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Abstract

This paper theoretically investigates the role of the tax system in sustaining the public debt. The paper explicitly derives the critical level of the public debt-to-GDP ratio that is compatible with a balanced growth path. If the ratio exceeds this critical level at time 0, then it diverges to $+\infty$ as time passes. Analyzing a situation where the government marginally increases the consumption tax rate, the paper reveals the extent to which the government can then cut the income tax rate while maintaining the sustainability of public debt. Tax rates that are compatible with the balanced growth are also derived as a function of the initial level of debt-to-GDP ratio.

Key Words: sustainability of public debt, tax system, balanced growth path, dynamic general equilibrium

JEL Classification Numbers: E62, H6

1 Introduction

Along with many other developed countries, Japan faces high levels of accumulated public debt. The Organization for Economic Cooperation and Development (OECD) predicted that Japan's public debt-to-gross domestic product (GDP) ratio, which is monotonically increasing, would exceed 2.30 in 2016, (OECD "Economic Outlook 98", November 2015).[1] To avoid fiscal bankruptcy, the Japanese government has established an overriding objective: Japan's primary balance should be in surplus until 2020. To achieve this objective, the government has discussed implementing fiscal reforms (including increasing the consumption tax rate) to secure its financial resources.

The purpose of this paper is to theoretically study the role of the tax system in sustaining public debt. For that aim, this paper explicitly derives a critical level of public debt-to-GDP ratio that is compatible with a balanced growth path (BGP). If the ratio exceeds this critical level at time 0, then it diverges to $+\infty$ as time passes, which means that the public debt cannot be sustained. If the public debt-to-GDP ratio is at the critical level at time 0, then it remains constant over time; this can be interpreted as the public debt being sustainable. This paper highlights how the critical level of the public debt-to-GDP ratio depends on the tax system. The tax system includes the labor income and interest income tax rates, and the taxation on the consumption behavior of households.

The analysis reveals the "marginal rate of substitution of the tax rates". That is, if the government increases the consumption tax rate at the marginal level, then to what extent can it cut the income tax rate while maintaining a sustainable public debt. The analysis may be of interest to policy makers in those countries with high levels of accumulated public debt. Further, the approach used explicitly derives a tax rate that is compatible with the sustainability of public debt as a function of the initial level of debt-to-GDP ratio, given other parameters. For example, if the public debt-to-GDP ratio increases from 2.30 to 2.40, then it is possible to calculate the extent to which the government must increase the consumption tax rate to avoid fiscal bankruptcy.

Many studies examine the sustainability of public debt. Bräuninger (2005), Yakita (2008) and Arai (2011) theoretically study the relationship between a deficit ratio and economic growth rates, by using overlapping gen-

[1] http://www.paris-oecd.diplo.de/contentblob/4658486/Daten/6022232/Economic_Outlook.pdf

erations models. These studies find that there is a critical level of public debt-to-GDP ratio for sustainable public debt. If the public debt is greater than the critical level at the initial time point, then it is no longer sustainable. Greiner (2011) shows the existence of a critical level of public debt that is sustainable in a case where the primary surplus-to-GDP ratio is independent of the public debt-to-GDP ratio. This critical level depends on the difference between the real interest rate and the economic growth rate. Although Kondo (2007, 2012) and Kondo and Kitaura (2009) explicitly derive the critical level of public debt, they assume a lump-sum tax. Hence, they do not consider the interaction between different taxations for the sustainability of public debt. Kondo's (2016) model accounts for income and consumption taxes, and derives the critical level of the public debt-to-GDP ratio. However, that paper looks at properties of consumption tax in comparison with other taxes. Hence, the tax rates for labor income and interest income are supposed to be same. The present paper identifies the tax rates for the labor income and the interest income, to reflect the complex tax system of the real world.

Many empirical studies focus on the sustainability of public debt. Greiner, Koeller and Semmler (2007) and Fincke and Greiner (2011, 2012) test whether governments are properly subject to Bohn's (1998) rule for fiscal sustainability; i.e., governments should increase the primary surplus-to-GDP ratio to maintain a healthy fiscal status if public debt-to-GDP ratio increases. Greiner, Koeller and Semmler (2007) and Fincke and Greiner (2012) primarily focus on European countries, while Fincke and Greiner (2011) focuses on the United States, Germany and Japan. Bökemeier (2015) investigates the relationship between public deficits and economic growth rates for central and eastern European countries.

Many studies consider the tax system from a macroeconomics perspective. Schmitt-Grohe and Uribe (1997) demonstrate that under the balanced-budget rule, economic fluctuations driven from expectation are more likely to arise than under a case in which public deficit is counted in. This is because under the balanced-budget rule, it is impossible to make fine adjustments of the public expenditure in response to economic fluctuations. They assume that public expenditures are exclusively financed from income taxation. Giannitsarou (2007) shows that if a government fully finances its expenditure by consumption taxation, then such fluctuations will disappear. While Schmitt-Grohe and Uribe (1997) and Giannitsarou (2007) assume constant tax rates, Nourry *et al.* (2013) and Greiner and Bondarev (2015) include a state-dependent consumption tax, in which the consumption tax rate increases in line with the consumption level of consumers. However, those studies focus on the local indeterminacy of the equilibrium paths near steady states, and do not highlight the sustainability of public debt. Nourry *et al.* (2013) study the case of balanced budgets, and Greiner and Bondarev (2015) postulate Bohn's (1998) rule, for the public debt to be sustainable.

