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Thermodynamic structure of a macroeconomic model

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Abstract

This paper reports a novel mathematical structure of economic models with rational agents. Taking a cash-in-advance (CIA) model as an example, I show that macroscopic conservation and irreversibility laws, which are similar to those in thermodynamics, hold for the model. These properties allow for defining internal energy, Helmholtz free energy, entropy, and temperature. Thermodynamic relations among these quantities are also proven for the model. Possible extensions to and implications for macroeconomic models are discussed.

1 Introduction

Social sciences have long aimed to describe the society using models. Among the social sciences, economics is the most successful in establishing such models. Economics focuses on quantities in society and establishes relations among them. Among the established relations are identities such as the savings–investment identity that states that the amount of savings equals the amount of investment. These identities are derived from the balance between different sectors of an economy. The identity relationship between different quantities is an important principle that supports economics as a quantitative science.

In addition to balance, the rationality of agents is another important principle for developing economic models. In the framework of the rational expectation hypothesis (Muth, 1961; Lucas, 1972), agents in a system are assumed to forecast and behave rationally. Although it is unrealistic to assume that all agents perfectly know every detail of the economy and choose the optimal action (Tversky and Kahneman, 1974; Hommes, 2011), assuming rational agents is quite useful in constructing and analyzing models. Consequently, majority of macroeconomic models are based on this hypothesis. In these models, rational agents choose actions that maximize their utility, that is, total satisfaction. Micro-founded macroeconomic models describe the collective behavior of utility-maximizing agents. These models have been successfully applied in economic forecasting and policy experiments (Del Negro and Schorfheide, 2012). Hence, the mathematical structure of macroeconomic models with rational agents deserves precise investigation.

Let us speculate over the mathematical structure that can emerge from the assumption of rational agents. Here, I present an intuitive argument and leave a detailed proof of a model to the following sections. Rational agents act optimally to maximize their utility. In a perfectly competitive market, a non-optimal action leads to a decrease in utility, which is accompanied by a loss of money. In other words, a not-so-rational agent, X, in an economy with rational agents cannot earn money because all other agents take advantage of the non-optimality of agent X. Strictly speaking, the profit of agent X trading with other agents in the economy is nonpositive if the net amount of capital goods owned by agent X does not change. To further examine the consequence of a loss, let us define the state of the economy by the amount of money and capital owned by agent X. Even if agent X can change the state of the economy from state A to state B, this does not necessarily mean that agent X can reverse the change. On the contrary, as we see in the following sections, most changes are irreversible because agent X cannot regain the loss. This irreversibility of the loss implies that the set of states can be regarded as an ordered set. There exists a function $S(\cdot)$, and $S(A) \leq S(B)$ is satisfied if agent X can change the state of the economy from state A to state B.

In economics, nonpositivity and irreversibility have attracted less attention than identity the relationship. However, the combination of identity and irreversibility is a fundamental concept of thermodynamics (Callen, 1985; Tasaki and Paquette, 2018), which is one of the most universal theories in physics (Einstein and Schilpp, 1979). Identity and irreversibility in macroeconomic models can be regarded as conservation and irreversibility laws, respectively, in thermodynamics.

The objective of thermodynamics is to describe the macroscopic properties of a system. As an example, consider gas confined by a piston in a cylinder. We can measure the volume and temperature of the gas and push and pull the piston to change them. Pushing the piston decreases the volume and increases the temperature, and pulling the piston increases the volume and decreases the temperature. The gas is the system. The volume and temperature are the variables describing the state of the system. Pushing and pulling the piston and the resulting change in the state of the system are called a process. A thermodynamic process can be isothermal or adiabatic. In an isothermal process, the system is in equilibrium with a temperature reservoir or an isothermal environment. The system exchanges heat with the environment, and the temperature of the system converges to that of the environment. In an adiabatic process, the system is isolated from the environment.

When the piston is pushed and pulled, the system does work on the piston and the instruments connected to it. The first law of thermodynamics is related to the conservation of energy: the amount of work done by the system in an adiabatic process equals the amount of heat that the system loses. The second law of thermodynamics states the irreversibility of some processes: a cyclic process of quickly pushing the piston and pulling it back to the same position can raise the temperature of the system without an overall change in the volume. In an adiabatic process, we cannot reduce the temperature of the system without decreasing the volume. We can reduce the temperature only by bringing the system into contact with a system or isothermal environment with a lower temperature. A temperature increase in an adiabatic cycle means that work on the system is converted to heat and, consequently, that the work done by the system in an adiabatic cycle is nonpositive. Whether we can change the state of a system from A to B in an adiabatic process is determined by comparing a quantity called entropy between these two states. A quasi-static adiabatic process—an infinitely slow adiabatic process—does not change the entropy of the system and, consequently, is reversible.

These considerations suggest an analogy between thermodynamic systems and macroeconomic models. Nonpositive work in thermodynamic systems corresponds to nonpositive profit in economic models. Thermodynamics describes the macroscopic properties of a system that is microscopically a collection of molecules. The macroscopic properties of an economy which is microscopically a collection of agents might be described using a thermodynamic framework. This paper provides a novel mathematical approach to macroeconomic models with rational agents based on identities among quantities and the nonpositivity of profit or the irreversibility of loss. This approach allows for defining thermodynamic functions for a simple macroeconomic model and deriving previously unknown relations.

This paper is organized as follows. Section 2 introduces the model and derives its equilibrium solution. Section 3 proves that the profit of a not necessarily rational agent, which is called the government for convenience, is nonpositive. Section 4 defines infinitely slow trade between the government and the economy, which is a quasi-static adiabatic process in terms of thermodynamics, and derives a quantity kept constant in this process, with which entropy is defined. Section 5 examines the behavior of the model in a constant-price environment and shows that this is equivalent to an isothermal environment in thermodynamics. Section 6 shows that the profit of the government that trades with a system of multiple economies is nonpositive if these economies belong to a model class. Section 7 derives thermodynamic functions including internal energy, entropy, Helmholtz free energy, and temperature. Thermodynamic relations among them are also proven. Section 8 evaluates the work done in a Carnot cycle, which exemplifies the consistency of the derived thermodynamic functions. Section 9 summarizes the results and discusses possible extensions and economic implications.

2 CIA model

I consider a cash-in-advance (CIA) model (Lucas, 1980, 1982) with money, goods, and land. The model contains a representative household, a representative firm, a government, and a central bank.

