Global Bond Market Interaction: An Arbitrage-free Dynamic Nelson Siegel Modeling Approach

Takeshi Kobayashi

NUCB Business School

1-3-1 Nishiki Naka-ku,Nagoya 460-0003 JAPAN

T +81 52 203 8111 F +81 52 221 5221

This revision:Aug 29, 2019

Abstract

We have extended Diebold, Li and Yue (2008) to an arbitrage-free setting, proposing a global factor model in which country yield-level and slope factors may depend on global-level, slope and curvature factors as well as country-specific factors. Using a monthly dataset of government bond yields for Germany, Japan, the US, and the UK from January 1995 to November 2018, we extracted global and country-specific factors for both the full sample. The results indicate strongly that global yield-level, slope, and curvature factors do indeed exist and are economically important, accounting for a significant fraction of variation in country bond yields with interesting differences across countries. Moreover, the global yield factors appear linked to global macroeconomic fundamentals. We show, in particular, that curvature factors are key to explaining term premium dynamics, and appear more important in the second sub-sample.

JELCodes: C13,C32,E43

Keywords:Term Structure Model, Global Bond Market, Global Factor

1 Introduction

The yield curve is of great interest to both academics and market practitioners. Hence, yield curve modeling has generated a huge literature, spanning many decades, particularly as regards the term structure of government bond yields. There are compelling reasons to assert that global shocks affect cross-country government yield curves. The recent credit crisis, for instance, shows that macrofinance shocks can be crucially transmitted internationally. As a consequence of financial integration, a sizable amount of domestic government debt is held by foreigners in global capital markets. Thus, positions on foreign bonds are naturally affected by home macrofinance conditions, and vice versa. Despite these important stylized facts, little is known, however, about whether global yield factors are operative and, more generally, about the nature of dynamic cross-country bond yield interactions. One might naturally conjecture the existence of global bond yield factors. This paper takes up this challenge and investigates the role of global factors in the yield curves of several industrialized countries. Global yield curve modeling mainly has two strands. One is Nelson-Siegel modeling approach,and another is no-arbitrage affine term structure modeling approach.

Diebold et al. (2008) (hereafter DLY) extend the research of Diebold and Li (2006) to a global context by modeling a potentially large set of country yield curves for Germany, Japan, the UK, and the US in a framework that allows for both global and country-specific factors. They show that two global factors, "global level and global slope," are largely responsible for the co-movements of yield curves in industrialized economies. Coroneo, Garrett and Sanhueza (2018) analyze the relationship between the yield curves of the USA, the UK and Germany using global and local factors. They disentangle the latent global and local factors contained in country factors, based on the DL parametrization of Nelson and Siegel (1987) three factor model and a quasi-maximum likelihood approach. The results indicate that global factors explain on average 55% of the variance of yields. Using impulse response analysis, they examine the effects of shocks to the factors on yields. They find that the response of yields to shocks to global factors is larger and longer-lasting than the response to shocks to local factors. The DLY approach is also applied to the term structure of other countries. Bae and Kim (2011) empirically evaluated the developments of bond markets in East Asia based on the DLY methodology of dynamic cross-country bond yield interactions. Morita and Bueno (2008) used the DLY framework to extract common factors related to sovereign bonds from investment-grade emerging markets.

Abbritti, Dell'Erba, Moreno, Sola et al. (2018) introduce unspanned global factors within a FAVAR framework in a flexible reduced-form affine term structure model. They show that, together with the first and second global factors, a third factor is also important in explaining the dynamics of interest rates. Kaminska, Meldrum and Smith (2013) also extends a popular no-arbitrage affine term structure model to model jointly bond markets and exchange rates across the United Kingdom, United States and euro area. Their results are similar to previous work. But they are able to decompose interest rates into risk-free rates and risk premia. Additionally, they are able to study the implications for exchange rates.

We have extended DLY approach to an arbitrage-free setting. Concretely, we use the arbitrage-free dynamic version of the Nelson-Siegel yield curve (hereafter, AFNS), which is derived from Christensen, Diebold and Rudebusch (2011) to model government bond yields of developed countries.An AFNS model blends two important and successful approaches to yield curve modeling: the empirically based dynamic Nelson-Siegel (hereafter, DNS) approach and the no-arbitrage theoretically based one. Models in the DNS tradition fit and forecast well, but they lack theoretical rigor as they may admit arbitrage possibilities, while models in the arbitrage-free tradition are theoretically rigorous as they enforce absence of arbitrage, but they may fit and forecast poorly. AFNS bridges with a DNS model, which enforces absence of arbitrage.

This article differs from DLY in three important aspects. First, and more importantly, we show that together with the global level and slope factor, a curvature factor is also important in explaining the dynamics of the interest rates. We show that this factor, which turns out to be especially important for explaining long-run variations in interest rates and the term premium, is related to financial and policy risks and precedes the financial instability after the world financial crisis period. Second, our approach is to model international yield curves jointly in a no-arbitrage framework. Third, this paper also introduces a strategy that allows us to estimate all unknown parameters via a one-step approach using panel data of bond yield curves.

This paper proceeds as follows. In section 2 we present the data In section 3 we discuss the NS model, its parametrization by Diebold and Li (2006) and some descriptive evidence in support of the presence of global factors. Section 4 describes the building blocks of our term structure model. Section 5 explains the estimation methodology, and section 6 discusses our main results and section 7 concludes.

2 Data

2.1 Government bond yield curve data

The monthly zero-coupon yields of maturities 3, 6, 12, 24, 36, 48, 60, 72, 84, 96, 108, and 120 months were considered for the period from January 1995 to November 2018. The zero-coupon yields were obtained from Blomberg. Figure (1) shows the time series of the term structure of yields of ten countries, including the US, Germany, Japan, and the UK. Apparently, each yield curve displays significant level movements. Cross-country comparison of yield curves suggests some cross-country commonality.

— Figure (1) —

3 The Nelson-Siegel (NS) and AFNS Model and Preliminary Analysis

3.1 Single country yield curve

The Nelson and Siegel model fits the yield curve at any point in time with the simple functional form

$$y(\tau) = L + S\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + C\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) \tag{1}$$

where $y(\tau)$ is the zero-coupon yield with τ months to maturity, and L,S, and C are model parameters. The three parameters can be interpreted as factors. Due to its flexibility, this model is able to provide a good fit of the cross-section of yields at a given point in time, which is the primary reason for its popularity in the financial market.

Although such a static representation appears useful for some purposes, a dynamic version is required to understand the evolution of the bond market over time. Diebold and Li (2006) achieve this by introducing time-varying parameters.

$$y_{i,t}(\tau) = L_{i,t} + S_{i,t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + C_{i,t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) + v_{i,t}(\tau) \tag{2}$$

where $L_{i,t}$,$S_{i,t}$,$C_{i,t}$ are interpreted as level, slope, and curvature factors (given their associated Nelson-Siegel factor loadings in Figure(2)). $v_{i,t}(\tau)$ is a disturbance with standard deviation $\sigma_i(\tau)$

— Figure (2) —

Figure (3) shows estimates of the local level and slope factors, obtained via step one of the Diebold and Li (2006), using the monthly yield curve from January 1995 to November 2018.

— Figure (3) —

Figure (4) demonstrates the principal component analysis of estimated factors by Diebold and Li (2006). The PCA shows over 90% variation in the level and 50% variation in the slope, and the curvature is driven by the first principal components.

— Figure (4) —

3.2 The AFNS model for single country yield curve

We choose to build on the affine three-factor arbitrage-free approximation of the well-known Nelson-Siegel term structure model presented in Christensen et al. (2011). Their strategy is to find the affine arbitrage free model with factor loadings that match Nelson-Siegel exactly. This is a three-factor model where the latent state variables $X_t = (L_t, S_t, C_t)$ can be interpreted as the level, slope, and curvature with the imposition of a fixed set of restrictions on the Q-dynamics of a general three-factor affine Gaussian term structure model.