The rest of this paper is organized as follows. Section 2 builds a model on which the subsequent analysis is based. Section 3 analyzes an equilibrium path, with a particular focus on a BGP. Section 4 presents the main results. Section 5 briefly concludes the paper.

2 Model

This section sets up a basic model. Think of an economy with an infinite time horizon, in which economic activities are conducted at each time point $t \in [0, \infty)$[2]. The economy consists of households, firms and a government.

2.1 Household

The economy contains many identical households. A representative household maximizes a discounted integral of utilities from now to the future. The instantaneous utility function is represented by a utility function $u(C) = (C^{1-\sigma} - 1) / (1 - \sigma)$, where $C\,(\geq 0)$ stands for consumption. The parameter $\sigma > 0$ $(\sigma \neq 1)$ represents the inverse of the elasticity of consumption, which is independent from time. It is well-known that the logarithmic utility function is potentially included as the case of $\sigma = 1$. The household supplies the labor service L^S to a representative firm. The labor service is assumed to be inelastically supplied and thus, it becomes constant over time. The wage rate is denoted by $w(t)$. The household purchases consumption goods, and partly saves its income for a future consumption at each time point. The total real asset is denoted by

$$W(t) = B(t) + K(t) \tag{1}$$

where $B(t)\,(\in \mathbb{R})$ and $K(t)\,(\geq 0)$ are the public debt and the physical capital, respectively. Their initial levels $B(0)$ and $K(0)$ are given for the house-

[2]The symbol $[0, \infty)$ stands for the set $\{x \in \mathbb{R} : 0 \leq x < \infty\}$, where $\mathbb{R}$ is the set of real numbers.

hold. The saving bears real interest, which is denoted by $r(t)$. The consumption behavior and the income of the household are levied by the government. The consumption tax rate is $\tau_C > 0$, and the labor income and interest income tax rates are denoted by $\tau_L \in (0,1)$ and $\tau_W \in (0,1)$, respectively. These tax rates are supposed to be constants. The feasible path of consumption and saving stream is subject to budget constraints and the no Ponzi-game (NPG) condition. The household's behavior can be summarized as the following maximizing problem:

$$\max_{C(t),\, W(t)} \int_0^\infty e^{-\rho t} \frac{C(t)^{1-\sigma} - 1}{1-\sigma} dt \tag{2}$$

subject to

$$(1+\tau_C) C(t) + \dot{W}(t) = (1-\tau_L) w(t) L^S + (1-\tau_W) r(t) W(t) \tag{3}$$

$$\lim_{t\to\infty} e^{-(1-\tau_W)\int_0^t r(s)ds} W(t) = 0 \tag{4}$$

where $\rho > 0$ stands for the subjective discount rate. The time derivative is represented by the dot symbol: $\dot{(\ \ )} = d/dt$. In what follows, the time index is often omitted if no ambiguity arises.

2.2 Firms

The representative firm produces output Y, by using labor L^D and physical capital K. The production process is influenced by the economy-wide capital level $\overline{K}$. Although each firm takes $\overline{K}$ as given, the condition $\overline{K} = K$ is required in an equilibrium. The production function regarding the firm is given by

$$Y = AK^\alpha \left(\overline{K} L^D\right)^{1-\alpha}, \tag{5}$$

where $\alpha \in (0,1)$ and $A\,(>0)$ are constant parameters that represent a capital share and a technology level, respectively. The representative firm maximizes its static profit at each time point. Its behavior is summarized as the following maximizing problem:

$$\max_{K,\, L^D \geq 0} AK^\alpha \left(\overline{K} L^D\right)^{1-\alpha} - rK - wL^D \tag{6}$$

$$\overline{K}\,(\geq 0) : \text{given}$$

2.3 Government

The government obtains tax revenues from taxation on consumption, labor income, and interest income. It also obtains revenues from issuing public bonds B^G. The government's budget constraint is given by

$$\dot{B}^G = rB^G - (T - G) \tag{7}$$

where

$$T = \tau_C C + \tau_L wL + \tau_W rW \tag{8}$$

is the total tax revenue and G is the public spending. Equation (7) can be regarded as an accounting identity. For simplicity, this paper assumes that the public spending is wastefully used. I also assume that the government uses a constant fraction of GDP Y for the public spending, i.e.,

$$G = gY \tag{9}$$

where $g = [0, 1)$ is a constant. The initial level of the public debt $B^G(0) \, (= B(0))$ is given for the government.

