)(K_X + \Lambda_X^n) - \mathcal{R}_n, \quad \Sigma_n(0) = 0, \tag{2.21}$$

$$\frac{d\sigma_n(\tau)}{d\tau} = \left(\Sigma_n(\tau) - (K_X + \Lambda_X)'\right)\sigma_n(\tau) - \Sigma_n(\tau)\lambda_X - \rho_n, \quad \sigma_n(0) = 0. \tag{2.22}$$

(ii) The j-th index:

$$\frac{d(S_{nt}^{j}\varepsilon_{nt})+D_{nt}^{j}\varepsilon_{nt}dt}{S_{nt}^{j}\varepsilon_{nt}}=\left\{r_t+\left(\begin{pmatrix}\sigma_{nj}+\Sigma_{nj}X_t\\ O\end{pmatrix}+\begin{pmatrix}\Lambda_t^X-\Lambda_{nt}^X\\ \Lambda_t^Y-\Lambda_{nt}^Y\end{pmatrix}\right)'\begin{pmatrix}\Lambda_t^X\\ \Lambda_t^Y\end{pmatrix}\right\}dt$$
$$+\left(\begin{pmatrix}\sigma_{nj}+\Sigma_{nj}X_t\\ O\end{pmatrix}+\begin{pmatrix}\Lambda_t^X-\Lambda_{nt}^X\\ \Lambda_t^Y-\Lambda_{nt}^Y\end{pmatrix}\right)'dB_t, \quad (2.23)$$

where

$$\Sigma_{nj}^2-(K_X+\Lambda_X)'\Sigma_{nj}+\frac{1}{2}(\Delta_{nj}-\mathcal{R}_n)=0, \quad (2.24)$$

$$\sigma_{nj}=(K_X+\Lambda_X^n-\Sigma_{nj})'^{-1}(\delta_{nj}-\rho_n-\Sigma_{nj}\lambda_X^n). \quad (2.25)$$

Proof. See Appendix A.2. □

Remark 2. *Λ_t^Y does not appear in domestic security return rate processes, but appears as the market price of risk in the exchange rate process (i.e., eq. (2.6)) and in foreign security return rate processes denominated in the domestic currency. Thus, we call Λ_t^Y the domestic market price of currency-specific risk. Similarly, we call Λ_{nt}^Y the n-th foreign market price of currency-specific risk.*

Remark 3. *In eq. (2.6), difference $\Lambda_t^X-\Lambda_{nt}^X$ between domestic and foreign market prices of global risk and difference $\Lambda_t^Y-\Lambda_{nt}^Y$ between domestic and foreign market prices of currency-specific risk are volatilities in the exchange rate. Then, it follows from no arbitrage condition that the exchange rate's expected return rate depends not only on the difference between domestic and foreign instantaneous interest rate but also on the difference between domestic and foreign market prices of these risks. As a result, these market price differences also appear in volatilities and in the expected return rate in the n-th foreign security return rate processes denominated in the domestic currency.*

2.4 International Asset Allocation Problem

Let Φ_t^j and Φ_{nt}^j denote portfolio weights on domestic and j-th foreign indices. Regarding the default-free bond, let $\varphi_t(\tau)$ and $\varphi_{nt}(\tau)$ denote the densities of portfolio weights on domestic and n-th foreign bonds with τ-time to maturity.[2]

[2]We assume that the functional space of densities of portfolio weights on bonds include the set of distributions.

We define Ψ_t as:

$$\Psi_t = \begin{pmatrix} \Psi_t^X \\ O \end{pmatrix} + \begin{pmatrix} \hat{\Psi}_t^X \\ \hat{\Psi}_t^Y \end{pmatrix}, \tag{2.26}$$

where

$$\Psi_t^X = \int_0^{\bar{\tau}} \varphi_t(\tau)(\sigma(\tau) + \Sigma(\tau)X_t)\, d\tau + \sum_{j=1}^{J} \Phi_t^j(\sigma_j + \Sigma_j X_t),$$

$$\begin{aligned}\hat{\Psi}_t^X = \sum_{n=1}^{N} \int_0^{\tau_n} \varphi_{nt}(\tau) \left(\sigma_n(\tau) + \Sigma(\tau)X_t + \Lambda_t^X - \Lambda_{nt}^X\right) d\tau \\ + \sum_{n=1}^{N} \sum_{j=1}^{J_n} \Phi_{nt}^j \left(\sigma_{nj} + \Sigma_j X_t + \Lambda_t^X - \Lambda_{nt}^X\right),\end{aligned}$$

$$\hat{\Psi}_t^Y = \sum_{n=1}^{N} \int_0^{\bar{\tau}_n} \varphi_{nt}(\tau)\, d\tau\, (\Lambda_t^Y - \Lambda_{nt}^Y) + \sum_{n=1}^{N} \sum_{j=1}^{J_n} \Phi_{nt}^j(\Lambda_t^Y - \Lambda_{nt}^Y).$$

We call Ψ_t investment control and let c_t denote consumption plan.

Let W_t denote the real wealth process and $u_t = (c_t, \Psi_t)$. Then, the agent's budget-constraint is expressed in the following lemma.

Lemma 3. *Under Assumptions 1–3 and no arbitrage condition, given a control u_t, the budget-constraint satisfies*

$$\frac{dW_t}{W_t} = \left\{ \left(r_t - i_t + \Psi_t' \Lambda_t\right) - \frac{c_t}{W_t} \right\} dt + \Psi_t'\, dB_t. \tag{2.27}$$

Proof. See Appendix A.3. □

Budget-constraint (2.27) shows that the real wealth process is determined by a control $u_t = (c_t, \Psi_t)$.

Assumption 4. *The agent maximizes the following CRRA utility over a finite time horizon under budget-constraint (2.27):*

$$U(c) = \mathrm{E}\left[\alpha \int_0^T e^{-\beta t} \frac{c_t^{1-\gamma}}{1-\gamma} dt + (1-\alpha)\, e^{-\beta T} \frac{W_T^{1-\gamma}}{1-\gamma} \right], \tag{2.28}$$

where $\alpha \in [0,1]$, $\beta > 0$, and $\gamma > 1$.

Let $\mathbf{X}_t = (W_t, X_t', Y_t')'$. We call a control satisfying budget-constraint (2.27) with initial state $\mathbf{X}_0 = (W_0, X_0', Y_0')'$ as the admissible control and denote by $\mathcal{B}(\mathbf{X}_0)$ the set of admissible controls.