In maximizing consumption and minimizing labor to earn the money needed for consumption, the household maximizes its own utility. Let

$$U = \sum_{t=1}^{\infty} \beta^t [u(C_t) - d(L_t)] \tag{1}$$

be the lifetime utility function of the representative household, where C_t and L_t are the consumption and labor during period t, respectively, $0 < \beta < 1$ is the discount factor, and $u(\cdot)$ and $d(\cdot)$ are increasing concave and increasing convex functions, respectively; that is,

$$u'(C) \geq 0, \tag{2}$$

$$u''(C) \leq 0, \tag{3}$$

$$d'(L) \geq 0, \tag{4}$$

$$d''(L) \geq 0. \tag{5}$$

The household owns money M_{t-1} at the beginning of period t, earns money $W_t L_t$, consumes $P_t C_t$, and saves money M_t for consumption in the next period, where P_t, M_t, and W_t are the price level, the nominal amount of money, and the nominal wage, respectively. In addition, the household owns land A_t that is lent to the firm at rental price R_t and used in production, and purchases it from the government or sells it to the government at land price Q_t at the beginning of period t. The government does not use the land it owns. Land is capital that does not depreciate. Thus, the budget constraint of the household in period t is

$$P_t C_t + Q_t (A_t - A_{t-1}) + M_t = W_t L_t + R_t A_t + M_{t-1}, \tag{6}$$

and the CIA constraint is

$$P_tC_t + Q_t(A_t - A_{t-1}) \leq M_{t-1}. \tag{7}$$

The cash which the government uses to purchase land in period t can be spent by the household later in the period.

The central bank provides the money needed for government purchases of land. The nominal money supply is

$$M_t = M_{t-1} + Q_t(A_{t-1} - A_t). \tag{8}$$

Although for simplicity I refer to the agent purchasing land and increasing debt as the government and the central bank, the present model applies to any agent that can purchase land only by increasing debt. In the following, I characterize what this agent can do in the economy.

The production function of the representative firm is assumed to be

$$Y_t = A_t y(l_t), \tag{9}$$

where $l_t = L_t/A_t$ and $y(l)$ is a positive increasing concave function of l, that is,

$$y(0) \geq 0, \tag{10}$$

$$y'(l) \geq 0, \tag{11}$$

$$y''(l) \leq 0. \tag{12}$$

Because the profit of the firm is $P_tY_t - W_tL_t - R_tA_t$, the first-order condition of optimality is given by

$$W_t = P_t\frac{\partial Y_t}{\partial L_t} = P_t y'(l_t), \tag{13}$$

$$R_t = P_t\frac{\partial Y_t}{\partial A_t} = P_t\hat{y}(l_t), \tag{14}$$

where I define $\hat{y}(l) = y(l) - ly'(l)$. Note that $W_tL_t + R_tA_t = P_tY_t$. R_t is an increasing nonnegative function of l_t because

$$\hat{y}(0) = y(0) \geq 0, \tag{15}$$

$$\frac{\mathrm{d}\hat{y}(l)}{\mathrm{d}l} = y'(l) - y'(l) - ly''(l) \geq 0. \tag{16}$$

Because the model describes a closed economy, the goods market clearing condition is

$$C_t = Y_t. \tag{17}$$

The Lagrangian of the optimization problem for the household is given by

$$\Lambda = \sum_{t=1}^{\infty} \beta^t \{u(C_t) - d(L_t) + \lambda_t[W_tL_t + R_tA_t + M_{t-1} - P_tC_t - Q_t(A_t - A_{t-1}) - M_t]$$

$$+ \xi_t[M_{t-1} - P_tC_t - Q_t(A_t - A_{t-1})]\}. \tag{18}$$

The first-order condition of optimality is

$$\frac{\partial \Lambda}{\partial C_t} = u'(C_t) - P_t\lambda_t - P_t\xi_t = 0, \tag{19}$$

$$\frac{\partial \Lambda}{\partial L_t} = W_t\lambda_t - d'(L_t) = 0, \tag{20}$$

$$\frac{\partial \Lambda}{\partial M_t} = \beta\lambda_{t+1} - \lambda_t + \beta\xi_{t+1} = 0, \tag{21}$$

$$\frac{\partial \Lambda}{\partial A_t} = R_t\lambda_t - Q_t\lambda_t + \beta Q_{t+1}\lambda_{t+1} - Q_t\xi_t + \beta Q_{t+1}\xi_{t+1} = 0. \tag{22}$$

The equations are rearranged as

$$d'(L_t) = \frac{\beta W_t u'(C_{t+1})}{P_{t+1}}, \tag{23}$$

$$\frac{Q_t u'(C_t)}{P_t} = \frac{\beta(Q_{t+1} + R_t)u'(C_{t+1})}{P_{t+1}}, \tag{24}$$

$$\lambda_t = \frac{\beta u'(C_{t+1})}{P_{t+1}}, \tag{25}$$

$$\xi_t = \frac{u'(C_t)}{P_t} - \frac{\beta u'(C_{t+1})}{P_{t+1}}. \tag{26}$$

By replacing variables and functions with A, M, $\tilde{C}(A)$, $\tilde{L}(A)$, $\tilde{P}(A, M)$, $\tilde{Q}(A, M)$, $\tilde{R}(A, M)$, $\tilde{W}(A, M)$, $\tilde{\lambda}(A, M)$, and $\tilde{\xi}(A, M)$ in Eqs. 7, 9, 23, 24, 25, and 26, we obtain the steady-state equations

$$\tilde{P}(A, M) = \frac{M}{\tilde{C}(A)}, \tag{27}$$

$$\tilde{C}(A) = Ay(\tilde{l}(A)), \tag{28}$$

$$d'(\tilde{L}(A)) = \beta y'(\tilde{l}(A))u'(\tilde{C}(A)), \tag{29}$$

$$\tilde{Q}(A, M) = \frac{\beta\tilde{R}(A, M)}{1-\beta} = \frac{\beta M\hat{y}(\tilde{l}(A))}{(1-\beta)\tilde{C}(A)} \equiv M\tilde{q}(A), \tag{30}$$

$$\tilde{\lambda}(A, M) = \frac{\beta u'(\tilde{C}(A))\tilde{C}(A)}{M}, \tag{31}$$

$$\tilde{\xi}(A, M) = \frac{(1-\beta)u'(\tilde{C}(A))\tilde{C}(A)}{M}, \tag{32}$$

where $\tilde{l}(A) = \tilde{L}(A)/A$. The CIA constraint binds at the steady state because $\tilde{\xi}(A, M) > 0$. In the following, steady-state values are indicated by tilde.