$$\begin{aligned} r_t &= \delta_0 + \delta_1' X_t \\ dX_t &= K^Q(\theta^Q - X_t)dt + \Sigma_n dW_t^Q \end{aligned} \tag{3}$$

where $\delta_0 \in \boldsymbol{R}^{3\times1}$ and $\delta_1 \in \boldsymbol{R}^{3\times3}$ are coefficients. $W_t^Q \in \boldsymbol{R}^{3\times3}$ is a standard Brownian motion. $\theta^Q \in \boldsymbol{R}^{3\times1}$ is the drifts and $K^Q \in \boldsymbol{R}^{3\times1}$ is the mean-reversion matrix. $\Sigma_n \in \boldsymbol{R}^{3\times3}$ is the volatility matrix. The first critical assumption is to define the instantaneous risk-free rate as the sum of the level and the slope factors:

$$r_t = L_t + S_t \tag{4}$$

The second critical assumption is that the mean-reversion matrix under the Q-matrix must have the following simple form:

$$K^{Q,T} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & \lambda \end{pmatrix} \tag{5}$$

where λ will be identical to λ in the standard Nelson-Siegel model in Equation (1). These two assumptions also highlight the role of the curvature factor C_t. This factor is absent from the instantaneous risk-free rate and therefore does not impact short-term yields. Instead, its sole role (under the Q-measure) is to act as a stochastic mean for the slope factor S_t^l. Finally, we follow Christensen et al. (2011) and fix the mean vector under the Q-measure at zero, $\theta^{Q,T} = 0$. They show that this identification method can be applied without loss of generality. Imposing the above structure on the general affine model default risk-free zero-coupon yields

$$y_t^T(\tau) = L_t + \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau}\right) S_t + \left[\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right] C_t - \frac{A^T(\tau)}{\tau} \tag{6}$$

where

$$\begin{aligned}\frac{A_y(t,T)}{T-t} =& (\sigma_{11})^2 \frac{(T-t)^2}{6} \\ &+ (\sigma_{22})^2 \left[\frac{1}{2\lambda^2} - \frac{1}{\lambda^3}\frac{1-e^{-\lambda(T-t)}}{(T-t)} + \frac{1}{4\lambda^3}\frac{1-e^{-2\lambda(T-t)}}{(T-t)}\right] \\ &+ (\sigma_{33})^2 \left[\frac{1}{2\lambda^2} + \frac{1}{\lambda^2}e^{-\lambda(T-t)} - \frac{1}{4\lambda}(T-t)e^{-2\lambda(T-t)} - \frac{3}{4\lambda^2}e^{-2\lambda(T-t)} - \frac{2}{\lambda^3}\frac{1-e^{-\lambda(T-t)}}{(T-t)} + \frac{5}{8\lambda^3}\frac{1-e^{-2\lambda(T-t)}}{(T-t)}\right]\end{aligned} \tag{7}$$

In AFNS mode we recognize the Nelson-Siegel factor loadings for the level, slope, and curvature factors. In addition, the yield function contains the maturity-dependent term $\frac{A^T(\tau)}{T-t}$, which arises from imposing the absence of arbitrage on the dynamic Nelson-Siegel model.

4 AFNS Global Term Structure Model

DLY expands the study of the Diebold and Li (2006) model in a global context, while modeling a potentially large set of country yield curves in a framework that allows for both common and country-specific factors. Through an empirical analysis of the term structures of government bond yields in Germany, Japan, the UK, and the US, they find that common yield factors do exist and that such factors are economically important, as they explain significant fractions of country yield curve dynamics, with interesting differences across countries.

In this section, we extend the AFNS model to a multi-country environment, following the DLY approach. Thus, one may adapt a single-country model to an N-country approach, with a similar space-state framework. The appendix A describes the details. We make the following two assumptions. First, the instantaneous short rate of country i is assumed to be a function of the global level L_t^g and the slope factor S_t^g in addition to country-specific risk factor $L_t^{l(i)}$,$S_t^{l(i)}$ for each country $i = 1, \ldots, N$.

$$r_t^i = \beta_L^i L_t^g + \beta_S^i S_t^g + L_t^{l(i)} + S_t^{l(i)} \tag{8}$$

where β_L^i and β_S^i are loadings on global factors Second, the dynamics of the $3(N+1)$ factors under the risk-neutral pricing measure must be given by the solution to the following SDE: This is a $3(N+1)$-factor model where the latent state variables $X_t = (L_t^g, S_t^g, C_t^g, L_t^{l(i)}, S_t^{l(i)}, C_t^{l(i)})$.

For that reason, the dynamics of the global risk factors under the Q measure must be assumed to be given by

$$
\begin{pmatrix} dL_t^g \\ dS_t^g \\ dC_t^g \\ dL_t^{l(i)} \\ dS_t^{l(i)} \\ dC_t^{l(i)} \end{pmatrix} = - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^g & -\lambda^g & 0 & 0 & 0 \\ 0 & 0 & \lambda^g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^{l(i)} & -\lambda^{l(i)} \\ 0 & 0 & 0 & 0 & 0 & \lambda^{l(i)} \end{pmatrix} \begin{pmatrix} L_t^g \\ S_t^g \\ C_t^g \\ L_t^{l(i)} \\ S_t^{l(i)} \\ C_t^{l(i)} \end{pmatrix} dt
$$

$$
+ \begin{pmatrix} \sigma_L^g & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_S^g & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_C^g & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_L^{l(i)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_S^{l(i)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_C^{l(i)} \end{pmatrix} \begin{pmatrix} dW_t^{L^g,Q} \\ dW_t^{S^g,Q} \\ dW_t^{C^g,Q} \\ dW_t^{L^{l(i)},Q} \\ dW_t^{S^{l(i)},Q} \\ dW_t^{C^{l(i)},Q} \end{pmatrix} \tag{9}
$$

Given the assumption of independent factors, the zero-coupon yield function for a representative bond from country i and $T-t$ years to maturity is equal to

$$
\begin{aligned}
y^i(t,T) = {} & \beta_L^i L_t^g + \beta_S^i \left(\frac{1 - e^{-\lambda^g (T-t)}}{\lambda^g (T-t)} \right) S_t^g + \beta_S^i \left(\frac{1 - e^{-\lambda^g (T-t)}}{\lambda^g (T-t)} - e^{-\lambda^g (T-t)} \right) C_t^g \\
& + L^{l(i)} + \left(\frac{1 - e^{-\lambda (T-t)}}{\lambda (T-t)} \right) S_t^{l(i)} + \left(\frac{1 - e^{-\lambda (T-t)}}{(\lambda^{l(i)})\tau} - e^{-(\lambda^{l(i)})(T-t)} \right) C_t^{l(i)} - \frac{A_y^i(t,T)}{T-t}
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\frac{A_y^i(t,T)}{T-t} = {} & (\sigma_L^g)^2 (\beta_L^i)^2 \frac{(T-t)^2}{6} \\
& + (\sigma_S^g)^2 (\beta_S^i)^2 \left[\frac{1}{2(\lambda^g)^2} - \frac{1}{(\lambda^g)^3} \frac{1 - e^{-\lambda^g (T-t)}}{(T-t)} + \frac{1}{4(\lambda^g)^3} \frac{1 - e^{-2\lambda^g (T-t)}}{(T-t)} \right] \\
& + (\sigma_C^g)^2 (\beta_S^i)^2 \left[\frac{1}{2(\lambda^g)^2} + \frac{1}{(\lambda^g)^2} e^{-(\lambda^g)(T-t)} - \frac{1}{4\lambda^g} (T-t) e^{-2\lambda^g (T-t)} - \frac{3}{4(\lambda^g)^2} e^{-2\lambda^g n} - \frac{2}{(\lambda^g)^3} \frac{1 - e^{-\lambda^g (T-t)}}{(T-t)} + \frac{5}{8(\lambda^g)^3} \frac{1 - e^{-2\lambda^g (T-t)}}{(T-t)} \right] \\
& + (\sigma_L^{l(i)})^2 \frac{(T-t)^2}{6} \\
& + (\sigma_S^{l(i)})^2 \left[\frac{1}{2(\lambda^{l(i)})^2} - \frac{1}{(\lambda^{l(i)})^3} \frac{1 - e^{-\lambda^{l(i)} (T-t)}}{(T-t)} + \frac{1}{4(\lambda^{l(i)})^3} \frac{1 - e^{-2\lambda^{l(i)} (T-t)}}{(T-t)} \right] \\
& + (\sigma_C^{l(i)})^2 \left[\frac{1}{2(\lambda^{l(i)})^2} + \frac{1}{(\lambda^{l(i)})^2} e^{-\lambda^{l(i)} (T-t)} - \frac{1}{4\lambda^{l(i)}} (T-t) e^{-2\lambda^{l(i)} (T-t)} - \frac{3}{4(\lambda^{l(i)})^2} e^{-2\lambda^{l(i)} (T-t)} \right. \\
& \left. - \frac{2}{(\lambda^{l(i)})^3} \frac{1 - e^{-\lambda^{l(i)} (T-t)}}{(T-t)} + \frac{5}{8(\lambda^{l(i)})^3} \frac{1 - e^{-2\lambda^{l(i)} (T-t)}}{(T-t)} \right]
\end{aligned} \tag{11}
$$

There are no restrictions on the dynamic drift components under the empirical P-measure. Therefore, beyond the requirement of constant volatility, we are free to choose the dynamics under the P-measure. However, to facilitate the empirical implementation we limit our focus to the essentially affine risk premium specification introduced in Duffee (2002). In the Gaussian framework that we are working within this specification implies that the risk premiums are given by

$$\Gamma_t = \gamma_0 + \gamma_1 X_t \tag{12}$$

where $\gamma_0 \in \boldsymbol{R}^{3(N+1)\times 1}$and $\gamma_1 \in \boldsymbol{R}^{3(N+1)\times 3(N+1)}$contain unrestricted parameters. Thus, in general, we can write the P-dynamics of the state variables as

$$dX_t = K^P(\theta^P - X_t)dt + \Sigma_n dW_t^P \tag{13}$$

where both K^P and θ^P are allowed to vary freely relative to their counterparts under the Q measure.