Then, the indirect utility function is defined by:

$$J(t, \mathbf{X}_t^u) = \mathrm{E}_t \left[\alpha \int_t^T e^{-\beta t} \frac{c_t^{1-\gamma}}{1-\gamma} dt + (1-\alpha)\, e^{-\beta T} \frac{W_T^{1-\gamma}}{1-\gamma} \right], \qquad \forall t \in [0, T]. \tag{2.29}$$

The agent's consumption and international asset allocation problem and the value function are defined by:

$$\mathbf{V}(\mathbf{X}_0) = \sup_{u \in \mathcal{B}(\mathbf{X}_0)} J(0, \mathbf{X}_0). \tag{2.30}$$

3 Semi-Analytical Solution and Optimal Control

Here, we first derive the PDE for an unknown function constituting the indirect utility function from the HJB equation. Then, we derive the semi-analytical solution of the PDE and present the optimal consumption and investment.

3.1 PDE for the Indirect Utility Function

Let E_n denote an $n \times n$ identity matrix. The HJB equation is expressed as

$$\sup_{u \in \mathcal{B}(\mathbf{X}_0)} \left\{ J_t(t, \mathbf{X}^u) + \mu(t)' J_{\mathbf{X}}(t, \mathbf{X}^u) + \frac{1}{2} \operatorname{tr} \left[\mathbf{\Sigma}(t) \mathbf{\Sigma}(t)' J_{\mathbf{XX}}(t, \mathbf{X}^u) \right] + \alpha e^{-\beta t} \frac{c^{1-\gamma}}{1-\gamma} \right\} = 0 \tag{3.1}$$

$$\text{s.t.} \quad J(T, \mathbf{X}_T^u) = (1-\alpha)\, e^{-\beta T} \frac{W_T^{1-\gamma}}{1-\gamma},$$

where

$$\mu(t) = \begin{pmatrix} W_t(r_t - i_t + \Psi_t' \Lambda_t) - c_t \\ -K_X X_t \\ -K_Y Y_t \end{pmatrix}, \qquad \mathbf{\Sigma}(t) = \begin{pmatrix} W_t(\Psi_t^X)' & W_t(\Psi_t^Y)' \\ E_M & O \\ O & E_N \end{pmatrix}.$$

It is straightforward to see that the optimal control $u^* = (c^*, \Psi^*)$ satisfies the following:

$$c_t^* = \alpha^{\frac{1}{\gamma}} e^{-\frac{\beta}{\gamma} t} J_W^{-\frac{1}{\gamma}}, \tag{3.2}$$

$$\Psi_t^* = \frac{\psi_t}{W_t^{*2} J_{WW}}, \tag{3.3}$$

where

$$\psi_t = -W_t^* \left\{ J_W \begin{pmatrix} \Lambda_t^X \\ \Lambda_t^Y \end{pmatrix} + \begin{pmatrix} J_{XW} \\ J_{YW} \end{pmatrix} \right\}. \tag{3.4}$$

Then, the consumption-related terms in HJB eq. (3.1) are computed as:

$$-c_t^* J_W + \alpha e^{-\beta t} \frac{c_t^{*1-\gamma}}{1-\gamma} = \frac{c_t^*}{1-\gamma} \left\{ (\gamma-1) J_W + \alpha e^{-\beta t} c_t^{*-\gamma} \right\} = \frac{\gamma}{1-\gamma} c_t^* J_W. \tag{3.5}$$

It also follows from eq. (3.3) that the third term in HJB eq. (3.1) is expanded as:

$$\begin{aligned}
&\mathrm{tr}\left[\boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}(t)' J_{\mathbf{XX}}(t, \mathbf{X}^u)\right] \\
&= \mathrm{tr}\left[\begin{pmatrix} W_t^*(\varPsi_t^{X*})' & W_t^*(\varPsi_t^{Y*})' \\ E_M & O \\ O & E_N \end{pmatrix} \begin{pmatrix} W_t^*(\varPsi_t^{X*})' & W_t^*(\varPsi_t^{Y*})' \\ E_M & O \\ O & E_N \end{pmatrix}' \begin{pmatrix} J_{WW} & J_{WX} & J_{WY} \\ J_{XW} & J_{XX} & J_{XY} \\ J_{YW} & J_{YX} & J_{YY} \end{pmatrix} \right] \\
&= \mathrm{tr}\left[\begin{pmatrix} W_t^{*2}\Big((\varPsi_t^{X*})'\varPsi_t^{X*} + (\varPsi_t^{Y*})'\varPsi_t^{Y*}\Big) & W_t^*(\varPsi_t^{X*})' & W_t^*(\varPsi_t^{Y*})' \\ W_t^*\varPsi_t^{X*} & E_M & O \\ W_t^*\varPsi_t^{Y*} & O & E_N \end{pmatrix} \begin{pmatrix} J_{WW} & J_{WX} & J_{WY} \\ J_{XW} & J_{XX} & J_{XY} \\ J_{YW} & J_{YX} & J_{YY} \end{pmatrix} \right] \\
&= \mathrm{tr}\left[J_{XX} + J_{YY}\right] - \frac{\psi_t' \psi_t}{W_t^{*2} J_{WW}} - 2W_t^* J_W (\varPsi_t^*)' \Lambda_t. \qquad (3.6)
\end{aligned}$$

Substituting optimal control (3.2) and (3.3) into HJB eq. (3.1) and using eqs. (3.5) and (3.6) yields the following PDE for J:

$$\begin{aligned}
&J_t + \frac{1}{2}\,\mathrm{tr}\left[J_{XX} + J_{YY}\right] - \frac{\psi_t' \psi_t}{2W_t^{*2} J_{WW}} \\
&+ (r_t - i_t) W_t^* J_W + (-K_X X_t)' J_X + (-K_Y Y_t)' J_Y + \frac{\gamma}{1-\gamma} c_t^* J_W = 0. \qquad (3.7)
\end{aligned}$$

From the above PDE, we guess that the indirect utility function takes the following form:

$$J(t, \mathbf{X}_t) = e^{-\beta t} \frac{W_t^{1-\gamma}}{1-\gamma} \big(G(t, X_t, Y_t)\big)^{\gamma}. \tag{3.8}$$

where $G(t, X_t, Y_t)$ is a function of (t, X_t, Y_t).