First, let us examine the dependence of $\tilde{l}(A)$ on land A. Rearranging Eq. 29 yields

$$d'(\tilde{l}(A)A) = \beta y'(\tilde{l}(A))u'(Ay(\tilde{l}(A))). \tag{33}$$

Both increasing A and increasing $\tilde{l}(A)$ increase the left-hand side and decrease the right-hand side of Eq. 33 (Eqs. 3, 5, and 12). Thus, if $A = A_0$ and $\tilde{l}(A) = l_0$ satisfy Eq. 33, $l \geq l_0$ satisfies Eq. 33 for $A \leq A_0$, and $l \leq l_0$ satisfies Eq. 33 for $A \geq A_0$; that is,

$$\frac{\mathrm{d}\tilde{l}(A)}{\mathrm{d}A} = \frac{\mathrm{d}}{\mathrm{d}A}\left(\frac{\tilde{L}(A)}{A}\right) \leq 0. \tag{34}$$

Second, let us examine the dependence of $\tilde{q}(A)$ on A. The derivative of the denominator of Eq. 30 with respect to A is

$$\frac{\mathrm{d}}{\mathrm{d}A}[Ay(\tilde{l}(A))] = y(\tilde{l}(A)) - Ay'(\tilde{l}(A))\frac{\mathrm{d}\tilde{l}(A)}{\mathrm{d}A} \geq 0, \tag{35}$$

which follows from Eqs. 10, 11, and 34. The numerator is a decreasing function of A,

$$\frac{\mathrm{d}}{\mathrm{d}A}\hat{y}(\tilde{l}(A)) \leq 0 \tag{36}$$

which follows from Eqs. 16 and 34. Thus, we have

$$\frac{\mathrm{d}\tilde{q}(A)}{\mathrm{d}A} \leq 0. \tag{37}$$

Last, let us examine what happens when the government purchases or sells land. Let us assume that the household solves the optimization problem in period t' as follows: $A_{t'-1}$ is the initial condition; $A_{t'}$ is determined by the government's purchase in period t'; and the household expects that $A_t = A_{t'} = A$ for $t \geq t'$. In other words, the household assumes that the government does not purchase or sell land in period $t > t'$. The optimality conditions (Eqs. 23, 24, 25, and 26) are satisfied by

$$\begin{aligned}
C_t =& \tilde{C}(A), && (38)\\
L_t =& \tilde{L}(A), && (39)\\
P_t =& \tilde{P}(A, M), && (40)\\
Q_t =& \tilde{Q}(A, M), && (41)\\
\lambda_t =& \tilde{\lambda}(A, M), && (42)\\
\xi_t =& \tilde{\xi}(A, M) && (43)
\end{aligned}$$

for $t \geq t'$ under the condition $M_{t'} = M$. Hence, the model household instantaneously jumps to the steady state in response to the government's purchase or sale of land.

3 The government's profit cannot be positive

Here, I prove that the government's profit is equal to or less than zero in a sequence of purchases and sales if the amount of land owned by the government at the beginning of the sequence, $t = 0$, equals that at the end of the sequence, $t = \tau$. Let us assume that the time course of the government's purchases is $A_{t-1} - A_t$ $(1 \le t \le \tau)$ and that $A_0 = A_\tau$. Inserting Eq. 30 with $A = A_t$ and $M = M_t$ into Eq. 8 yields

$$M_t = \frac{M_{t-1}}{1 - \tilde{q}(A_t)(A_{t-1} - A_t)}. \tag{44}$$

The government's profit in the sequence is

$$\begin{aligned}\Pi =& M_0 - M_\tau \\ =& M_0 - M_0 \prod_{t=1}^{\tau} \frac{1}{1 - \tilde{q}(A_t)(A_{t-1} - A_t)} \\ =& M_0 - M_0 \exp I,\end{aligned} \tag{45}$$

where

$$I = \sum_{t=1}^{\tau} f[\tilde{q}(A_t)(A_t - A_{t-1})] \tag{46}$$

and $f(x) = -\log(1 + x)$. Because $f(x)$ is a convex function, we have

$$f(x) \ge f(0) + f'(0)x = -x. \tag{47}$$

Let us define $a(t)$ by

$$a(t) = A_{\lfloor t \rfloor} + (A_{\lfloor t \rfloor + 1} - A_{\lfloor t \rfloor})(t - \lfloor t \rfloor), \tag{48}$$

where $\lfloor t \rfloor$ is the greatest integer less than or equal to t. Using these functions and Eq. 37, we obtain

$$\begin{aligned} I =& \int_0^\tau f\left(\tilde{q}(A_{\lfloor t \rfloor + 1}) \frac{\mathrm{d}}{\mathrm{d}t} a(t)\right) \mathrm{d}t \\ \ge& -\int_0^\tau \tilde{q}(A_{\lfloor t \rfloor + 1}) \frac{\mathrm{d}}{\mathrm{d}t} a(t) \, \mathrm{d}t \\ \ge& -\int_0^\tau \tilde{q}(a(t)) \frac{\mathrm{d}}{\mathrm{d}t} a(t) \, \mathrm{d}t \\ =& -\int_{a(0)}^{a(\tau)} \tilde{q}(a) \, \mathrm{d}a = 0. \end{aligned} \tag{49}$$

Therefore, combining it with Eq. 45, we have

$$\Pi \le 0. \tag{50}$$

Planck's principle, a version of the second law of thermodynamics, states that the work done by a system in an adiabatic cycle is nonpositive. If profit in the present model corresponds to work in thermodynamics, we can regard a sequence of purchases and sales with $A_0 = A_\tau$ as an adiabatic cycle. Moreover, Eq. 50 implies that a sequence of purchases and sales of land between the government and the economy can be regarded as an adiabatic process even if $A_0 = A_\tau$ is not satisfied. Because an adiabatic process occurs in a system isolated from an external environment, it is reasonable to regard a sequence of purchases and sales in an economy isolated from other economies as an adiabatic process.

In the following, I specify the economy by assuming

$$u'(C) = C^{-\eta}, \tag{51}$$

$$d'(L) = \mu L^{\gamma}, \tag{52}$$

$$y(l) = l^{1-\alpha}, \tag{53}$$

where η, γ, μ, and α are constants satisfying $0 < \eta \leq 1$, $\gamma > 0$, $\mu > 0$, and $0 < \alpha < 1$. Equation 53 defines the Cobb-Douglas production function. The steady-state solution is

$$\tilde{C}(A) = \theta A^{\nu}, \tag{54}$$

$$\tilde{L}(A) = \theta^{1/(1-\alpha)} A^{\kappa}, \tag{55}$$

$$\tilde{P}(A, M) = \theta^{-1} M A^{-\nu}, \tag{56}$$

$$\tilde{Q}(A, M) = \zeta M A^{-1}, \tag{57}$$

where

$$\zeta = \frac{\alpha\beta}{1-\beta}, \tag{58}$$

$$\nu = \frac{\alpha(\gamma+1)}{\alpha+\gamma+\eta(1-\alpha)}, \tag{59}$$

$$\kappa = \frac{\alpha(1-\eta)}{\alpha+\gamma+\eta(1-\alpha)}, \tag{60}$$

$$\theta = \left(\frac{\beta}{\mu}(1-\alpha)\right)^{\frac{1-\alpha}{\alpha+\gamma+\eta(1-\alpha)}}. \tag{61}$$

We note that $0 < \zeta < 1$, $0 < \nu < 1$, $0 < \kappa < 1$, and $\theta > 0$. For the model specified by Eqs. 51, 52, and 53, combining Eq. 44 and $\tilde{q}(A) = \zeta A^{-1}$ gives

$$M_t = \frac{M_{t-1}}{1 - \zeta(\frac{A_{t-1}}{A_t} - 1)}. \tag{62}$$

4 Quasi-Static adiabatic process

Although the government's profit cannot be positive, the profit may be close to zero if the government does its best not to change the land price, which can be achieved by infinitely slow trade between the government and the economy. In this case, the relation $f(x) = -x$ exactly holds, and

$$I = -\int_{A_0}^{A_\tau} \tilde{q}(a)\,\mathrm{d}a. \tag{63}$$

Thus, we have

$$M_{\text{final}} = M_{\text{initial}} \exp\left(-\int_{A_{\text{initial}}}^{A_{\text{final}}} \tilde{q}(A)\,\mathrm{d}A\right). \tag{64}$$