5 Estimation Strategy

We estimate the models introduced in the previous subsections using the Kalman filter. Since all models considered in this paper are affine Gaussian models, the Kalman filter is an efficient and consistent estimator. In addition, the Kalman filter requires a minimum of assumptions about the observed data, and it easily handles missing data. This motivates our use of the standard Kalman filter in this setting. Unlike DLY, we introduces a strategy that allows us to estimate all unknown parameters via a one-step approach using panel data of bond yield curves. The measurement equation for the panel data of bond yields is given by

$$\begin{aligned}
\begin{pmatrix} y^1(\tau_1) \\ y^1(\tau_2) \\ \cdots \\ y^1(\tau_J) \\ \cdots \\ y^N(\tau_J) \end{pmatrix}_{NJ\times 1} &= \boldsymbol{B}\begin{pmatrix} L_t^g \\ S_t^g \\ C_t^g \end{pmatrix} + \boldsymbol{A}\begin{pmatrix} L_t^{l(1)} \\ S_t^{l(1)} \\ C_t^{l(1)} \\ \cdots \\ L_t^{l(N)} \\ S_t^{l(N)} \\ C_t^{l(N)} \end{pmatrix} - \begin{pmatrix} \frac{A_y^1(t,T)}{\tau_1} \\ \frac{A_y^1(t,T)}{\tau_2} \\ \cdots \\ \frac{A_y^1(t,T)}{\tau_J} \\ \cdots \\ \frac{A_y^N(t,T)}{\tau_J} \end{pmatrix} + \begin{pmatrix} v_{1,t}(\tau_1) \\ v_{1,t}(\tau_2) \\ \cdots \\ v_{1,t}(\tau_J) \\ \cdots \\ v_{N,t}(\tau_J) \end{pmatrix} \\
&= \boldsymbol{C}\begin{pmatrix} L_t^g \\ S_t^g \\ C_t^g \\ L_t^{l(1)} \\ S_t^{l(1)} \\ C_t^{l(1)} \\ \cdots \\ L_t^{l(N)} \\ S_t^{l(N)} \\ C_t^{l(N)} \end{pmatrix} - \begin{pmatrix} \frac{A_y^1(t,T)}{\tau_1} \\ \frac{A_y^1(t,T)}{\tau_2} \\ \cdots \\ \frac{A_y^1(t,T)}{\tau_J} \\ \cdots \\ \frac{A_y^N(t,T)}{\tau_J} \end{pmatrix} + \begin{pmatrix} v_{1,t}(\tau_1) \\ v_{1,t}(\tau_2) \\ \cdots \\ v_{1,t}(\tau_J) \\ \cdots \\ v_{N,t}(\tau_J) \end{pmatrix}
\end{aligned} \tag{14}$$

where N is the number of countries and J is the number of maturities.

$$\underset{NJ\times 3N}{\boldsymbol{A}} = \begin{pmatrix} 1 & \left(\frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1}\right) & \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau_1} - e^{-\lambda\tau_1}\right) & 0 & \cdots & 0 & 0 & 0 \\ 1 & \left(\frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2}\right) & \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau_1} - e^{-\lambda\tau_1}\right) & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \left(\frac{1-e^{-\lambda\tau_J}}{\lambda\tau_J}\right) & \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau_1} - e^{-\lambda\tau_1}\right) \end{pmatrix} \tag{15}$$

$$\underset{NJ\times 3}{\boldsymbol{B}} = \begin{pmatrix} \beta_L^1 & \beta_S^1\left(\frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1}\right) & \beta_S^1\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau_1} - e^{-\lambda\tau_1}\right) \\ \beta_L^1 & \beta_S^1\left(\frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2}\right) & \beta_S^1\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau_2} - e^{-\lambda\tau_1}\right) \\ \cdots & & \cdots \\ \beta_L^N & \beta_S^N\left(\frac{1-e^{-\lambda\tau_J}}{\lambda\tau_J}\right) & \beta_S^N\left(\frac{1-e^{-\lambda\tau_J}}{\lambda\tau_J} - e^{-\lambda\tau_J}\right) \end{pmatrix} \tag{16}$$

$$\underset{C=[BA]}{\boldsymbol{C}} = \left(\begin{array}{c:c} \boldsymbol{B} & 0 \\ \hdashline 0 & \boldsymbol{A} \end{array}\right) \tag{17}$$

The global factor and β_S^i,$\beta_L^i$are not identified separately in equation (14). Therefore,we assume that the US loadings on the global factors are positive, that is, $\beta_n^i > 0, n = S, L$ to identify the signs of factors and factor loadings, We also fixed the λ_g and $\lambda_{l(i)}$ as 0.6 which attains the maximum value for curvature factor at maturity of 2.5 years. We initialize the estimates of the global factor using the first principal component for each factors estimated using Diebold and Li (2006) methods. We initialized the estimates of the local factor using the residual by regressing NS factors of each country on the the first principal component on the NS factors.

The numbers of estimated parameters are $6+20N = 86$. In the measurement equation,there are $2N$ parameters to be estimated (β_L^i, β_S^i), four for each country. In the state equations, there are $3+3N$ parameters, three mean-reversion parameters relative to the global factors (Kp_{11}^g,Kp_{22}^g,Kp_{33}^g) and three mean reversion parameters for the local factors for each country ($Kp_{i,11}$,$Kp_{i,22}$,$Kp_{i,33}$) to be estimated. The standard deviation is considered constant over time. For local factors of state equation, there are $3(N+1)$ standard deviations for each country. For local factors of measurement equation, there are $12N$ standard deviations for each country.

6 Estimation Results

In this section we report the empirical results obtained for our global factor term structure model. First we show the three estimated global factors and provide an intuitive macroeconomic explanation for each of them. We then assess the specification of our global factor model and evaluate its fit in terms of how well it can replicate yield curves across different maturities for different countries.

6.1 Estimates of the Global Factors

We report estimates of the term structure model, obtained using the monthly yield curve from January 1995 to November 2018. Table (1) indicates the estimated parameters and standard errors. Although we need to estimate many parameters most parameters estimated by our one-step approach are statistically significant.

— Table (1) —

We show the estimated common factor as a solid line and the first principal component as a dashed line, January 1995 through November 2018 in Figure (5).The close link between the common factor and the first principal component of level and slopes is confirmed. The correlation between the common factor and the first principal component of level, slope, and curvature are 0.90, 0.97, and 0.97, respectively. Overall, these results point to the crucial importance of global factors in explaining domestic yield curves. DLY showed the importance of two global yield factors related to global inflation and economic activity. Our results suggest that a third global factor also needs to be taken into account.

— Figure (5) —

6.2 Links to the global macroeconomy

Several studies, for example, Ang and Piazzesi (2003) and Diebold, Rudebusch and Aruoba (2006), show that latent country yield factors are linked to, and interact dynamically with, domestic macroeconomic factors. The key factors, moreover, are inflation and real activity, which are linked to the yield curve level and slope, respectively. In a parallel fashion, our extracted global level and slope factors reflect the major developments in global inflation and real activity during the past 20 years. The temporal decline in the global-level factor reflects the reduction of inflation in the industrialized countries. The correlation between our extracted global level factor and average G-7 inflation during our sample is 0.3. Similarly, movements in the global slope factor reflect the global business cycle, with the global slope factor peaking just before the two global recessions of the early 1990s and in early 2009. The correlation between our extracted global slope factor and average G-7 GDP annual growth during our sample is 0.4.

— Figure (6) —

Finding an economic interpretation for the global curvature factor is, however, a novel task. Figure (7) plots the global curvature factor together with several related macro-finance series: a financial stress index published by the Federal Reserve Bank of St. Louis, the U.S. policy uncertainty index derived by Baker, Bloom and Davis (2016), and the U.S. term premium computed by Bauer, Rudebusch and Wu (2014). The correlation between our extracted global curvature factor and a financial stress index, the U.S. policy uncertainty index and the U.S. term premium during our sample is -0.48,0.48 and 0.65 respectively. Overall, the dynamics of the global curvature factor shows low level in the first part of the sample (up to the middle of 2000) and an increase in the second part, especially around $2007-2013$. The graphs show that the peak of the global curvature in $2010-2013$ coincides with a sharp increase in the policy uncertainty index. It also precedes the peak in the financial stress index and thus anticipates this financial/liquidity risk episode. Finally, a positive shock to the global curvature factor produces an increase in term premiums across all countries. Interestingly, these results are consistent with the bottom-right panel of Figure (7), which plots the global curvature factor against the U.S. term premium implied by the model in Bauer et al. (2014). It reveals that the global curvature factor captures the dynamics of the term premium very closely. This is especially the case at the end of the sample, coinciding with the onset of the financial crisis.

— Figure (7) —

6.3 Loading of yields on Global Factors

We regress yields on the global and local factors to gauge the shape of the implied loadings. As figure (8) shows, the loadings on the global level factor are relatively flat and persistent, resembling a classical global level factor. The loadings on the global slope factor differ across countries, exhibiting either increasing or decreasing patterns. Finally, loadings on the curvature factor are all decreasing and, in most cases, convex. The results indicate that global yield-level, slope, and curvature factors account for a significant fraction of variation in country bond yields with interesting differences across countries.