Then the sufficient condition for optimization in the left-hand side of the HJB equation is confirmed since the following Hessian $\mathbf{H}$ is negative definite for any control $(c, \Psi) \in \mathbb{R}_+ \times \mathbb{R}^N$:

$$\mathbf{H} = \begin{pmatrix} -\alpha\gamma e^{-\beta t} c^{-\gamma-1} & 0 & \cdots & 0 \\ 0 & -\gamma e^{-\beta t} W_t^{1-\gamma} G^{\gamma} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\gamma e^{-\beta t} W_t^{1-\gamma} G^{\gamma} \end{pmatrix}. \tag{3.9}$$

Inserting eqs. (3.2) and (3.3) and the partial derivatives of J into the PDE (3.7), we obtain the following proposition.

Proposition 1. *Under Assumptions 1–4 and no arbitrage condition, the indirect utility function, optimal consumption, and optimal investment for problem (2.30) satisfy eqs. (3.8), (3.10), and (3.12), respectively. Function* $G(t, X_t, Y_t)$ *constituting the indirect utility function is a solution of PDE (3.13).*

$$c_t^* = \alpha^{\frac{1}{\gamma}} \frac{W_t^*}{G}, \tag{3.10}$$

where

$$W_t^* = W_0 \exp\left(\int_0^t \left(r_s + (\Psi_s^*)'\Lambda_s - \frac{\alpha^{\frac{1}{\gamma}}}{G} - \frac{1}{2}(\Psi_s^*)'\Psi_s^* \right) ds + (\Psi_s^*)' \, dB_s \right), \tag{3.11}$$

$$\Psi_t^* = \frac{1}{\gamma} \begin{pmatrix} \Lambda_t^X \\ \Lambda_t^Y \end{pmatrix} + \frac{1}{G} \begin{pmatrix} G_X \\ G_Y \end{pmatrix}, \tag{3.12}$$

$$\frac{\partial}{\partial t} G(t, X_t, Y_t) + \mathcal{L}G(t, X_t, Y_t) + \alpha^{\frac{1}{\gamma}} = 0, \qquad G(T, X_T, Y_T) = (1-\alpha)^{\frac{1}{\gamma}}, \tag{3.13}$$

where $\mathcal{L}$ *is a linear differential operator defined by*

$$\begin{aligned} \mathcal{L}G = &\frac{1}{2} \operatorname{tr}\left[G_{XX} + G_{YY}\right] \\ &+ \left(-K_X X - \frac{\gamma-1}{\gamma}(\lambda_X + \Lambda_X X) \right)' G_X + \left(-K_Y Y - \frac{\gamma-1}{\gamma}(\lambda_Y + \Lambda_Y Y) \right)' G_Y \\ &- \left\{ \frac{\gamma-1}{2\gamma^2} \Big((\lambda_X + \Lambda_X X)'(\lambda_X + \Lambda_X X) + (\lambda_Y + \Lambda_Y Y)'(\lambda_Y + \Lambda_Y Y) \Big) \right. \\ &\left. + \frac{\gamma-1}{\gamma} \left(\bar{\rho} - \bar{\iota} + (\rho - \iota)'X + \frac{1}{2} X'(\mathcal{R} - \mathcal{I})X \right) + \frac{\beta}{\gamma} \right\} G. \end{aligned} \tag{3.14}$$

Proof. See Appendix A.4. □

3.2 Semi-Analytical Solution

A non-homogeneous term $\alpha^{\frac{1}{\gamma}}$ appears in the PDE (3.13), making it difficult to derive an analytical solution. Liu [10] presents a method to derive a semi-analytical solution by exploiting an analytical solution for a homogeneous PDE that abandons the non-homogeneous term. Following his method, we examine homogeneous PDE (3.15).

$$\frac{\partial}{\partial \tau} g(\tau, X, Y) = \mathcal{L} g(\tau, X, Y), \qquad g(0, X, Y) = 1. \tag{3.15}$$

An analytical solution to PDE (3.15) is expressed as:

$$g(\tau, Z) = \exp\left(\bar{a}(\tau) + a'(\tau) Z + \frac{1}{2} Z' A(\tau) Z\right), \tag{3.16}$$

where

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}, \qquad a(\tau) = \begin{pmatrix} a_X(\tau) \\ a_Y(\tau) \end{pmatrix}, \qquad A(\tau) = \begin{pmatrix} A_X(\tau) & A_{XY}(\tau) \\ A'_{XY}(\tau) & A_Y(\tau) \end{pmatrix},$$

and $A_X(\tau)$, $A_Y(\tau)$ is a symmetric matrix.

Then, it follows from the linearity of $\mathcal{L}$ that, under the interchange of differentiation and integration operators, a semi-analytical solution for PDE (3.13) is expressed as:

$$G(t, Z) = \alpha^{\frac{1}{\gamma}} \int_0^{T-t} g(s, Z)\, ds + (1-\alpha)^{\frac{1}{\gamma}} g(T-t, Z). \tag{3.17}$$

We use the following notations.

$$\lambda = \begin{pmatrix} \lambda_X \\ \lambda_Y \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \Lambda_X & O \\ O & \Lambda_Y \end{pmatrix}, \quad L = \begin{pmatrix} K_X + \frac{\gamma-1}{\gamma}\Lambda_X & O \\ O & K_Y + \frac{\gamma-1}{\gamma}\Lambda_Y \end{pmatrix}.$$

Substituting g and its derivatives into PDE (3.15) and paying attention to $A' = A$ and $Z'L'AZ = Z'ALZ$, we obtain:

$$\begin{aligned}
\frac{d\bar{a}}{d\tau} + Z'\frac{da}{d\tau} + \frac{1}{2} Z' \frac{dA}{d\tau} Z = {} & \frac{1}{2}(a'a + \mathrm{tr}[A]) + Z'Aa + \frac{1}{2} Z' A^2 Z \\
& - \frac{\gamma-1}{\gamma}\lambda' a - \frac{\gamma-1}{\gamma} Z' A\lambda - Z'L'a - \frac{1}{2} Z'L'AZ - \frac{1}{2} Z'ALZ \\
& - \frac{\gamma-1}{2\gamma^2}\lambda'\lambda - \frac{\gamma-1}{\gamma^2} Z'\Lambda'\lambda - \frac{\gamma-1}{2\gamma^2} Z'\Lambda'\Lambda Z \\
- \frac{\gamma-1}{\gamma}(\bar{\rho} - \bar{\iota}) - \frac{\gamma-1}{\gamma} Z' \begin{pmatrix} \rho - \iota \\ O \end{pmatrix} & - \frac{\gamma-1}{2\gamma} Z' \begin{pmatrix} \mathcal{R} - \mathcal{I} & O \\ O & O \end{pmatrix} Z - \frac{\beta}{\gamma}.
\end{aligned} \tag{3.18}$$