Performing infinitely slow purchases and sales from A_{initial} to A_{final} and from A_{final} to A_{initial} in sequence results in the money supply

$$M_{\text{initial}} \exp\left(-\int_{A_{\text{initial}}}^{A_{\text{final}}} \tilde{q}(A)\,\mathrm{d}A - \int_{A_{\text{final}}}^{A_{\text{initial}}} \tilde{q}(A)\,\mathrm{d}A\right) = M_{\text{initial}}, \tag{65}$$

that is, the money supply and the price level do not change, which means that the government's profit in this cycle equals zero. Hence, this can be regarded as a quasi-static adiabatic process in thermodynamics because work done by a system in a quasi-static adiabatic cycle equals zero. Rearranging the terms of Eq. 64 yields

$$M_{\text{final}} \exp\left(\int_{A_{\text{reference}}}^{A_{\text{final}}} \tilde{q}(A)\,\mathrm{d}A\right) = M_{\text{initial}} \exp\left(\int_{A_{\text{reference}}}^{A_{\text{initial}}} \tilde{q}(A)\,\mathrm{d}A\right), \tag{66}$$

where $A_{\text{reference}}$ is an arbitrary reference point. This suggests that

$$s(A, M) = M \exp\left(\int_{A_{\text{reference}}}^{A} \tilde{q}(a)\,\mathrm{d}a\right) \tag{67}$$

is conserved in a quasi-static adiabatic process.

For the model specified by Eqs. 51, 52, and 53, we have

$$M_{\text{final}} = M_{\text{initial}} \left(\frac{A_{\text{initial}}}{A_{\text{final}}}\right)^{\zeta}, \tag{68}$$

$$\begin{aligned} s(A, M) &= M A^{\zeta} \\ &= \theta P(A, M) A^{\nu+\zeta}, \end{aligned} \tag{69}$$

where I used Eq. 56 and set $A_{\text{reference}}$ to 1.

5 CIA model in a constant-price environment

Because adiabatic processes have been characterized in previous sections, I next consider isothermal processes (Callen, 1985; Tasaki and Paquette, 2018). In thermodynamics, a system is in an environment with a fixed temperature at the beginning and end of an isothermal process. In the present model, to determine what corresponds to an isothermal process, we must define an environment with a fixed temperature.

If a small economy trades goods with an infinitely large economy, the government's purchases of land in the small economy minimally affect the price level of goods. Let us assume that the goods are traded between two economies, and that land is not. The large economy can be regarded as an environment or a reservoir to keep the price level constant. The price level in the small economy is assumed to be fixed to the price level of the large economy, that is, $P_t = P$. Hence, I define the environment as a large economy in which the price level is constant. This is equivalent to regarding the price level as temperature. As we will see in the following sections, this equivalency can be justified for some models.

An isothermal process is different from an adiabatic process in two ways. First, in an isothermal process, consumption in the small economy does not necessarily equal its production. The small economy may sell goods to the large economy or buy goods from the large economy. Therefore, the market clearing condition (Eq. 17) does not hold. Second, in an isothermal process, two economies trade goods between them so that the amount of money owned by the household of the small economy is not constrained by Eq. 8. The money issued by the central bank may be passed from the small economy to the large economy, and the small economy may pass money from the large economy to the government.

However, in the steady state, a small economy trading goods with a large economy consumes the same amount of goods as that produced by itself. In other words, the small economy is identical to an isolated economy or an economy in an adiabatic process. The steady state of a small economy trading goods with a large economy is given by Eqs. 28 and 29 and

$$\tilde{M}(A,P) = P\tilde{C}(A), \tag{70}$$

$$\tilde{Q}(A,P) = \frac{\beta\tilde{R}(A,P)}{1-\beta} = \frac{\beta P\hat{y}(\tilde{l}(A))}{1-\beta}. \tag{71}$$

The utility function and budget constraint of the household of the small economy are the same as Eqs. 1 and 6, respectively. If the CIA constraint binds for period t', that is, $\xi_{t'} > 0$, and $A_t = A$ for $t \geq t'$, then

$$C_t = \begin{cases} \dfrac{M_{t'-1} + Q_{t'}(A_{t'-1} - A)}{P} & t = t' \\ C_t = \tilde{C}(A) & t \geq t'+1 \end{cases}, \tag{72}$$

$$L_t = \tilde{L}(A) \qquad t \geq t', \tag{73}$$

$$M_t = \tilde{M}(A,P) \qquad t \geq t', \tag{74}$$

$$Q_t = \begin{cases} \dfrac{\tilde{Q}(A,P)u'(\tilde{C}(A))}{u'(C_{t'})} & t = t' \\ \tilde{Q}(A,P) & t \geq t'+1 \end{cases} \tag{75}$$

is the solution. Thus, once the CIA constraint binds, the economy jumps to the steady-state solution in the next period. If the CIA constraint does not bind for period t', that is, $\xi_{t'} = 0$, we see from Eq. 26

$$u'(C_{t'}) = \beta u'(C_{t'+1}). \tag{76}$$

Combining this equation with Eq. 24 reveals

$$Q_{t'} = Q_{t'+1} + R_{t'}. \tag{77}$$

From Eq. 43, whether the CIA constraint binds for period t is determined by the sign of

$$\xi_t = \frac{u'(C_t) - \beta u'(C_{t+1})}{P}. \tag{78}$$

Thus, the CIA constraint does not bind if $C_t > u'^{-1}(\beta u'(C(A))) > \tilde{C}(A)$. For the model specified by Eqs. 51, 52, and 53, the CIA constraint does not bind if $M_{t-1} > \beta^{-1/\eta}\tilde{M}(A,P) > \tilde{M}(A,P)$.

Let us assume that the government purchases or sells land in period t' and that the household expects $A_t = A_{t'} = A$ for $t \geq t'$. Also let us assume that $t'' \geq t'$ is the first period during which the CIA constraint binds. From Eq. 77, we obtain

$$M_{t''} + P\sum_{t=t'}^{t''}(C_t - Y_t) = M_{t'-1} + Q_{t'}(A_{t'-1} - A) \equiv \hat{M}, \tag{79}$$

where

$$Q_{t'} = Q_{t''} + \sum_{t=t'}^{t''-1} R_t \geq Q_{t''} = \frac{\tilde{Q}(A,P)u'(\tilde{C}(A))}{u'(C_{t''})} \geq \tilde{Q}(A,P) \tag{80}$$

if $C_{t''} \geq \tilde{C}(A)$.