— Figure (8) —

6.4 Model Performance

To evaluate the fit of the model, Figure(8) shows actual mean yield vs estimated mean yield from global factor model. The fit of the model is relatively good for all countries.

— Figure (9) —

7 Concluding Remarks and Extensions

We have extended Diebold et al. (2008) to an arbitrage-free setting, proposing a global factor model in which country yield-level and slope factors may depend on global-level, slope and curvature factors as well as country-specific factors. Using a monthly dataset of government bond yields for Germany, Japan, the US, and the UK from January 1995 to November 2018, we extracted global and country-specific factors for both the full sample. The results indicate strongly that global yield-level, slope, and curvature factors do indeed exist and are economically important, accounting for a significant fraction of variation in country bond yields. Moreover, the global yield factors appear linked to global macroeconomic fundamentals. We show, in particular, that curvature factors are key to explaining term premium dynamics, and appear more important in the second sub-sample. In future work, we intend to examine the term premium dynamics and the forecasting ability of the model.

8 Appendix A: The analytical solution of the zero coupon bond yield function

All models analyzed in this paper are nested versions of the $3(N+1)$-factor model of zero-coupon bond prices introduced in Section 4. The $3(N+1)$ state variables consist of a global level, slope, and curvature factor in addition to a local level, slope, and curvature factor.That is, $X_t = (L_t^g, S_t^g, C_t^g, L_t^{l(i)}, S_t^{l(i)}, C_t^{l(i)})'$ In order to obtain a Nelson-Siegel factor loading structure for global and local factors in the zero-coupon bond yield function of the form, three assumptions must be imposed. First, we define a factor model for the yield curve that considers both global and local factors. We decompose the factors in the factor model into the sum of two orthogonal components:the global factor,which we denote by X_t^g and local factors demoted by X_t^l. Second, the instantaneous short rate is assumed to be an affine function of the state variables

$$r_t^i = \rho_0^i + \rho_1'(\beta X_t^g + X_t^l) \tag{18}$$

$$= \beta_L^i L_t^g + \beta_S^i S_t^g + L_t^{l(i)} + S_t^{l(i)} \tag{19}$$

This means that the vector of factor loadings in the discount rate are given by

$$\rho_0^i = 0, \rho_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} \beta_L^i \\ \beta_S^i \\ 0 \end{pmatrix} \tag{20}$$

Third, the dynamics of the $3(N+1)$ factors under the risk-neutral pricing measure must be given by the solution to the following SDE

$$\begin{pmatrix} dL_t^g \\ dS_t^g \\ dC_t^g \\ dL_t^{l(i)} \\ dS_t^{l(i)} \\ dC_t^{l(i)} \end{pmatrix} = - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^g & -\lambda^g & 0 & 0 & 0 \\ 0 & 0 & \lambda^g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^{l(i)} & -\lambda^{l(i)} \\ 0 & 0 & 0 & 0 & 0 & \lambda^{l(i)} \end{pmatrix} \begin{pmatrix} L_t^g \\ S_t^g \\ C_t^g \\ L_t^{l(i)} \\ S_t^{l(i)} \\ C_t^{l(i)} \end{pmatrix} dt$$
$$+ \begin{pmatrix} \sigma_L^g & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_S^g & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_C^g & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_L^{l(i)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_S^{l(i)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_C^{l(i)} \end{pmatrix} \begin{pmatrix} dW_t^{L^g,Q} \\ dW_t^{S^g,Q} \\ dW_t^{C^g,Q} \\ dW_t^{L^{l(i)},Q} \\ dW_t^{S^{l(i)},Q} \\ dW_t^{C^{l(i)},Q} \end{pmatrix} \tag{21}$$

Note here that all $3(N+1)$ factors are Gaussian processes with constant volatility matrix. From Duffie and Kan (1996) it now follows that the price at time t of a representative zero-coupon bond from country i with maturity at T is given by

$$P^i(t,T) = E^Q[e^{-\int_t^T r_u^i du}]$$
$$= \exp(B_{L^g}^i(t,T)L^g + B_{S^g}^i(t,T)S^g + B_{C^g}^i(t,T)C^g + B_{L^l}^i(t,T)L^{l(i)} + B_{S^l}^i(t,T)S^{l(i)} + B_{C^l}^i(t,T)C^{l(i)} + A_y^i(t,T)) \tag{22}$$

where $B_{L^g}^i(t,T)$, $B_{S^g}^i(t,T)$, $B_{C^g}^i(t,T)$,$B_{L^l}^i(t,T)$, $B_{S^l}^i(t,T)$, $B_{C^l}^i(t,T)$ and $A_y^i(t,T)$ are the unique solutions to the following set of ODEs.

$$\frac{dB^i(t,T)}{dt} = \rho_1^i + (K^Q(i))'B^i(t,T), \;\; B(T,T) = 0 \tag{23}$$

$$\frac{dA_y^i(t,T)}{dt} = \rho_0^i - \frac{1}{2}\sum_{j=1}^{6}(\Sigma^i)'B^i(s,T)B^i(s,T)'\Sigma^i)_{j,j}, \;\; A_y^i(T,T) = 0 \tag{24}$$

Now, we solve these two systems of ODEs. We start with the system for $B^i(t,T)$ Since

$$\frac{d}{dt}[e^{(K^Q(i)(T-t)}B^i(t,T)] = e^{(K^Q(i)(T-t)}\frac{dB^i(t,T)}{dt} - (K^Q(i))'e^{(K^Q(i)(T-t)}B^i(t,T) \tag{25}$$

we can use the system of ODEs for $B^i(t,T)$ to obtain

$$\int_t^T \frac{d}{dt}[e^{(K^Q(i))'(T-t)}B^i(t,T)]ds = \int_t^T e^{(K^Q(i))'(T-s)}\rho_1^i ds \tag{26}$$

or, equivalently, using the boundary conditions

$$B^i(t,T) = -e^{(K^Q(i))'(T-t)}\int_t^T e^{(K^Q(i))'(T-s)}\rho_1^i ds \tag{27}$$

Here, it is easy to show that

$$e^{(K^Q(i))'(T-s)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\lambda^g(T-t)} & 0 & 0 & 0 & 0 \\ 0 & -\lambda^g(T-t)e^{\lambda^g(T-t)} & e^{\lambda^g(T-t)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda^{l(i)}(T-t)} & 0 \\ 0 & 0 & 0 & 0 & -\lambda^{l(i)}(T-t)e^{\lambda^{l(i)}(T-t)} & e^{\lambda^{l(i)}(T-t)} \end{pmatrix} \tag{28}$$

and

$$e^{-(K^Q(i))'(T-s)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-\lambda^g(T-t)} & 0 & 0 & 0 & 0 \\ 0 & \lambda^g(T-t)e^{\lambda^g(T-t)} & e^{-\lambda^g(T-t)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\lambda^{l(i)}(T-t)} & 0 \\ 0 & 0 & 0 & 0 & \lambda^{l(i)}(T-t)e^{\lambda^{l(i)}(T-t)} & e^{-\lambda^{l(i)}(T-t)} \end{pmatrix} \tag{29}$$

Inserting this into the expression for $B^i(t,T)$ we obtain

$$\begin{aligned} B^i(t,T) = &-\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-\lambda^g(T-t)} & 0 & 0 & 0 & 0 \\ 0 & \lambda^g(T-t)e^{\lambda^g(T-t)} & e^{-\lambda^g(T-t)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\lambda^{l(i)}(T-t)} & 0 \\ 0 & 0 & 0 & 0 & \lambda^{l(i)}(T-t)e^{\lambda^{l(i)}(T-t)} & e^{-\lambda^{l(i)}(T-t)} \end{pmatrix} \\ &\times \int_t^T \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\lambda^g(T-t)} & 0 & 0 & 0 & 0 \\ 0 & -\lambda^g(T-t)e^{\lambda^g(T-t)} & e^{\lambda^g(T-t)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda^{l(i)}(T-t)} & 0 \\ 0 & 0 & 0 & 0 & -\lambda^{l(i)}(T-t)e^{\lambda^{l(i)}(T-t)} & e^{\lambda^{l(i)}(T-t)} \end{pmatrix} \begin{pmatrix} \beta_L^i \\ \beta_S^i \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} ds \end{aligned} \tag{30}$$

This can be rewritten as

$$B^i(t,T) = -\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-\lambda^g(T-t)} & 0 & 0 & 0 & 0 \\ 0 & \lambda^g(T-t)e^{\lambda^g(T-t)} & e^{-\lambda^g(T-t)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\lambda^{l(i)}(T-t)} & 0 \\ 0 & 0 & 0 & 0 & \lambda^{l(i)}(T-t)e^{\lambda^{l(i)}(T-t)} & e^{-\lambda^{l(i)}(T-t)} \end{pmatrix}$$

$$\times \int_t^T \begin{pmatrix} \beta_L^i \\ \beta_S^i e^{\lambda^g(T-t)} \\ -\beta_S^i \lambda^g(T-t)e^{\lambda^g(T-t)} \\ 1 \\ e^{\lambda^{l(i)}(T-t)} \\ -\lambda^{l(i)}(T-t)e^{\lambda^{l(i)}(T-t)} \end{pmatrix} ds \tag{31}$$