Since the above equations are identical on Z, the following system of ODEs for $(\bar{a}, a, A)$ is derived:

$$\frac{d\bar{a}}{d\tau} = \frac{1}{2}(a'\,a+\mathrm{tr}[A]) - \frac{\gamma-1}{\gamma}\lambda' a - \frac{1}{\gamma^2}\left\{(\gamma-1)\left(\frac{1}{2}\lambda'\lambda + \gamma(\bar{\rho}-\bar{\iota})\right) + \beta\gamma\right\},\ \bar{a}(0) = 0, \tag{3.19}$$

$$\frac{da}{d\tau} = (A - L')a - \frac{\gamma-1}{\gamma}A\lambda - \frac{\gamma-1}{\gamma^2}\begin{pmatrix}\Lambda_X'\lambda_X + \gamma(\rho-\iota)\\ \Lambda_Y'\lambda_Y\end{pmatrix}, \qquad a(0) = 0, \tag{3.20}$$

$$\frac{dA}{d\tau} = A^2 - L'A - AL - Q, \qquad A(0) = 0, \tag{3.21}$$

where

$$Q = \frac{\gamma-1}{\gamma^2}\begin{pmatrix}\Lambda_X'\Lambda_X + \gamma(\mathcal{R}-\mathcal{I}) & O\\ O & \Lambda_Y'\Lambda_Y\end{pmatrix}. \tag{3.22}$$

We should note that Q is a positive-definite symmetric matrix under Assumption 3, and eq. (3.21) is Riccati matrix differential equation.

We also use the following notations:

$$\begin{aligned} a^*(t, Z_t) &= \frac{\alpha^{\frac{1}{\gamma}}\int_0^\tau g(s,Z_t)a(s)\,ds + (1-\alpha)^{\frac{1}{\gamma}}g(\tau,Z_t)a(\tau)}{\alpha^{\frac{1}{\gamma}}\int_0^\tau g(s,Z_t)\,ds + (1-\alpha)^{\frac{1}{\gamma}}g(\tau,Z_t)},\\ A^*(t, Z_t) &= \frac{\alpha^{\frac{1}{\gamma}}\int_0^\tau g(s,Z_t)A(s)\,ds + (1-\alpha)^{\frac{1}{\gamma}}g(\tau,Z_t)A(\tau)}{\alpha^{\frac{1}{\gamma}}\int_0^\tau g(s,Z_t)\,ds + (1-\alpha)^{\frac{1}{\gamma}}g(\tau,Z_t)}. \end{aligned}$$

Then, we have Proposition 2.

Proposition 2. *Under Assumptions 1–4 and no arbitrage condition, an optimal control for problem (2.30) satisfies*

$$c_t^* = \frac{\alpha^{\frac{1}{\gamma}}W_t^*}{\alpha^{\frac{1}{\gamma}}\int_0^\tau g(s,Z_t)\,ds + (1-\alpha)^{\frac{1}{\gamma}}g(\tau,Z_t)}, \tag{3.23}$$

where W_t^* *is given by eq. (3.11), and*

$$\begin{aligned} \Psi_t^* &= \frac{1}{\gamma}\big(\lambda + \Lambda Z_t\big) + a^*(t,Z_t) + A^*(t,Z_t)Z_t\\ &= \frac{1}{\gamma}\begin{pmatrix}\lambda_X + \Lambda_X X_t\\ \lambda_Y + \Lambda_Y Y_t\end{pmatrix}' + \begin{pmatrix}a_X^*(t,Z_t) + A_X^*(t,Z_t)X_t + A_{XY}^*(t,Z_t)Y_t\\ a_Y^*(t,Z_t) + A_{XY}^*(t,Z_t)'X_t + A_Y^*(t,Z_t)Y_t\end{pmatrix}, \end{aligned} \tag{3.24}$$

where $(\bar{a}, a, A)$ *is given by eqs. (3.25)–(3.27).*

$$\bar{a}(\tau) = \int_0^\tau \left\{ \frac{1}{2}(a(s)'a(s) + \mathrm{tr}[A(s)]) - \frac{\gamma - 1}{\gamma}\lambda' a(s) - \frac{1}{\gamma^2}\left\{(\gamma - 1)\left(\frac{1}{2}\lambda'\lambda + \gamma(\bar{\rho} - \bar{\iota})\right) + \beta\gamma\right\}\right\} ds, \quad (3.25)$$

$$a(\tau) = \exp\left(\int_0^\tau (A(s) - L')ds\right) \times \int_0^\tau \left(-\frac{\gamma - 1}{\gamma}A(s)\lambda - \frac{\gamma - 1}{\gamma^2}\begin{pmatrix}\Lambda_X'\lambda_X + \gamma(\rho - \iota)\\ \Lambda_Y'\lambda_Y\end{pmatrix}\right) e^{-\int_0^s (A(s) - L')dt} ds, \quad (3.26)$$

$$A(\tau) = C_2(\tau)C_1^{-1}(\tau), \quad (3.27)$$

where

$$\begin{pmatrix}C_1(\tau)\\ C_2(\tau)\end{pmatrix} = \exp\left(\tau\begin{pmatrix}L & -E_{\mathbf{N}}\\ -Q & -L'\end{pmatrix}\right)\begin{pmatrix}E_{\mathbf{N}}\\ O\end{pmatrix}. \quad (3.28)$$

Proof. See Appendix A.5. □

3.3 Optimal Asset Allocation Example

Let I and I_n denote the number of domestic bonds (or bond groups) and of the n-th foreign bonds (or bond groups), respectively. Assume $\mathbf{N} = I + J + \sum_{n=1}^N (I_n + J_n)$. Then we can uniquely determine the optimal investment strategy.

Let Φ_t^P and Φ_t^S denote the portfolio weights on the domestic bonds and indices, respectively. Let V_t^P and V_t^S denote the volatilities of the domestic bonds and indices, respectively. Similarly, let Φ_{nt}^P and Φ_{nt}^S denote the portfolio weights on the n-th foreign bonds and indices, respectively. Let V_{nt}^P and V_{nt}^S denote the volatilities of the n-th foreign bonds and indices, respectively.

Assume that the number of domestic bond groups and those of all foreign bond groups is one, that is, $I = I_1 = \cdots = I_N = 1$, and each country's bond index is incorporated into the portfolio. Let $\omega_t(\tau)$ and $\omega_{nt}(\tau)$ denote densities of the domestic and the n-th foreign incorporation ratios of bonds with time τ to maturity.