Next, I examine whether or not increasing $M_{t'-1}$ from its steady-state value is advantageous in increasing the government's profit, which is maximized if $\hat{M}$ is minimized. Because setting $M_{t'-1}$ to a value larger than its steady-state value could increase $Q_{t'}$, the government might profit from increased $M_{t'-1}$ by selling land. On the contrary, in the following, I prove that $\hat{M}$ is an increasing function of $M_{t'-1}$, implying that $M_{t'-1}$ should be minimized to maximize the government's profit. Therefore, the government's profit is maximized when the government purchases and sells land in small amounts, in keeping with the CIA constraint binding.

C_t ($t' < t \leq t''$) is uniquely determined by $C_{t'}$ with Eq. 76. Note that t'' is the first period during which the sign of Eq. 78 is positive. Also, L_t ($t' \leq t < t''$) is determined by C_{t+1} and, thus, by $C_{t'}$ with Eq. 23. Because $d'(L_t)$ and W_t are increasing and decreasing functions of L_t, Eq. 23 has a unique solution for L_t. For $t = t''$, $L_{t''} = \tilde{L}(A)$ follows from Eq. 73. Thus, $C_{t'}$ uniquely determines $Q_{t'} = Q_{t''} + \sum_{t=t'}^{t''-1} R_t$. C_t is an increasing function of $C_{t'}$; thus, it uniquely determines the behavior of the household from period t' to t''. Next, let us examine the dependence of $C_t - Y_t$ ($t' \leq t \leq t''$) on C_{t+1}. Differentiating it yields

$$\frac{\partial}{\partial C_{t+1}}[C_t - Y_t] = \frac{\beta u''(C_{t+1})}{u''(C_t)} - y'\left(\frac{L_t}{A}\right)\frac{\partial L_t}{\partial C_{t+1}}, \tag{81}$$

where I used Eq. 76. Combining Eqs. 13 and 23 and differentiating it, we have

$$\frac{\partial L_t}{\partial C_{t+1}} = \frac{\beta y'\left(\frac{L_t}{A}\right)u''(C_{t+1})}{d''(L_t) - \beta\frac{1}{A}y''\left(\frac{L_t}{A}\right)u'(C_{t+1})} < 0. \tag{82}$$

For $t = t''$, this equals zero. Hence, $C_t - Y_t$ ($t' \leq t \leq t''$) is an increasing function of C_{t+1} and, thus, an increasing function of $C_{t'}$. $M_{t''} = \tilde{M}(A,P)$ holds from Eq. 74. Therefore, $\hat{M}$ is an increasing function of $C_{t'}$ if t'' is fixed. If $u'(C_{t''}) - \beta u'(\tilde{C}(A)) = 0$, both t'' and $t''+1$ can be regarded as the first period during which the CIA constraint binds. Because the value of $\hat{M}$ is identical in both cases, $\hat{M}$ is a continuous function of $C_{t'}$ even at the point at which t'' changes. Thus, $\hat{M}$ is a monotonically increasing function of $C_{t'}$.

Let us assume that two different values of $M_{t'-1}$ correspond to the same value of $\hat{M}$, resulting in a unique value of $C_{t'}$ and, thus, $Q_{t'}$, which contradicts with non-unique $M_{t'-1}$. Thus, $\hat{M}$ is a monotonic function of $M_{t'-1}$. If $t' = t''$, rearranging Eq. 79 yields

$$\hat{M} = M_{t'-1} + \frac{\tilde{Q}(A,P)u'(\tilde{C}(A))}{u'(\hat{M}/P)}(A_{t'-1} - A). \tag{83}$$

For $A > A_{t'-1}$, this equation indicates that $\hat{M}$ is an increasing function of $M_{t'-1}$ because $u(\cdot)$ is a concave function. Because $\hat{M}$ is a monotonic function of $M_{t'-1}$ even for $t' \neq t''$, $\hat{M}$ is an increasing function of $M_{t'-1}$ for any $M_{t'-1}$. Therefore, increasing $M_{t'-1}$ does not increase the government's profit from selling land.

In the following, I show that the government's profit cannot be positive in the constant-price environment. The analysis concentrates on the case in which $M_{t'-1}$ equals its minimal, that is, its steady-state value. Although solving Eq. 75, that is,

$$Q_t u'\left(\frac{M_{t-1} + Q_t(A_{t-1} - A_t)}{P}\right) = \tilde{Q}(A,P)u'(\tilde{C}(A)), \tag{84}$$

is needed to calculate the profit, this equation cannot be solved in an explicit form. However, the profit

$$\Pi = \sum_{t=1}^{\tau} Q_t(A_t - A_{t-1}) \tag{85}$$

can be evaluated by using

$$\hat{\Pi} = \sum_{t=1}^{\tau} \hat{Q}_t(A_t - A_{t-1}) \tag{86}$$

where $\hat{Q}_t$ is the solution to

$$\hat{Q}_t u'\left(\frac{M_{t-1}}{P}\right) = \tilde{Q}(A,P)u'(\tilde{C}(A)) \tag{87}$$

or

$$\hat{Q}_t u'(\tilde{C}(A_{t-1})) = \tilde{Q}(A_t,P)u'(\tilde{C}(A_t)). \tag{88}$$

Because $u'(\cdot)$ is a monotonically decreasing function, $\hat{Q}_t \leq Q_t$ when $A_{t-1} - A_t \geq 0$, and $\hat{Q}_t \geq Q_t$ when $A_{t-1} - A_t \leq 0$. Thus, we have

$$\Pi \leq \hat{\Pi}. \tag{89}$$

Next, I prove that $\hat{\Pi} \leq 0$. The land price in the steady state in the constant-price environment, $\tilde{Q}(A,P)$, is given by Eq. 71, whose differentiation with respect to A is nonpositive from Eq. 36. From Eq. 35, the differentiation of the consumption in the steady state (Eq. 28), $\tilde{C}(A)$, with respect to A is nonnegative . If $A_{t-1} \leq a \leq A_t$, because $u'(\tilde{C}(A_t)) \leq u'(\tilde{C}(A_{t-1}))$, we obtain $\hat{Q}_t \leq \tilde{Q}(A_t, P) \leq \tilde{Q}(a, P)$ from Eq. 88. Similarly, if $A_t \leq a \leq A_{t-1}$, because $u'(\tilde{C}(A_{t-1})) \leq u'(\tilde{C}(A_t))$, we obtain $\tilde{Q}(a, P) \leq \tilde{Q}(A_t, P) \leq \hat{Q}_t$ from Eq. 88. Hence, defining $a(t)$ by Eq. 48, we have

$$\begin{aligned}\hat{\Pi} &= \int_0^\tau \hat{Q}_{\lfloor t \rfloor+1} \frac{\mathrm{d}}{\mathrm{d}t} a(t)\, \mathrm{d}t \\ &\leq \int_0^\tau \tilde{Q}(a(t), P) \frac{\mathrm{d}}{\mathrm{d}t} a(t)\, \mathrm{d}t \\ &= \int_{a(0)}^{a(\tau)} \tilde{Q}(a, P)\, \mathrm{d}a = 0,\end{aligned} \tag{90}$$

which leads to $\Pi \leq 0$. Therefore, the government cannot profit from an economy that trades goods with an infinitely large economy. In terms of thermodynamics, this statement is equivalent to Kelvin's principle, another version of the second law, which states that work done by a system in an isothermal cycle cannot be positive.