Since

$$\int_t^T ds = T - t \tag{32}$$

and

$$\int_t^T e^{\lambda(T-s)} ds = \left[-\frac{1}{\lambda}e^{\lambda(T-s)}\right]_t^T = -\frac{1-e^{\lambda(T-t)}}{\lambda} \tag{33}$$

and

$$\begin{aligned} \int_t^T -\lambda(T-t)e^{\lambda(T-s)} ds &= \frac{1}{\lambda}\int_{\lambda(T-t)}^0 e^x dx = \frac{1}{\lambda}[xe^x]_{\lambda(T-t)}^0 - \frac{1}{\lambda}\int_{\lambda(T-t)}^0 e^x \\ &= -(T-t)e^{\lambda(T-t)} - \frac{1}{\lambda}[e^x]_{\lambda(T-t)}^0 = -(T-t)e^{\lambda(T-t)} - \frac{1-e^{\lambda(T-t)}}{\lambda} \end{aligned} \tag{34}$$

the system of ODEs can be reduced to

$$B^i(t,T) = -\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-\lambda^g(T-t)} & 0 & 0 & 0 & 0 \\ 0 & \lambda^g(T-t)e^{\lambda^g(T-t)} & e^{-\lambda^g(T-t)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\lambda^{l(i)}(T-t)} & 0 \\ 0 & 0 & 0 & 0 & \lambda^{l(i)}(T-t)e^{\lambda^{l(i)}(T-t)} & e^{-\lambda^{l(i)}(T-t)} \end{pmatrix}$$

$$\times \int_t^T \begin{pmatrix} \beta_L^i \\ \beta_S^i \frac{1-e^{\lambda^g(T-t)}}{\lambda^g} \\ -\beta_S^i \left[(T-t)e^{\lambda^g(T-t)} - \frac{1-e^{\lambda^g(T-t)}}{\lambda^g}\right] \\ 1 \\ \frac{1-e^{\lambda^{l(i)}(T-t)}}{\lambda^{l(i)}} \\ (T-t)e^{\lambda^{l(i)}(T-t)} - \frac{1-e^{\lambda^{l(i)}(T-t)}}{\lambda} \end{pmatrix} \tag{35}$$

This system can be reduced even further whereby the final solution is obtained as

$$B^i(t,T) = \begin{pmatrix} \beta_L^i \\ \beta_S^i \frac{1-e^{\lambda^g(T-t)}}{\lambda^g} \\ -\beta_S^i \left[(T-t)e^{\lambda^g(T-t)} - \frac{1-e^{\lambda^g(T-t)}}{\lambda^g}\right] \\ 1 \\ \frac{1-e^{\lambda^{l(i)}(T-t)}}{\lambda^{l(i)}} \\ (T-t)e^{\lambda^{l(i)}(T-t)} - \frac{1-e^{\lambda^{l(i)}(T-t)}}{\lambda} \end{pmatrix} \tag{36}$$

Thus, the yield-to-maturity of the zero-coupon bond is given by

$$\begin{aligned} y^i(t,T) &= -\frac{1}{T-t}\ln P^i(t,T) \\ &= \beta_L^i L_t^g + \beta_S^i \left(\frac{1-e^{-\lambda^g(T-t)}}{\lambda^g(T-t)}\right) S_t^g + \beta_S^i \left(\frac{1-e^{-\lambda^g(T-t)}}{\lambda^g(T-t)} - e^{-\lambda^g(T-t)}\right) C_t^g \\ &+ L_t^{l(i)} + \left(\frac{1-e^{-\lambda(T-t)}}{\lambda(T-t)}\right) S_t^{l(i)} + \left(\frac{1-e^{-\lambda(T-t)}}{(\lambda^{l(i)})\tau} - e^{-(\lambda^{l(i)})(T-t)}\right) C_t^{l(i)} - \frac{A_y^i(t,T)}{T-t} \end{aligned} \tag{37}$$

In order to obtain a full analytical formula for the corporate bond yields we need to look into the details of the maturity adjustment term $\frac{A_y^i(t,T)}{T-t}$.That is the focus of the following section.

8.1 The maturity adjustment term

The formula for the maturity adjustment term is given by

$$\frac{A_y^i(t,T)}{T-t} = \frac{1}{2}\frac{1}{T-t}\int_t^T \sum_{j=1}^{6}((\Sigma^i)' B^i(s,T) B^i(s,T)'\Sigma^i)_{j,j} ds \tag{38}$$

where $B^i(t,T) = (B_{L^g}(t,T), B_{s^g}(t,T), B_{C^g}(t,T), B_{L^l}^i(t,T), B_{S^l}^i(t,T), B_{C^l}^i(t,T))$ Since all factors are assumed to be independent, the volatility matrix is diagonal

$$\Sigma^i = \begin{pmatrix} \sigma_L^g & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_S^g & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_C^g & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_L^{l(i)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_S^{l(i)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_C^{l(i)} \end{pmatrix} \tag{39}$$

Thus, the maturity adjustment term reduces to

$$\begin{aligned} \frac{A_y^i(t,T)}{T-t} &= \frac{1}{2}\frac{1}{T-t}\left[(\sigma_L^g)^2 \int_t^T B_{L^g}^i(t,T)^2 ds + \frac{1}{2}\frac{1}{T-t}(\sigma_S^g)^2 \int_t^T B_{S^g}^i(t,T)^2 ds + \frac{1}{2}\frac{1}{T-t}(\sigma_C^g)^2 \int_t^T B_{C^g}^i(t,T)^2 ds\right] \\ &+ \frac{1}{2}\frac{1}{T-t}\left[(\sigma_L^{l(i)})^2 \int_t^T B_{L^l}^i(t,T)^2 ds + \frac{1}{2}\frac{1}{T-t}(\sigma_S^{l(i)})^2 \int_t^T B_{S^l}^i(t,T)^2 ds + \frac{1}{2}\frac{1}{T-t}(\sigma_C^{l(i)})^2 \int_t^T B_{C^l}^i(t,T)^2 ds\right] \end{aligned} \tag{40}$$

All these integrals have already been solved in Christensen et al. (2011), so we are able to immediately present the final result.

8.2 Summary

Given the assumption of independent factors, the zero-coupon yield function for a representative bond from country i and $T-t$ years to maturity is equal to

$$\begin{aligned}
\frac{A_y^i(t,T)}{T-t} &= (\sigma_L^g)^2(\beta_L^i)^2\frac{(T-t)^2}{6} \\
&+ (\sigma_S^g)^2(\beta_S^i)^2\left[\frac{1}{2(\lambda^g)^2} - \frac{1}{(\lambda^g)^3}\frac{1-e^{-\lambda^g(T-t)}}{(T-t)} + \frac{1}{4(\lambda^g)^3}\frac{1-e^{-2\lambda^g(T-t)}}{(T-t)}\right] \\
&+ (\sigma_C^g)^2(\beta_S^i)^2\left[\frac{1}{2(\lambda^g)^2} + \frac{1}{(\lambda^g)^2}e^{-(\lambda^g)(T-t)} - \frac{1}{4\lambda^g}(T-t)e^{-2\lambda^g(T-t)} - \frac{3}{4(\lambda^g)^2}e^{-2\lambda^g n} - \frac{2}{(\lambda^g)^3}\frac{1-e^{-\lambda^g(T-t)}}{(T-t)} + \frac{5}{8(\lambda^g)^3}\frac{1-e^{-2\lambda^g(T-t)}}{(T-t)}\right] \\
&+ (\sigma_L^{l(i)})^2\frac{(T-t)^2}{6} \\
&+ (\sigma_S^{l(i)})^2\left[\frac{1}{2(\lambda^{l(i)})^2} - \frac{1}{(\lambda^{l(i)})^3}\frac{1-e^{-\lambda^{l(i)}(T-t)}}{(T-t)} + \frac{1}{4(\lambda^{l(i)})^3}\frac{1-e^{-2\lambda^{l(i)}(T-t)}}{(T-t)}\right] \\
&+ (\sigma_C^{l(i)})^2\left[\frac{1}{2(\lambda^{l(i)})^2} + \frac{1}{(\lambda^{l(i)})^2}e^{-\lambda^{l(i)}(T-t)} - \frac{1}{4\lambda^{l(i)}}(T-t)e^{-2\lambda^{l(i)}(T-t)} - \frac{3}{4(\lambda^{l(i)})^2}e^{-2\lambda^{l(i)}(T-t)}\right. \\
&\left. - \frac{2}{(\lambda^{l(i)})^3}\frac{1-e^{-\lambda^{l(i)}(T-t)}}{(T-t)} + \frac{5}{8(\lambda^{l(i)})^3}\frac{1-e^{-2\lambda^{l(i)}(T-t)}}{(T-t)}\right]
\end{aligned} \tag{41}$$