Note that

$$\int_0^{\bar{\tau}} \omega_t(\tau)d\tau = \int_0^{\bar{\tau}} \omega_{nt}(\tau)d\tau = 1, \quad \forall n \in \{1, \cdots, N\},$$

$$V_t^P = \int_0^{\bar{\tau}} \omega_t(\tau)(\sigma(\tau) + \Sigma(\tau) X_t)' d\tau, \quad V_t^S = \begin{pmatrix} (\sigma_1 + \Sigma_1 X_t)' \\ (\sigma_2 + \Sigma_2 X_t)' \\ \vdots \\ (\sigma_J + \Sigma_J X_t)' \end{pmatrix},$$

$$V_{nt}^P = \int_0^{\tau_n} \omega_{nt}(\tau)(\sigma_n(\tau) + \Sigma_n(\tau) X_t)' d\tau, \quad V_{nt}^S = \begin{pmatrix} (\sigma_{n1} + \Sigma_{n1} X_t)' \\ (\sigma_{n2} + \Sigma_{n2} X_t)' \\ \vdots \\ (\sigma_{nJ_n} + \Sigma_{nJ_n} X_t)' \end{pmatrix},$$

for all $n \in \{1, \cdots, N\}$.

Let $\mathit{\Phi}_t$ and V_t denote the portfolio choice vector and its volatility matrix defined by

$$\mathit{\Phi}_t = \begin{pmatrix} \mathit{\Phi}_t^P \\ \mathit{\Phi}_t^S \\ \mathit{\Phi}_{1t}^P \\ \mathit{\Phi}_{1t}^S \\ \vdots \\ \mathit{\Phi}_{Nt}^P \\ \mathit{\Phi}_{Nt}^S \end{pmatrix}, \quad V_t = \begin{pmatrix} V_t^P \\ V_t^S \\ V_{1t}^P \\ V_{1t}^S \\ \vdots \\ V_{Nt}^P \\ V_{Nt}^S \end{pmatrix}.$$

Note that $\mathit{\Phi}_t$ is an $\mathbf{N} \times 1$ vector and that V_t is an $\mathbf{N} \times M$ matrix.

Then, it follows from eqs. (2.26) and (3.24) that optimal portfolio choice $\mathit{\Phi}_t$ is calculated as:

$$\begin{aligned} \mathit{\Phi}_t = & \frac{1}{\gamma} \begin{pmatrix} V_t' + \Delta\Lambda_t^X \\ \Delta\Lambda_t^Y \end{pmatrix}^{-1} \begin{pmatrix} \lambda_X + \Lambda_X X_t \\ \lambda_Y + \Lambda_Y Y_t \end{pmatrix}' \\ & + \begin{pmatrix} V_t' + \Delta\Lambda_t^X \\ \Delta\Lambda_t^Y \end{pmatrix}^{-1} \begin{pmatrix} a_X^*(t, Z_t) + A_X^*(t, Z_t) X_t + A_{XY}^*(t, Z_t) Y_t \\ a_Y^*(t, Z_t) + A_{XY}^*(t, Z_t)' X_t + A_Y^*(t, Z_t) Y_t \end{pmatrix}', \end{aligned} \tag{3.29}$$

where

$$\begin{pmatrix} \Delta\Lambda_t^X \\ \Delta\Lambda_t^Y \end{pmatrix} = \begin{pmatrix} 0_{M\times(1+J)} & \Delta\Lambda_{1t}^X & \Delta\Lambda_{2t}^X & \cdots & \Delta\Lambda_{Nt}^X \\ 0_{N\times(1+J)} & \Delta\Lambda_{1t}^Y & \Delta\Lambda_{2t}^Y & \cdots & \Delta\Lambda_{Nt}^Y \end{pmatrix},$$

where $\Delta\Lambda_{nt}^X$ is an $M \times (1 + J_n)$ matrix, and $\Delta\Lambda_{nt}^Y$ is an $N \times (1 + J_n)$ matrix, given by:

$$\begin{pmatrix} \Delta\Lambda_{nt}^X \\ \Delta\Lambda_{nt}^Y \end{pmatrix} = \begin{pmatrix} \Lambda_t^X - \Lambda_{nt}^X & \Lambda_t^X - \Lambda_{nt}^X & \cdots & \Lambda_t^X - \Lambda_{nt}^X \\ \Lambda_t^Y - \Lambda_{nt}^Y & \Lambda_t^Y - \Lambda_{nt}^Y & \cdots & \Lambda_t^Y - \Lambda_{nt}^Y \end{pmatrix},$$

for all $n \in \{1, \cdots, N\}$.

Remark 4. *To compare international asset allocation and domestic one, we consider a case of domestic asset allocation, that is,* $\mathbf{N} = I + J$ *and*

$$\tilde{\Phi}_t = \begin{pmatrix} \Phi_t^P \\ \Phi_t^S \end{pmatrix}, \qquad \tilde{V}_t = \begin{pmatrix} V_t^P \\ V_t^S \end{pmatrix}.$$

Then, the optimal domestic portfolio choice $\tilde{\Phi}_t$ *is given by*

$$\tilde{\Phi}_t = \frac{1}{\gamma}\tilde{V}_t'^{-1}\big(\lambda_X + \Lambda_X X_t\big) + \tilde{V}_t'^{-1}\big(a_X^*(t, Z_t) + A_X^*(t, Z_t)X_t\big). \tag{3.30}$$

Comparing eqs. (3.29) and (3.30), while in the optimal domestic portfolio choice, there exist the global factor and the domestic market price of global risk, in the optimal international portfolio choice, there also exist the currency-specific factor, domestic market price of currency-specific risk, the difference between the domestic and foreign market prices of global risk, and the difference between the domestic and foreign market prices of currency-specific risk. This indicates that, in international asset allocation, investors should always estimate the currency-specific factor, the domestic market price of currency-specific risk, the difference between the domestic and foreign market prices of global risk, and the difference between the domestic and foreign market prices of currency-specific risk, as well as the global factor, and the domestic market price of the global risk.