A quasi-static isothermal process can also be defined. The money that the government gains from the household in the limit of infinitely slow trading between the government and the economy is given by

$$\begin{aligned}\Pi &= \int_{A_{\mathrm{initial}}}^{A_{\mathrm{final}}} \tilde{Q}(A, P)\, \mathrm{d}A \\ &= \int_{A_{\mathrm{initial}}}^{A_{\mathrm{final}}} P\tilde{C}(A)\tilde{q}(A)\, \mathrm{d}A.\end{aligned} \tag{91}$$

This profit is the largest that the government can gain from the economy in a constant-price environment, which is proven as follows. Let us assume that the government can gain a larger profit $\Pi' > \Pi$ by trading land to change the amount of land that the household owns from A_{initial} to A_{final}. The profit of the isothermal cycle composed of the process with profit Π' starting from A_{initial} to A_{final} and the reversed quasi-static process from A_{final} to A_{initial} is $\Pi' - \Pi > 0$, which contradicts with the nonpositivity of the profit. Thus, Eq. 91 can be regarded as the maximal work done by a system in equilibrium with an isothermal environment.

For the model specified by Eqs. 51, 52, and 53, the steady-state solution is given by Eqs. 54, 55, and

$$\tilde{M}(A, P) = \theta P A^\nu, \tag{92}$$

$$\tilde{Q}(A, P) = \zeta\theta P A^{\nu-1}, \tag{93}$$

where I rearranged the terms of Eqs. 56 and 57. The maximal profit of the government is given by

$$\begin{aligned}\Pi =& \int_{A_{\text{initial}}}^{A_{\text{final}}} P\tilde{C}(A)\tilde{q}(A)\,\mathrm{d}A \\ =& \frac{\zeta}{\nu}\theta P(A_{\text{final}}^{\nu} - A_{\text{initial}}^{\nu}).\end{aligned} \tag{94}$$

6 Multi-economy system

Let us consider $n \geq 2$ economies trade with each other. We assume that these economies trade goods but not land and that people cannot move among economies. The budget and the CIA constraints are assumed for each economy. All economies share the same price level of goods. Like an economy trading with a large economy, in the steady state, each economy is identical to the isolated economy, that is, each economy consumes the same amount of goods as that produced by itself. If the CIA constraint binds for any economy in period t', the economies converge to the steady state in period $t'+1$. In this case, the land price in economy i is determined by

$$Q_{t,i} = \frac{\beta_i}{1-\beta_i} \frac{M_t \hat{y}_i(\tilde{l}_i(A_{t,i}))}{\sum_{j=1}^{n} A_{t,j} y_j(\tilde{l}_j(A_{t,j}))}, \tag{95}$$

where M_t is the total amount of money owned by households of the economies and

$$\hat{y}_i(l) = y_i(l) - l y_i'(l). \tag{96}$$

Assuming that the land owned by the household of economy i changes from $A_{t-1,i}$ to $A_{t,i}$, the total amount of money is given by

$$M_t = M_{t-1} + \sum_{i=1}^{n} Q_{t,i}(A_{t-1,i} - A_{t,i}) \tag{97}$$

which is rearranged to give

$$\begin{aligned}M_\tau =& M_0 \exp\left[\sum_{t=1}^{\tau} f\left(\sum_{i=1}^{n} \frac{Q_{t,i}}{M_t}(A_{t,i} - A_{t-1,i})\right)\right] \\ \geq& M_0 \exp\left[-\sum_{t=1}^{\tau}\sum_{i=1}^{n} \frac{Q_{t,i}}{M_t}(A_{t,i} - A_{t-1,i})\right],\end{aligned} \tag{98}$$

where Eq. 47 is used.

In the following, I consider the multi-economy system of the models specified by Eqs. 51, 52, and 53 with identical utility and production functions. Inserting Eq. 95 into Eq. 98, we have

$$\log \frac{M_\tau}{M_0} \geq -\zeta \sum_{t=1}^{\tau} \frac{\sum_{i=1}^{n} A_{t,i}^{\nu-1}(A_{t,i} - A_{t-1,i})}{\sum_{i=1}^{n} A_{t,i}^{\nu}}. \tag{99}$$

Defining $a_i(t)$ by

$$a_i(t) = A_{\lfloor t \rfloor,i} + (A_{\lfloor t \rfloor+1,i} - A_{\lfloor t \rfloor,i})(t - \lfloor t \rfloor), \tag{100}$$

we have

$$\begin{aligned}
\log \frac{M_\tau}{M_0} \geq & -\zeta \int_0^\tau \frac{\sum_{i=1}^n a_i(\lfloor t \rfloor + 1)^{\nu-1} \frac{\mathrm{d}a_i(t)}{\mathrm{d}t}}{\sum_{i=1}^n a_i(\lfloor t \rfloor + 1)^\nu} \mathrm{d}t \\
\geq & -\zeta \int_0^\tau \frac{\sum_{i=1}^n a_i(t)^{\nu-1} \frac{\mathrm{d}a_i(t)}{\mathrm{d}t}}{\sum_{i=1}^n a_i(t)^\nu} \mathrm{d}t \\
= & -\frac{\zeta}{\nu} \left[\log \left(\sum_{i=1}^n a_i(t)^\nu \right) \right]_0^\tau = 0
\end{aligned} \tag{101}$$

if $A_{0,i} = A_{\tau,i}$.

Hence, the government cannot profit from the model economies even if these economies trade goods. As with the single economy, infinitely slow trading between the government and the economies is reversible in the multi-economy system. In other words, if infinitely slow trading between the government and the economies with states (P, A_1), $\ldots$, (P, A_n) changes the amounts of land to A'_1, $\ldots$, A'_n, the price level at the end of the process, P', is unique. This statement is proven as follows. Let us assume that the price level can be P' and P''. Then, the government can change the set of states from (P', A'_1), $\ldots$, (P', A'_n) to (P'', A'_1), $\ldots$, (P'', A'_n) and vice versa. Therefore, the government can profit from the economies if $P' \neq P''$, which contradicts with Eq. 101.