9 Appendix B: Kalman filter estimation

We display the state-space representations of the AFNS global factor models. The state equation in the Kalman filter is given by

$$X_t = \Phi_t X_{t-1} + \eta_t \tag{42}$$

where $\Phi_t = \exp(-K^P \triangle t)$ and $\triangle t = T - t$ is the time between observations. In the Kalman filter estimations, all measurement errors are assumed to be i.i.d. white noise. Thus, the error structure is in general given by

$$\begin{pmatrix}\eta_t \\ v_t\end{pmatrix} \sim N\left[\begin{pmatrix}0 \\ 0\end{pmatrix}\begin{pmatrix}Q & 0 \\ 0 & H\end{pmatrix}\right] \tag{43}$$

where

$$Q = \int_0^{\triangle t} e^{-K^P s}\Sigma\Sigma' e^{-(K^P)' s} ds \tag{44}$$

where matrix H is diagonal, and the matrix Q is diagonal. Denote the yield curve information available at time t by $y_t = (y^1(\tau_1), y^1(\tau_1), \cdots y^N(\tau_J))$ and denote model parameters by ψ. Consider the period and suppose that the state update X_t and its mean square error matrix have been obtained. The prediction step is

$$X_{t|t-1} = \Phi_t X_{t-1|t-1} \tag{45}$$

$$P_{t|t-1} = \Phi_t P_{t-1|t-1}\Phi_t' + Q \tag{46}$$

In the time t update step, $X_{t|t-1}$ is improved by using the additional information contained in y_t. We have

$$X_{t|t} = X_{t|t-1} + P_{t|t-1}\boldsymbol{C}' F_{t|t-1}^{-1} v_{t|t-1} \tag{47}$$

$$P_{t|t} = (1 - P_{t|t-1}\boldsymbol{C}' F_{t|t-1}^{-1}\boldsymbol{C})P_{t|t-1} \tag{48}$$

where

$$v_{t|t-1} = y_t + \frac{A_y^i(t,T)}{T-t} - \boldsymbol{C}X_{t|t-1} \tag{49}$$

$$F_{t|t-1} = \boldsymbol{C}P_{t|t-1}\boldsymbol{C}' + H \tag{50}$$

$$H(\psi) = \text{diag}(\sigma_v^2(\tau_1), \dots, \sigma_v^2(\tau_J)) \tag{51}$$

At this point, the Kalman filter has delivered all ingredients needed to evaluate the Gaussian log likelihood, the prediction-error decomposition of which is

$$\log l(\psi) = \sum_{t=1}^{T}(-\frac{JN}{2}\log(2\pi) - \frac{1}{2}\log \mid F_t \mid -\frac{1}{2}v_t' F_t^{-1} v_t) \tag{52}$$

where JN is the number of observed yields. We numerically maximize the likelihood with respect to ψ. Upon convergence, we obtain standard errors from the estimated covariance matrix. The linear least-squares optimality of the Kalman filter require that the white noise transition and measurement errors be orthogonal to the initial state. Finally, the standard deviations of the estimated parameters are calculated as

$$\Omega(\hat{\psi}) = \frac{1}{T}\left[\frac{1}{T}\sum_{t=1}^{T}\frac{\partial \log l_t(\hat{\psi})}{\partial \psi}\frac{\partial \log l_t(\hat{\psi})'}{\partial \psi}\right]^{-1} \tag{53}$$

where $\hat{\psi}$ denotes the optimal parameter set.

References

Abbritti, Mirko, Salvatore Dell'Erba, Antonio Moreno, Sergio Sola et al., "Global Factors in the Term Structure of Interest Rates," *International Journal of Central Banking*, 2018, *14* (2), 301–339.

Ang, Andrew and Monika Piazzesi, "A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables," *Journal of Monetary economics*, 2003, *50* (4), 745–787.

Bae, Byung Yoon and Dong Heon Kim, "Global and regional yield curve dynamics and interactions: The case of some Asian countries," *International Economic Journal*, 2011, *25* (4), 717–738.

Baker, Scott R, Nicholas Bloom, and Steven J Davis, "Measuring economic policy uncertainty," *The quarterly journal of economics*, 2016, *131* (4), 1593–1636.

Bauer, Michael D, Glenn D Rudebusch, and Jing Cynthia Wu, "Term premia and inflation uncertainty: Empirical evidence from an international panel dataset: Comment," *American Economic Review*, 2014, *104* (1), 323–37.

Christensen, Jens HE, Francis X Diebold, and Glenn D Rudebusch, "The affine arbitrage-free class of Nelson–Siegel term structure models," *Journal of Econometrics*, 2011, *164* (1), 4–20.

Coroneo, Laura, Ian Garrett, and Javier Sanhueza, "Dynamic Linkages Across Country Yield Curves: The Effects of Global and Local Yield Curve Factors on US, UK and German Yields," in "New Methods in Fixed Income Modeling," Springer, 2018, pp. 205–222.

Diebold, Francis X and Canlin Li, "Forecasting the term structure of government bond yields," *Journal of econometrics*, 2006, *130* (2), 337–364.

_ , _ , and Vivian Z Yue, "Global yield curve dynamics and interactions: a dynamic Nelson–Siegel approach," *Journal of Econometrics*, 2008, *146* (2), 351–363.

_ , Glenn D Rudebusch, and S Boragan Aruoba, "The macroeconomy and the yield curve: a dynamic latent factor approach," *Journal of econometrics*, 2006, *131* (1-2), 309–338.

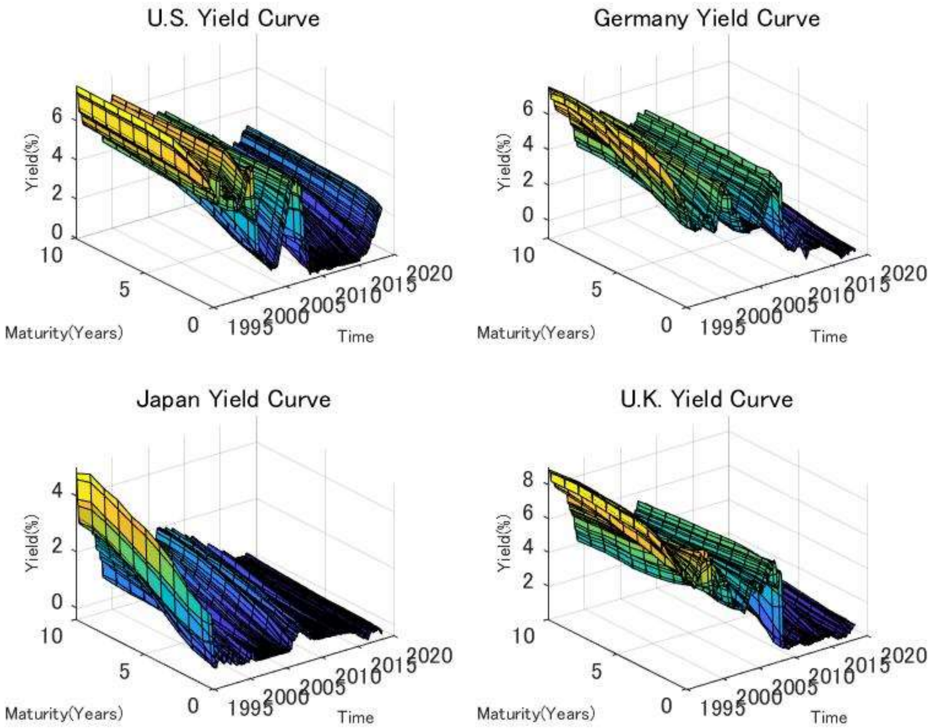
Duffee, Gregory R, "Term premia and interest rate forecasts in affine models," *The Journal of Finance*, 2002, *57* (1), 405–443.

Duffie, Darrell and Rui Kan, "A yield-factor model of interest rates," *Mathematical finance*, 1996, *6* (4), 379–406.

Kaminska, Iryna, Andrew Meldrum, and James Smith, "A Global Model of International Yield Curves: No-Arbitrage Term Structure Approach," *International Journal of Finance & Economics*, 2013, *18* (4), 352–374.

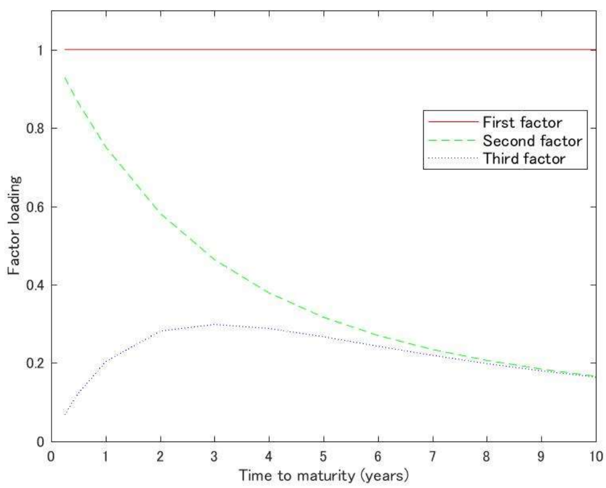
Morita, Rubens Hossamu and Rodrigo DLS Bueno, "Investment grade countries yield curve dynamics," in "XXX Meeting of the Brazilian Econometric Society" 2008.