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A Proofs

A.1 Proof of Lemma 1

Assume π_t^X is given by:

$$\frac{d\pi_t^X}{\pi_t^X} = \mu_t^X dt + (\sigma_t^X)' dB_t^X. \tag{A.1}$$

Let $\tilde{S}_t$ denote any dividend-included domestic security price process. Since $\tilde{S}_t$ does not depend on the currency-specific factor by Assumption 1, the no arbitrage price process satisfies

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = (r_t + \sigma_t' \Lambda_t^X) dt + \sigma_t' \, dB_t^X. \tag{A.2}$$

Therefore, by Assumptions 1 and 2, the product of the state-price deflator and security price $\tilde{S}_t$ satisfies:

$$\begin{aligned}
\frac{d(\pi_t \tilde{S}_t)}{\pi_t \tilde{S}_t} &= \frac{d\pi_t}{\pi_t} + \frac{d\tilde{S}_t}{\tilde{S}_t} + \left(\frac{d\pi_t}{\pi_t}\right)\left(\frac{d\tilde{S}_t}{\tilde{S}_t}\right) \\
&= \frac{d\pi_t^X}{\pi_t^X} + \frac{d\pi_t^Y}{\pi_t^Y} + \frac{d\tilde{S}_t}{\tilde{S}_t} + \left(\frac{d\pi_t^X}{\pi_t^X}\right)\left(\frac{d\tilde{S}_t}{\tilde{S}_t}\right) \\
&= \left(\mu_t^X + r_t + \sigma_t'(\sigma_t^X + \Lambda_t^X)\right) dt + (\sigma_t^X + \sigma_t) dB_t^X - \Lambda_t^Y dB_t^Y.
\end{aligned}$$

By the definition of the state-price deflator, the product of the state-price deflator and the dividend-included security price is an exponential martingale, which implies:

$$\mu_t^X + r_t + \sigma_t'(\sigma_t^X + \Lambda_t^X) = 0. \tag{A.3}$$

Since the above equation is identical on σ_t, we obtain $\mu_t^X = -r_t$ and $\Lambda_t^X = -\sigma_t^X$, *i.e.*, eq. (2.5).

Secondly, we prove eq. (2.6). Note the following holds by definition of state-price deflator:

$$\pi_t = \pi_{nt}\varepsilon_t^n. \tag{A.4}$$

Therefore, substituting eq. (2.3) into eq. (A.4) and taking the logarithm of both sides of the equation yields:

$$\log \varepsilon_t^n = \log \pi_{nt}^X + \log \pi_{nt}^Y - \log \pi_t^X - \log \pi_t^Y. \tag{A.5}$$

Differentiating the above equation and substituting eqs. (2.5) and (2.4), we obtain eq. (2.6).

A.2 Proof of Lemma 2

Following Kikuchi [8], it follows from Girsanov's theorem that process $\tilde{B}_t^X$ defined by

$$\tilde{B}_t^X = B_t^X + \int_0^t \Lambda_s^X \, ds, \tag{A.6}$$

is a standard Brownian motion under the risk-neutral measure. Then, the SDE for X_t under the risk-neutral measure is rewritten as

$$\begin{aligned} dX_t &= \left(-K_X X_t - \Lambda_t^X\right) dt + d\tilde{B}_t^X \\ &= \{-\lambda_X - (K_X + \Lambda_X)X_t\} \, dt + d\tilde{B}_t^X. \end{aligned}$$

We regard the default-free bond P_t^T as a derivative written on the instantaneous interest rate r_t. Since r_t is a quadratic function of X_t, P_t^T is expressed as an analytic function $f(X_t, t)$, that is,

$$P_t^T = f(X_t, t). \tag{A.7}$$

It follows from no arbitrage condition that f is a solution to the PDE:

$$f_t + \{-\lambda_X - (K_X + \Lambda_X)X_t\}' f_X + \frac{1}{2}\mathrm{tr}[f_{XX}] - \left(\bar{\rho} + \rho' X_t + \frac{1}{2}X_t' \mathcal{R} X_t\right) f = 0,$$
$$f(X_T, T) = 1. \tag{A.8}$$

Then, f is expressed as

$$f(X_t,t) = e^{\bar{\sigma}(\tau)+\sigma(\tau)'X_t+\frac{1}{2}X_t'\Sigma(\tau)X_t}, \qquad (\bar{\sigma}(0),\sigma(0),\Sigma(0)) = (0,0,0), \quad \text{(A.9)}$$

where $\bar{\sigma}(\tau),\sigma(\tau)$, and $\Sigma(\tau)$ are analytic functions of $\tau = T-t$ and $\Sigma(\tau)$ is a symmetric matrix. Differentiating (A.9) and inserting the result into eq. (A.8), we have

$$\begin{aligned}
-\frac{d\bar{\sigma}(\tau)}{d\tau} - X_t'\frac{d\sigma(\tau)}{d\tau} - \frac{1}{2}X_t'\frac{d\Sigma(\tau)}{d\tau}X_t + \{-\lambda_X-(K_X+\Lambda_X)X_t\}'(\sigma(\tau)+\Sigma(\tau)X_t)& \\
+\frac{1}{2}\big(\sigma(\tau)'\sigma(\tau)+\mathrm{tr}[\Sigma(\tau)]\big) + X_t'\Sigma(\tau)\sigma(\tau) + \frac{1}{2}X_t'\Sigma^2(\tau)X_t& \\
-\left(\bar{\rho}+\rho'X_t+\frac{1}{2}X_t'\mathcal{R}X_t\right) = 0.& \quad \text{(A.10)}
\end{aligned}$$

Since the above equation is identical on X_t, eq. (2.16) is obtained. Finally, differentiating eq. (A.9), we obtain eq. (2.14).

On the j-th index, Kikuchi [8] shows that S_t^j is given by:

$$S_t^j = \exp\left(\bar{\sigma}_j t + \sigma_j'X_t + \frac{1}{2}X_t'\Sigma_j X_t\right). \quad \text{(A.11)}$$

Hence, the dividend rate process is

$$\frac{D_t^j}{S_t^j} = \bar{\sigma}_j + \sigma_j'X_t + \frac{1}{2}X_t'\Sigma_j X_t. \quad \text{(A.12)}$$

Then, the following identical equation on X_t is obtained from eqs. (A.11) and (A.12) and no arbitrage condition that

$$\begin{aligned}
\bar{\sigma}_j + \{-\lambda_X-(K_X+\Lambda_X)X_t\}'(\sigma_j+\Sigma_j X_t) + \frac{1}{2}\big(\sigma_j'\sigma_j+\mathrm{tr}[\Sigma_j]\big) + X_t'\Sigma_j\sigma_j + \frac{1}{2}X_t'\Sigma_j^2X_t& \\
+\left(\bar{\delta}_j+\delta_j'X_t+\frac{1}{2}X_t'\Delta_jX_t\right) - \left(\bar{\rho}+\rho'X_t+\frac{1}{2}X_t'\mathcal{R}X_t\right) = 0.& \quad \text{(A.13)}
\end{aligned}$$

Thus, we have eq. (2.19).