The government's nonpositive profit allows us to discriminate whether a set of states of isolated economies can be made from another set of states by the government. In the following, the first set of states is denoted by $(P_1^{(1)}, A_1^{(1)})$, $\ldots$, $(P_n^{(1)}, A_n^{(1)})$ and the second set by $(P_1^{(2)}, A_1^{(2)})$, $\ldots$, $(P_n^{(2)}, A_n^{(2)})$. To compare two sets of states, let us assume the following process with infinitely slow trade between the government and the economies. First, the government sells or purchases land of isolated economies to set the price levels of all economies to $P^{(1)}$ and $P^{(2)}$ for the first and second sets, respectively. Now we have $(P^{(1)}, A_1^{(1)'})$, $\ldots$, $(P^{(1)}, A_n^{(1)'})$ and $(P^{(2)}, A_1^{(2)'})$, $\ldots$, $(P^{(2)}, A_n^{(2)'})$. Second, in allowing the second set of economies to trade goods with each other, the government sells and purchases land to set the amount of land of economy i of the second set to that of economy i in the first set, $A_i^{(1)}$. Then, the states of the second set of economies are $(P^{(2)'}, A_1^{(1)})$, $\ldots$, $(P^{(2)'}, A_n^{(1)})$. $P^{(2)'}$ is uniquely determined by the sets of states at the beginning of the process. If $P^{(1)} \geq P^{(2)'}$, the government can bring the states of the second set of economies to those of the first set of the economies. If $P^{(1)} \leq P^{(2)'}$, the government can bring the states of the first set of economies to those of the second set of economies. If the government can bring the economies from the first set of states to the second set of states and vice versa, it cannot profit from the circle from the first set to the first set through the second set because Eq. 101 holds.

7 Thermodynamic functions

We have seen that in both an economy isolated from a larger one and an economy trading with a larger one, the government's profit is nonpositive given that no net change occurs in the amount of land owned by the household. We have also seen that the government's profit is nonpositive in a multi-economy system under the same condition. These results lead us to the similarity of the work done by a thermodynamic system and the government's profit. The nonpositivity of work in a cycle underlies the thermodynamic functions (Tasaki and Paquette, 2018).

Having formulated adiabatic and isothermal processes, here I derive thermodynamic functions of the model specified by Eqs. 51, 52, and 53. Let us assume that $T = P^{1/\sigma}$ corresponds to the temperature of the system. In the following, I specify the state of the economy with T and A. Because the work done by a system is given by the difference in the internal energy (Callen, 1985; Tasaki and Paquette, 2018), using Eq. 56, the internal energy of the system can be defined by

$$U(T; A) = M + c = \theta T^{\sigma} A^{\nu} + c, \tag{102}$$

where c is a constant set to 0 in the following. Because the entropy of a system does not change in a quasi-static adiabatic process, Eq. 69 suggests that the entropy must be in the form of

$$S(T; A) = g(\theta T^{\sigma} A^{\nu+\zeta}). \tag{103}$$

Here I assume that $g(x) = \omega x^{\rho}$. Requiring the thermodynamic relation

$$\frac{\partial}{\partial T} U(T; A) = T \frac{\partial}{\partial T} S(T; A) \tag{104}$$

(Callen, 1985) to hold, we have

$$\rho = \frac{\nu}{\nu + \zeta}, \tag{105}$$

$$\sigma = 1 + \frac{\nu}{\zeta}, \tag{106}$$

$$\omega = \rho^{-1} \theta^{\frac{\zeta}{\nu+\zeta}}, \tag{107}$$

where $0 < \rho < 1$, $\sigma > 1$, and $\omega > 0$. Thus, the entropy is given by

$$S(T; A) = \left(1 + \frac{\zeta}{\nu}\right) \theta T^{\frac{\nu}{\zeta}} A^{\nu}. \tag{108}$$

The Helmholtz free energy is defined by

$$\begin{aligned} F(T; A) &= U(T; A) - TS(T; A) \\ &= -\frac{\zeta}{\nu} \theta T^{\sigma} A^{\nu} \end{aligned} \tag{109}$$

(Kittel and Kroemer, 1980; Callen, 1985; Tasaki and Paquette, 2018). The maximal work calculated from the difference in the Helmholtz free energy

$$\Pi = \frac{\zeta}{\nu}\theta T^{\sigma}(A_{\text{final}}^{\nu} - A_{\text{initial}}^{\nu}) \tag{110}$$

equals the maximal profit (Eq. 94) in the constant-price environment.

The entropy as a function of U and A is given by

$$S(U, A) = \omega U^{\rho} A^{\rho\zeta}. \tag{111}$$

This coincides with the fundamental relation for a photon gas, $S = \omega U^{3/4} V^{1/4}$ (Kittel and Kroemer, 1980), if $A = V$, $\rho = 3/4$, and $\zeta = 1/3$, which are achieved in the limit of $\gamma \to 0$ and $\eta \to 0$.

Introducing the variable N to represent the size of the system, which corresponds to the population in the economy, yields

$$S(U, A, N) = \omega N \left(\frac{U}{N}\right)^{\rho} \left(\frac{A}{N}\right)^{\rho\zeta}. \tag{112}$$

In terms of thermodynamics, P and T are intensive variables, and U, F, S, M, A, and N are extensive variables. It is easily proven that entropy monotonically increases with respect to U, is concave with respect to U, A, and N, and is a homogeneous function of degree 1, that is,

$$\frac{\partial}{\partial U} S(U, A, N) > 0, \tag{113}$$

$$S(\lambda U, \lambda A, \lambda N) = \lambda S(U, A, N), \tag{114}$$

$$S(U_1 + U_2, A_1 + A_2, N_1 + N_2) \geq S(U_1, A_1, N_1) + S(U_2, A_2, N_2), \tag{115}$$

which are major properties of thermodynamic functions. If $S(U_1, A_1) < S(U_2, A_2)$, the government can bring the state of the economy from (U_1, A_1) to (U_2, A_2), but not vice versa.

The entropy of an isolated system can also be used to judge whether the government can change the state of a multi-economy system from one to another. Comparing the summation of the entropies of a set of economies with that of another set of economies, we can determine whether the government can bring the system from one to another (Tasaki and Paquette, 2018). Let us define the entropy of two sets of states by

$$S^{(1)} = \sum_{i=1}^{n} S(U_i^{(1)}, A_i^{(1)}, N_i), \tag{116}$$

$$S^{(2)} = \sum_{i=1}^{n} S(U_i^{(2)}, A_i^{(2)}, N_i). \tag{117}$$

If $S^{(1)} \geq S^{(2)}$, the first set can be made from the second set, and if $S^{(1)} \leq S^{(2)}$, the second set can be made from the first set.

In thermodynamics, the Legendre transform of internal energy gives the Helmholtz free energy. Similarly, the Legendre transform of $U(S, A, N)$ as a function of S is defined by

$$\begin{aligned}\min_S[U(S,A,N)-TS] =& \min_S\left(\omega^{-1/\rho}N\left(\frac{S}{N}\right)^{1/\rho}\left(\frac{A}{N}\right)^{-\zeta}-TS\right)\\ =&-\frac{\zeta}{\nu}\theta T^{\sigma}N\left(\frac{A}{N}\right)^{\nu}\\ =&F(T;A,N),\end{aligned} \tag{118}$$

which recovers the Helmholtz free energy.