Figure (1): Yield curves across countries and time



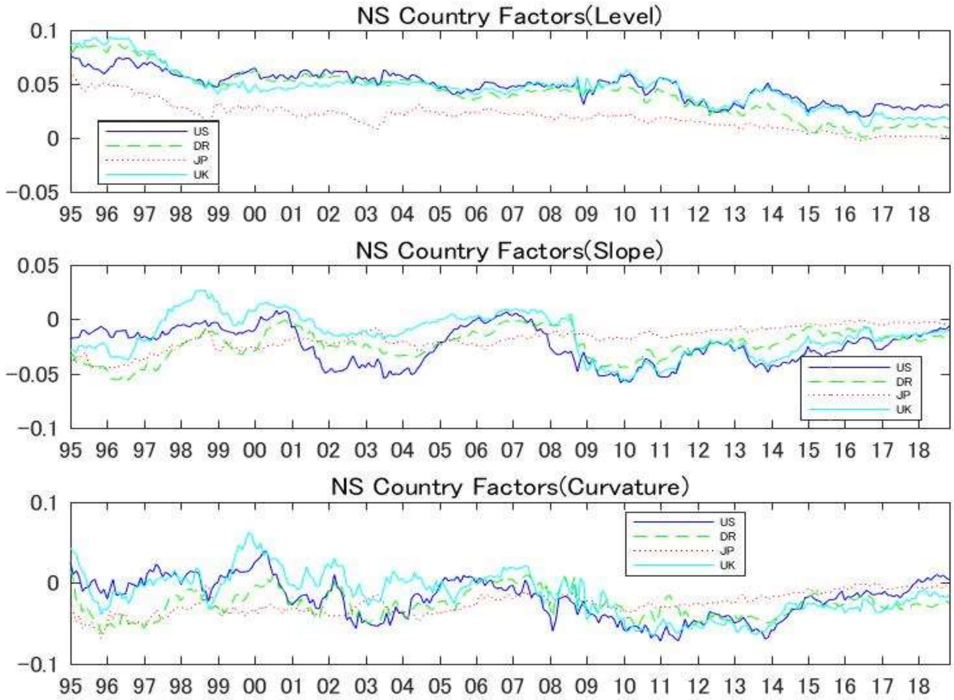
(*Note*): This figure shows the evolution of the yield curves across countries. All yield data are monthly, January 1995 through November 2018.

Figure (2): Factor loadings in the Nelson-Siegel yield function.



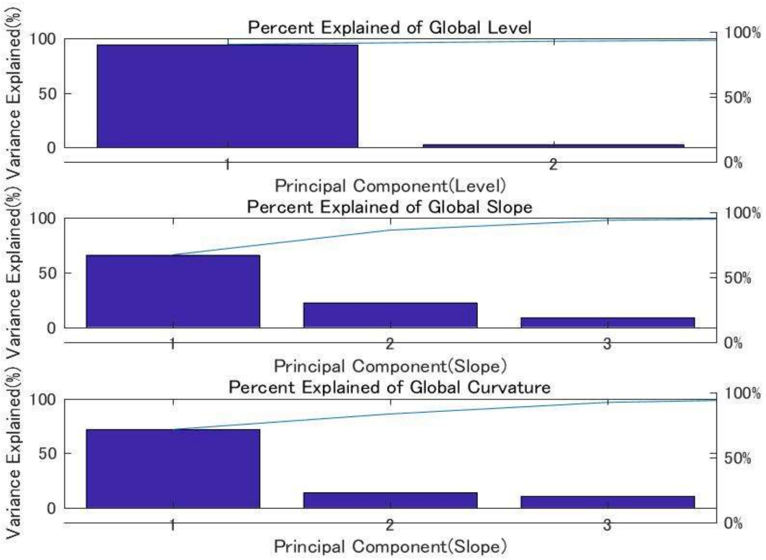
(*Note*): Illustration of the factor loadings on the three state variables in the Nelson-Siegel model. The value for λ is 0.60 and maturity is measured in years.

Figure (3): Term structure factors of Diebold Li model by country



(*Note*): This figure shows estimates of the company level and slope factor, obtained via step one of the Diebold and Li(2006), using monthly yield curve January 1995 to November 2018.

Figure (4): Principal Component Analysis



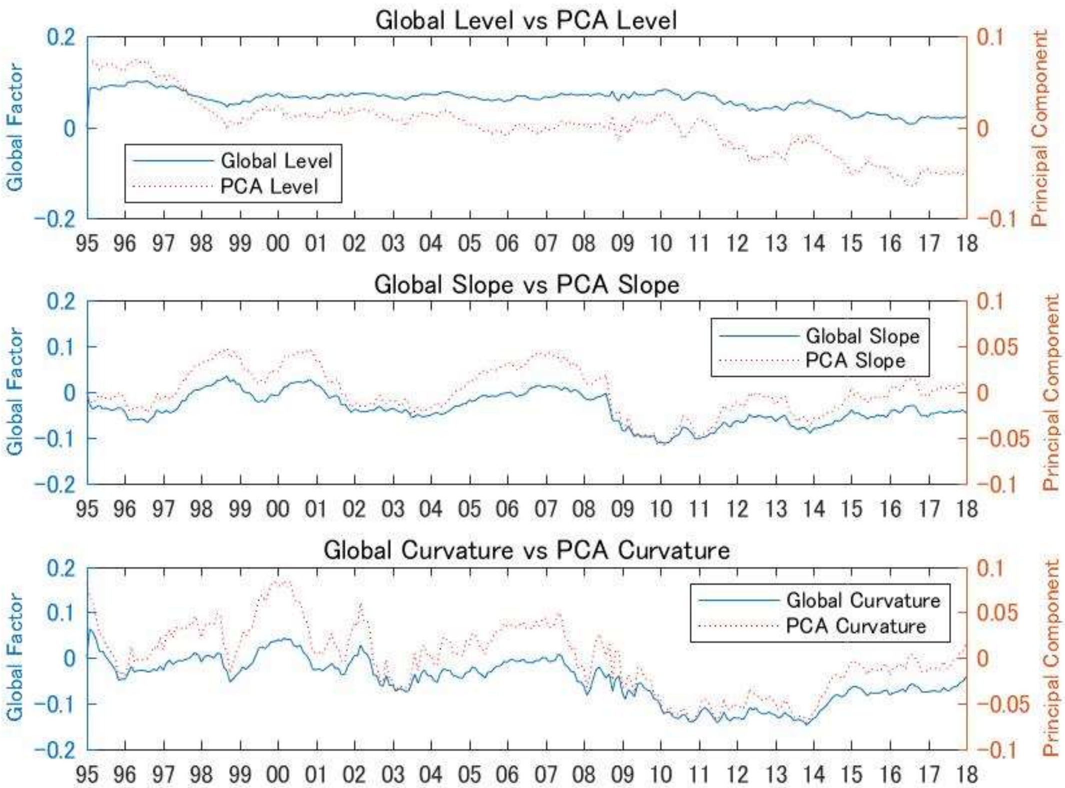
(*Note*) : Percent explained of Global factors are demonstrated by principal component analysis

Table (1): Estimates of the AFNS-Global Factor model

State equation											Mesurement equation
Mean reversion	Estimates	std									
Kp11_g	0.037	0.034									
Kp22_g	0.125	0.083									
Kp33_g	0.117	0.082									
Mean reversion	Estimates	std	Standard deviation of Idiosyncratic factor	Estimates	std	Standard deviation of Idiosyncratic factor	Estimates	std	loading on global factor	Estimates	std
kp11_us	0.056	0.032	σ_l_g	0.043%	0.002%	σ_l_jp	0.024%	0.007%	β_l_us	0.251	0.021
kp22_us	0.234	0.140	σ_s_g	0.011%	0.003%	σ_s_jp	0.100%	0.004%	β_s_us	0.436	0.000
kp33_us	0.761	0.477	σ_c_g	0.040%	0.002%	σ_c_jp	0.100%	0.005%	β_l_dr	0.621	0.006
kp11_dr	0.122	0.076	σ_l_us	0.030%	0.001%	σ_l_uk	0.419%	0.031%	β_s_dr	0.408	0.000
kp22_dr	0.112	0.064	σ_s_us	0.034%	0.001%	σ_s_uk	0.714%	0.055%	β_l_jp	0.342	0.014
kp33_dr	0.235	0.124	σ_c_us	0.001%	0.005%	σ_c_uk	1.659%	0.125%	β_s_jp	0.081	0.000
kp11_jp	0.608	0.263	σ_l_dr	0.043%	0.002%				β_l_uk	0.572	0.013
kp22_jp	0.111	0.067	σ_s_dr	0.090%	0.004%				β_s_uk	0.447	0.000
kp33_jp	0.169	0.087	σ_c_dr	0.100%	0.004%						
kp11_uk	0.127	0.067									
kp22_uk	0.545	0.297									
kp33_uk	0.914	0.649									

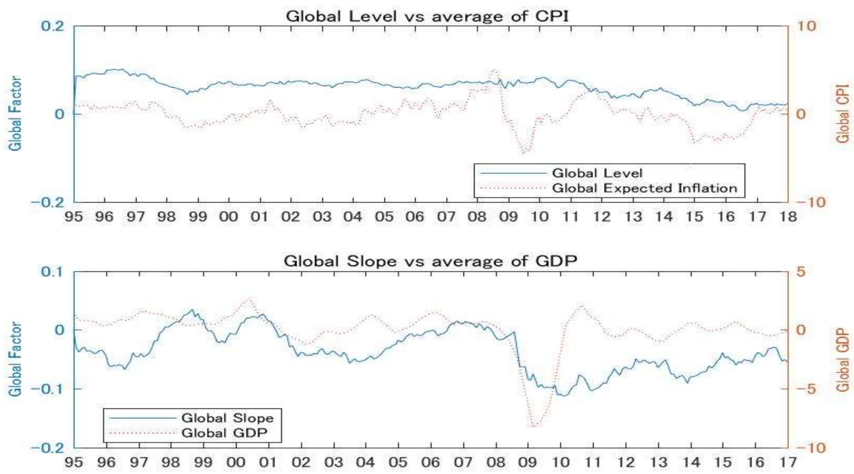
(*Note*): We report estimated parameters and standard errors of the term structure model, obtained using monthly yield curve during January 1995 through November 2018.