On the n-th foreign country's default-free bond, the following equation holds from the arbitrage-free condition:

$$\frac{dP_{nt}^T}{P_{nt}^T} = \big(r_{nt} + (\sigma_n(\tau)+\Sigma_n(\tau)X_t)'\Lambda_{nt}^X\big)\,dt + (\sigma_n(\tau)+\Sigma_n(\tau)X_t)'dB_t^X, \quad \text{(A.14)}$$

Then, we have eq. (2.20). In a similar way, we obtain eq. (2.23).

A.3 Proof of Lemma 3

Let $(\vartheta, (\vartheta(\tau)), (\vartheta^j), (\vartheta_n(\tau)), (\vartheta_n^j))$ denote a portfolio. The nominal wealth is given by:

$$\tilde{W}_t = \vartheta_t P_t + \int_0^{\bar{\tau}} \vartheta_t(\tau) P_t(\tau) d\tau + \sum_{j=1}^{J} \vartheta_t^j S_t^j + \sum_{n=1}^{N} \int_0^{\tau_n} \vartheta_{nt}(\tau) P_{nt}(\tau) d\tau + \sum_{n=1}^{N} \sum_{j=1}^{J_n} \vartheta_{nt}^j S_{nt}^j. \tag{A.15}$$

Then, given c_t, the self-financing portfolio $(\vartheta, (\vartheta(\tau)), (\vartheta^j), (\vartheta_n(\tau)), (\vartheta_n^j))$ satisfies

$$\begin{aligned}
\frac{d\tilde{W}_t}{\tilde{W}_t} &= \frac{1}{\tilde{W}_t}\Bigg\{\vartheta_t dP_t + \int_0^{\bar{\tau}} \vartheta_t(\tau) dP_t(\tau) d\tau + \sum_{j=1}^{J} \vartheta_t^j \Big(dS_t^j + D_t^j dt\Big) \\
&\quad + \sum_{n=1}^{N} \int_0^{\tau_n} \vartheta_{nt}(\tau) dP_{nt}(\tau) d\tau + \sum_{n=1}^{N} \sum_{j=1}^{J_n} \vartheta_{nt}^{*j} \Big(dS_{nt}^j + D_{nt}^j dt\Big) - \frac{p_t}{\tilde{W}_t} c_t dt\Bigg\} \\
&= \frac{\vartheta_t P_t}{\tilde{W}_t}\frac{dP_t}{P_t} + \int_0^{\bar{\tau}} \frac{\vartheta_t(\tau) P_t(\tau)}{\tilde{W}_t}\frac{dP_t(\tau)}{P_t(\tau)} d\tau + \sum_{j=1}^{J} \frac{\vartheta_t^j S_t^j}{\tilde{W}_t}\frac{dS_t^j + D_t^j dt}{S_t^j} \\
&\quad + \sum_{n=1}^{N} \int_0^{\tau_n} \frac{\vartheta_{nt}(\tau) P_{nt}(\tau)}{\tilde{W}_t}\frac{dP_{nt}(\tau)}{P_{nt}(\tau)} d\tau + \sum_{n=1}^{N} \sum_{j=1}^{J_n} \frac{\vartheta_{nt}^j S_{nt}^j}{\tilde{W}_t}\frac{dS_{nt}^j + D_{nt}^j dt}{S_{nt}^j} - \frac{c_t}{W_t} dt\Bigg\} \\
&= \left(1 - \int_0^{\bar{\tau}} \varphi_t(\tau) d\tau - \sum_{j=1}^{J} \Phi_t^j - \sum_{n=1}^{N} \int_0^{\tau_n} \varphi_{nt}(\tau) d\tau - \sum_{n=1}^{N} \sum_{j=1}^{J_n} \Phi_{nt}^j\right) \frac{dP_t}{P_t} \\
&\quad + \int_0^{\bar{\tau}} \varphi_t(\tau) \frac{dP_t(\tau)}{P_t(\tau)} d\tau + \sum_{j=1}^{J} \Phi_t^j \frac{dS_t^j + D_t^j dt}{S_t^j} \\
&\quad + \sum_{n=1}^{N} \int_0^{\tau_n} \varphi_{nt}(\tau) \frac{dP_{nt}(\tau)}{P_{nt}(\tau)} d\tau + \sum_{n=1}^{N} \sum_{j=1}^{J_n} \Phi_{nt}^j \frac{dS_{nt}^j + D_t^j dt}{S_{nt}^j} - \frac{c_t}{W_t} dt.
\end{aligned}$$

Thus, the SDE for the real wealth process is derived as

$$\begin{aligned}
\frac{dW_t}{W_t} &= \frac{d\tilde{W}_t}{\tilde{W}_t} - i_t dt \\
&= \left(1 - \int_0^{\bar{\tau}} \varphi_t(\tau) d\tau - \sum_{j=1}^{J} \Phi_t^j - \sum_{n=1}^{N} \int_0^{\tau_n} \varphi_{nt}(\tau) d\tau - \sum_{n=1}^{N} \sum_{j=1}^{J_n} \Phi_{nt}^j\right) \frac{dP_t}{P_t}
\end{aligned}$$

$$+\int_0^{\bar{\tau}} \varphi_t(\tau)\frac{dP_t(\tau)}{P_t(\tau)}d\tau + \sum_{j=1}^{J} \Phi_t^j \frac{dS_t^j + D_t^j dt}{S_t^j}$$
$$+\sum_{n=1}^{N}\int_0^{\tau_n} \varphi_{nt}(\tau)\frac{dP_{nt}(\tau)}{P_{nt}(\tau)}d\tau + \sum_{n=1}^{N}\sum_{j=1}^{J_n} \Phi_{nt}^j \frac{dS_{nt}^j + D_{nt}^j dt}{S_{nt}^j} - \frac{c_t}{W_t}dt.$$

Substituting eqs. (2.13), (2.14), (2.17), (2.20), and (2.23) into the above eq. and organizing the result yield eq. (2.27).

A.4 Proof of Proposition 1

First, the optimal consumption control (3.10) is obtained as follows:

$$c_t^* = \alpha^{\frac{1}{\gamma}} e^{-\frac{\beta}{\gamma}t} J_W^{-\frac{1}{\gamma}} = \alpha^{\frac{1}{\gamma}} e^{-\frac{\beta}{\gamma}t} \left\{ e^{-\beta t}(W_t^*)^{-\gamma} G^{\gamma} \right\}^{-\frac{1}{\gamma}} = \alpha^{\frac{1}{\gamma}} \frac{W_t^*}{G}.$$

Then, inserting c_t^* into budget-constraint (2.27) and solving the SDE, we have eq. (3.11).