8 Work done in a Carnot cycle

Let us evaluate the government's profit from purchasing land from an economy in environments with price levels P_H and P_L, where $P_H > P_L$ (Fig. 1). In the first step, the amount of land owned by the household of an economy is A_1, and the economy is in an environment with a price level P_H. In the second step, the government in this environment increases the household-owned land to A_2, where $A_2 > A_1$. In the third step, the economy is dissociated from the environment and the price level is changed from P_H to P_L by a quasi-static adiabatic process. Because Eq. 69 is constant in a quasi-static adiabatic process, the amount of land owned by the household, A_3, at the price level P_L is defined by

$$\theta P_H A_2^{\nu+\zeta} = \theta P_L A_3^{\nu+\zeta}. \tag{119}$$

In the fourth step, in the environment with the price level P_L, where $A_4 < A_3$, the government decreases the amount of household-owned land to A_4. In the fifth step, the government again dissociates the economy from the environment and changes the price level from P_L to P_H through a quasi-static adiabatic process. To restore the amount of household-owned land to A_1 at the end of the fifth step, A_4 must satisfy

$$\theta P_L A_4^{\nu+\zeta} = \theta P_H A_1^{\nu+\zeta}. \tag{120}$$

The state of the economy returns to the initial state, thus completing a Carnot cycle. The money that the economy gains from the environment with price level P_H is given by the summation of the government's profit and the increase in the amount of money owned by the economy from the beginning to the end of the process, that is,

$$\begin{aligned}\Delta M_H =&\frac{\zeta}{\nu}\theta P_H(A_2^{\nu}-A_1^{\nu})-\theta P_H A_1^{\nu}+\theta P_H A_2^{\nu}\\ =&\left(1+\frac{\zeta}{\nu}\right)\theta P_H(A_2^{\nu}-A_1^{\nu})>0.\end{aligned} \tag{121}$$

Similarly, the money that the economy loses to the environment with price level P_L is

$$\Delta M_L = \left(1+\frac{\zeta}{\nu}\right)\theta P_L(A_3^{\nu}-A_4^{\nu})>0. \tag{122}$$

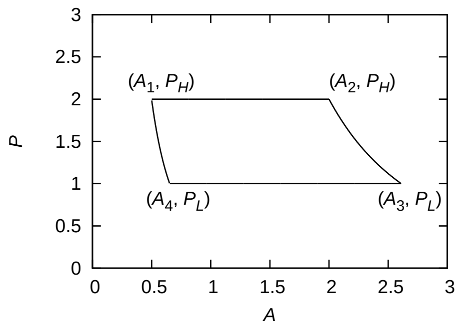

Figure 1: Carnot cycle. The following parameter values are used: $\gamma = 0.5$, $\beta = 0.8$, $\mu = 1$, $\alpha = 0.5$, and $\eta = 0.5$.

$\Delta M_H - \Delta M_L$ is the work done in the Carnot cycle and the amount of money that the government gains from the environment through the economy. This can also be calculated using the thermodynamic functions as

$$\begin{aligned}
&F(T_H; A_1) - F(T_H; A_2) + U(T_H; A_2) - U(T_L; A_3) \\
&\quad + F(T_L; A_3) - F(T_L; A_4) + U(T_L; A_4) - U(T_H; A_1) \\
=&\left(1 + \frac{\zeta}{\nu}\right) \theta T_H^\sigma (A_2^\nu - A_1^\nu) + \left(1 + \frac{\zeta}{\nu}\right) \theta T_L^\sigma (A_4^\nu - A_3^\nu) \\
=&\Delta M_H - \Delta M_L,
\end{aligned} \tag{123}$$

where we define $T_H = P_H^{1/\sigma}$ and $T_L = P_L^{1/\sigma}$. Using Eqs. 119 and 120, the ratio of the money gained and lost by the economy in environments with price levels P_H and P_L is

$$\begin{aligned}
\frac{\Delta M_H}{\Delta M_L} =& \frac{P_H (A_2^\nu - A_1^\nu)}{P_L (A_3^\nu - A_4^\nu)} \\
=& \frac{T_H}{T_L},
\end{aligned} \tag{124}$$

which coincides with the definition of temperature in thermodynamics.

9 Discussion

In the present model, the government debt equals the money supply (Table 1). This identity allows us to regard the money supply as the internal energy and the government's profit as the work. The conservation of the amount of money corresponds to the conservation of energy, that is, the first law of thermodynamics.

The second law of thermodynamics corresponds to the nonpositivity of profit. A non-optimal action of an agent in any economy with optimal agents leads to nonpositive profit

Table 1: Correspondence of thermodynamics and macroeconomics in the present model

Thermodynamics	Internal energy U	Work W	Temperature T
Macroeconomics	Money supply M	Profit Π	Power of price $P^{1/\sigma}$

taken advantage of by others. The nonpositivity of profit is found in a perfectly competitive market.

The conservation of a quantity in a system and the irreversibility of the flow of the quantity allow the formalization of thermodynamics (Tasaki and Paquette, 2018). Hence, thermodynamic functions should be derived for other macroeconomic models that satisfy these two properties. Let us note that, although thermodynamic functions are defined for equilibrium states, the system does not have to be in equilibrium at every instant of the process for the laws of thermodynamics to hold. For example, the process that first dissociates an economy from an environment, second purchases and sells goods in the economy, and third brings the economy in contact with the environment is an isothermal process that is not in equilibrium with the environment in the middle of the process. Even in this case, the government's maximal profit is given by the Helmholtz free energy. Because an agent cannot profit from an economy with rational agents, checking whether a model satisfies these two laws can be used to test the soundness of the model. This procedure is like testing the soundness of a statistical mechanical model by examining whether it is consistent with thermodynamics.

However, some agents may profit from real markets. Several reasons exist for the incompatibility between agents gaining profits and the results of the present model. First, these agents may be trading between two disconnected markets, but not in a single market. This situation allows agents to profit from the markets, as we have seen in the Carnot cycle. Second, the government can profit from economies trading with each other unless these economies are the models specified by Eqs. 51, 52, and 53 with the same parameter values. However, the government might not profit from a multi-economy system in which households have other expectations than $A_t = A$ for $t \geq t'$. Third, if an agent has information that other agents do not have, the agent can profit from the market. Markets are not necessarily perfectly competitive; information asymmetry is found in markets (Akerlof, 1970). In an analogy of information thermodynamics (Parrondo et al., 2015), the maximal profit might be a function of the amount of information that the agent has.

Examining how taxes and bonds affect the properties of the model will be of interest. Economies with multiple types of goods and capitals, which can be regarded as multicomponent systems, also are of interest. Although fluctuations in physical systems are ignored in thermodynamics, fluctuations in economic as well as physical systems represent an important problem to be addressed. Additionally, technology advances should be investigated by extending the model.

Although this paper focuses on an economic system, the law of conservation and irreversibility could be found in other fields of social sciences and may allow the thermodynamic formulation of systems. The irreversibility of these systems could be a manifestation of the rational and optimal behavior of agents.

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None

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