Figure (5): Global factor vs. first principal component of country level and slopes



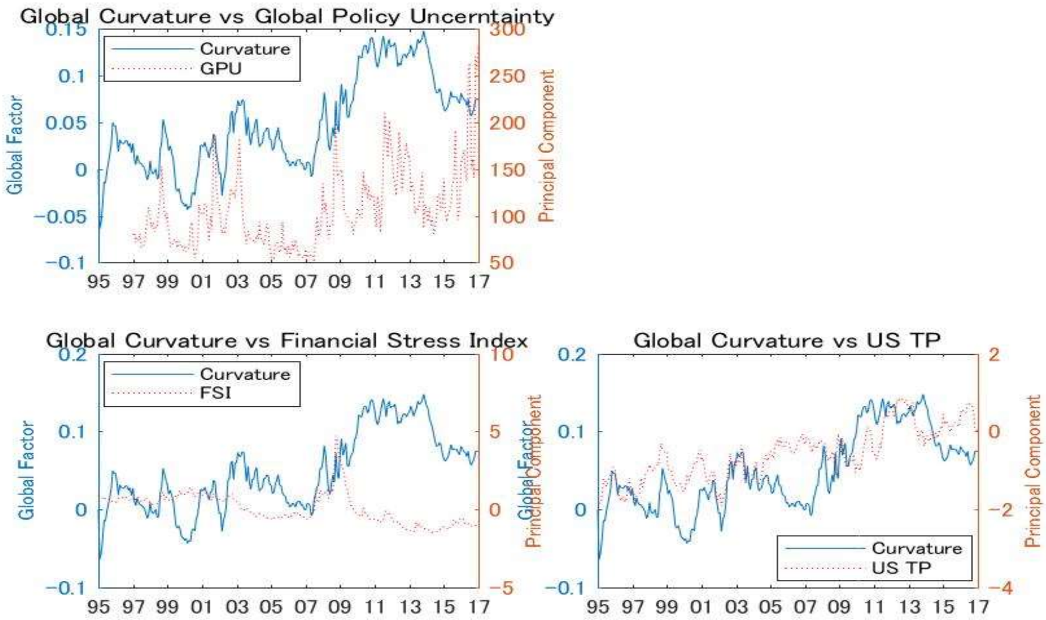
(*Note*): The estimated global factor as a solid line and the first principal component as a dashed line, January 1995 through November 2018. is presented.

Figure (6): Global Level and Slope Factor vs Macro Variables



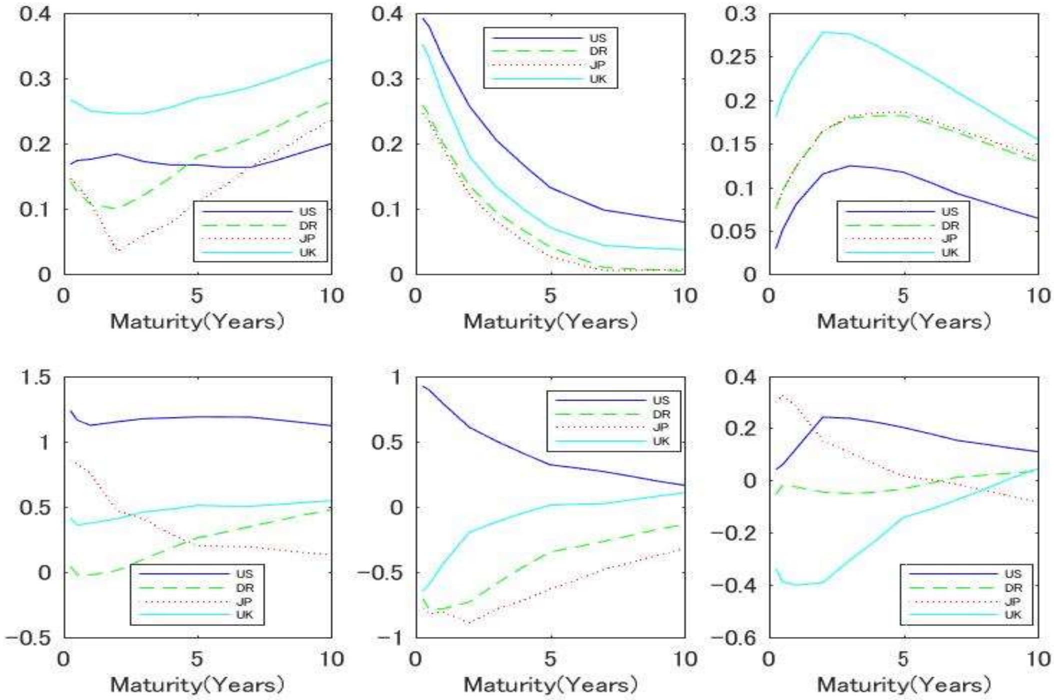
(*Note*): This figure shows the global level and slope factors plotted against their macroeconomic interpretations. The global level factor is plotted against the first principal component extracted from a matrix containing data on expected inflation for US, Germany, Japan and UK. The global slope factor is plotted against the first principal component extracted from a matrix containing data on GDP for above four countries.

Figure (7): Global Curvature Factor vs Macro Variables



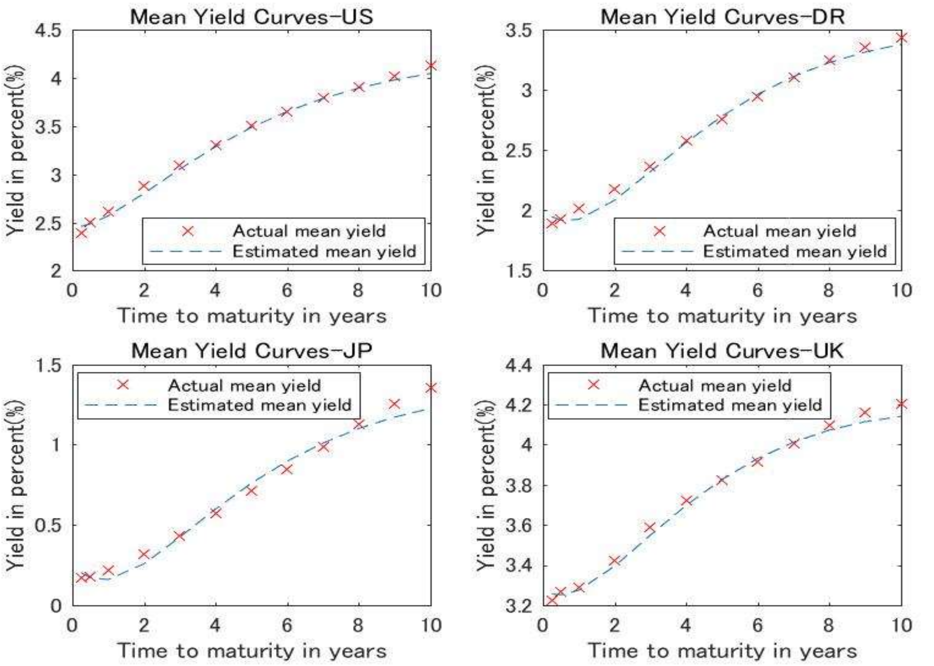
(*Note*): This figure plots the global curvature factor against several macro-finance time series: Top left: U.S. policy uncertainty index computed by Baker, Bloom, and Davis (2016). Bottom left: Financial stress index published by the Federal Reserve Bank of St. Louis. Bottom right: U.S. term premium, computed by Bauer, Rudebusch, and Wu (2014).

Figure (8): Loading of yields on Global Factors



(*Note*): This figure plots the factor loadings of regression of all yields on the estimated global(above) and local (below) factors.

Figure (9): Model Fit



(*Note*): This figure shows actual mean yield vs estimated mean yield from global factor model.