Second, the derivatives of J are given by

$$J_t = -\beta J, \qquad W_t J_W = (1-\gamma)J, \qquad J_X = \gamma J \frac{G_X}{G}, \qquad J_Y = \gamma J \frac{G_Y}{G},$$

$$W_t^2 J_{WW} = -\gamma(1-\gamma)J, \qquad W_t J_{XW} = \gamma(1-\gamma)J\frac{G_X}{G}, \qquad W_t J_{YW} = \gamma(1-\gamma)J\frac{G_Y}{G},$$

$$J_{XX} = \gamma J \left\{ (\gamma-1)\frac{G_X}{G}\frac{G'_X}{G} + \frac{G_{XX}}{G} \right\}, \qquad J_{YY} = \gamma J \left\{ (\gamma-1)\frac{G_Y}{G}\frac{G'_Y}{G} + \frac{G_{YY}}{G} \right\}.$$

Then, the numerator and the denominator on the right-hand side of eq. (3.12) are rewritten as:

$$\psi_t = J\left((\gamma-1)\begin{pmatrix} \Lambda_t^X \\ \Lambda_t^Y \end{pmatrix} + \gamma(\gamma-1)\frac{1}{G}\begin{pmatrix} G_X \\ G_Y \end{pmatrix} \right), \tag{A.16}$$

$$W_t^2 J_{WW} = \gamma(\gamma-1)J. \tag{A.17}$$

Therefore, inserting eqs. (A.16) and (A.17) into eq. (3.3), we obtain eq. (3.12). The second and third terms in eq. (3.7) are calculated from eqs. (A.16) and

(A.17) as:

$$
\begin{aligned}
&\frac{1}{2}\operatorname{tr}\left[J_{XX}+J_{YY}\right]-\frac{\psi_t'\psi_t}{2W_t^2J_{WW}}\\
&\quad=\frac{\gamma J}{2}\operatorname{tr}\left[\left((\gamma-1)\frac{G_X}{G}\frac{G_X'}{G}+\frac{G_{XX}}{G}\right)+\left((\gamma-1)\frac{G_Y}{G}\frac{G_Y'}{G}+\frac{G_{YY}}{G}\right)\right]\\
&\qquad-\frac{(\gamma-1)J}{2\gamma}\left(\begin{pmatrix}\Lambda_t^X\\ \Lambda_t^Y\end{pmatrix}+\frac{\gamma}{G}\begin{pmatrix}G_X\\ G_Y\end{pmatrix}\right)'\left(\begin{pmatrix}\Lambda_t^X\\ \Lambda_t^Y\end{pmatrix}+\frac{\gamma}{G}\begin{pmatrix}G_X\\ G_Y\end{pmatrix}\right)\\
&=\frac{\gamma J}{G}\left\{\frac{1}{2}\operatorname{tr}\left[G_{XX}+G_{YY}\right]-\frac{\gamma-1}{\gamma}\begin{pmatrix}\Lambda_t^X\\ \Lambda_t^Y\end{pmatrix}'\begin{pmatrix}G_X\\ G_Y\end{pmatrix}-\frac{\gamma-1}{2\gamma^2}\begin{pmatrix}\Lambda_t^X\\ \Lambda_t^Y\end{pmatrix}'\begin{pmatrix}\Lambda_t^X\\ \Lambda_t^Y\end{pmatrix}G\right\}.
\end{aligned}
\tag{A.18}
$$

The seventh term in eq. (3.7) is calculated from eq. (3.2) as

$$
\frac{\gamma}{1-\gamma}c_t^*J_W=\alpha^{\frac{1}{\gamma}}\frac{W_t^*}{G}\frac{\gamma J}{W_t^*}=\alpha^{\frac{1}{\gamma}}\frac{\gamma J}{G}. \tag{A.19}
$$

Substituting eqs. (A.18) and (A.19) into eq. (3.7), and dividing by $\gamma J/G$ yield eq. (3.13).

A.5 Proof of Proposition 2

It is straightforward to see that $\bar{a}(\tau)$ and $a(\tau)$ are expressed as eqs. (3.25) and (3.26). Following Theorem 5.2 in Arimoto [1], we prove $A(\tau)$ is expressed as eq. (3.27). We consider the following initial value problem of the linear differential equation for the $N\times N$ matrix-value functions $C_1(\tau)$ and $C_2(\tau)$:

$$
\frac{d}{d\tau}\begin{pmatrix}C_1(\tau)\\ C_2(\tau)\end{pmatrix}=\begin{pmatrix}L & -E_{\mathbf{N}}\\ -Q & -L'\end{pmatrix}\begin{pmatrix}C_1(\tau)\\ C_2(\tau)\end{pmatrix},\qquad \begin{pmatrix}C_1(0)\\ C_2(0)\end{pmatrix}=\begin{pmatrix}E_{\mathbf{N}}\\ 0_{\mathbf{N}}\end{pmatrix}. \tag{A.20}
$$

A solution to eq. (A.20) is given by eq. (3.28). Since we can prove $C_1(\tau)$ to be regular,[3] we define $A(\tau)$ by eq. (3.27). Then, noting that

$$
\frac{d}{d\tau}C_1^{-1}(\tau)=-C_1^{-1}(\tau)\left\{\frac{d}{d\tau}C_1(\tau)\right\}C_1^{-1}(\tau), \tag{A.21}
$$

we can derive

$$
\begin{aligned}
\frac{d}{d\tau}A(\tau) &= \left\{\frac{d}{d\tau}C_2(\tau)\right\}C_1^{-1}(\tau)+C_2(\tau)\frac{d}{d\tau}C_1^{-1}(\tau)\\
&= \left(-QC_1(\tau)-L'C_2(\tau)\right)C_1^{-1}(\tau)-A(\tau)\left(LC_1(\tau)-C_2(\tau)\right)C_1^{-1}(\tau)\\
&= A^2(\tau)-L'A(\tau)-A(\tau)L-Q,
\end{aligned}
$$

[3] See the proof for Theorem 5.2 in Arimoto [1].

and thus confirm that $A(\tau)$ satisfies matrix differential Riccati equation (3.21). For the uniqueness of the Riccati equation, see the proof of Theorem 5.2 in Arimoto [1]. Finally, for the symmetry of $A(\tau)$, taking the transposition of Riccati equation (3.21) for $A(\tau)$ yields the same equation for $A(\tau)'$, which implies that $A(\tau)' = A(\tau)$ because of the uniqueness of the Riccati equation.