In addition to (7), the government must be subject to the NPG condition:

$$\lim_{t \to \infty} e^{-(1-\tau_W)\int_0^t r(s)ds} B^G(t) = 0. \tag{10}$$

The condition (10) means that the present value of the public debt must converge to 0 in the remote future. This condition implies that the government plans to raise revenues by levying tax from now to the future to just compensate for their expenditure plan $\{G(t)\}$, and to repay the accumulated debt at the initial time point $B^G(0)$ (for this point, see Greiner (2013, 2015)).

2.4 Market Clearing Conditions

In an equilibrium, markets for the good, labor and the public bond are simultaneously cleared. That is, the following equations must hold:

$$Y = C + \dot{K} + gY, \quad L^S = L^D, \quad B^G = B, \tag{11}$$

for any $t \geq 0$. The equations in (11) are those for the good, the labor service and the public bonds, respectively. Henceforth, I simply denote the equilibrium value of labor and public debt by L and B, respectively. Further, as mentioned in Section 2.2, an average economy-wide level of capital is required to be equal to the capital level of the representative firm in an equilibrium:

$$\overline{K} = K. \tag{12}$$

As a consequence, it holds from (5) that

$$Y = AKL^{1-\alpha}. \tag{13}$$

3 Equilibrium

This section derives the equilibrium path, focusing on a BGP. Based on the discussions in this section, Section 4 investigates the initial level of public debt-to-capital (or debt-to-GDP) ratio that is compatible with the BGP.

3.1 Equilibrium Path

In an equilibrium, the following four conditions are satisfied:

(i) given the time paths of prices $\{w(t), r(t)\}$, the tax rates (τ_C, τ_L, τ_W) and the initial level of assets $(B(0), K(0))$, the households maximize (2) subject to (3) and (4),

(ii) given the time paths of prices $\{w(t), r(t)\}$ and an average level of economy-wide capital $\left\{\overline{K}(t)\right\}$, the firms maximize (6) for each t,

(iii) given $\{r(t)\}$ and the initial level of public debt $B^G(0)$, the government determines the policy variables $\left((\tau_C, \tau_L, \tau_W, g), \left\{B^G(t)\right\}\right)$ subject to (7) and (10),

(iv) all of the markets are simultaneously cleared (11), and an average level of economy-wide capital $\overline{K}$ coincides with the physical capital K for each t.

As first order conditions of the maximizing problem of the firm (6), it holds in the equilibrium that

$$r = \alpha AL^{1-\alpha}, \tag{14}$$
$$w = (1-\alpha) AKL^{-\alpha}. \tag{15}$$

Equation (14) is obtained from the condition $\overline{K} = K$.

Next, the consumption path in an equilibrium is investigated. The (present-value) Hamiltonian associating with the maximizing problem of the representative household (2)-(4) is defined by

$$\mathcal{H} = e^{-\rho t}\frac{C^{1-\sigma}-1}{1-\sigma} + \lambda\left[\left(1-\tau_L\right)wL^s + \left(1-\tau_W\right)rW - \left(1+\tau_C\right)C\right],$$

where λ is called a Hamilton multiplier. As conditions $\partial\mathcal{H}/\partial C = 0$ and $\partial\mathcal{H}/\partial W = -\dot{\lambda}$, the following holds:

$$\frac{\dot{C}}{C} = \frac{\left(1-\tau_W\right)\alpha A L^{1-\alpha} - \rho}{\sigma}. \tag{16}$$

Thus, it holds that

$$C\left(t\right) = C\left(0\right)e^{\theta t}, \tag{17}$$

where θ is defined as

$$\theta \equiv \frac{\left(1-\tau_W\right)\alpha A L^{1-\alpha} - \rho}{\sigma}. \tag{18}$$

The variable θ stands for the economic growth rate on the BGP. Note that among the tax system $\left(\tau_C, \tau_L, \tau_W\right)$, only the tax rate for the interest income τ_W influences the economic growth rate on the BGP. As seen in (4), the interest tax rate τ_W affects the discount rate for the future activities, and hence, affects the long-term growth rate.[3]

The dynamics of capital stock is derived from the first equation of the market clearing conditions (11):

$$\dot{K} = \left(1-g\right)AKL^{1-\alpha} - C.$$

It holds from (17) that

$$\dot{K} = \left(1-g\right)AKL^{1-\alpha} - C\left(0\right)e^{\theta t}.$$

[3] This point was also pointed out by Kondo (2016) although he deals with the case of $\tau_L = \tau_W$.

Solving this differential equation, I obtain

$$K = e^{(1-g)AL^{1-\alpha}t}\left[K(0) - C(0)\frac{1}{(1-g)AL^{1-\alpha}-\theta}\right] + \frac{C(0)}{(1-g)AL^{1-\alpha}-\theta}e^{\theta t}. \tag{19}$$

It is clear from (19) that the capital path is on the BGP if and only if the initial level of consumption is chosen at

$$C(0) = \left[(1-g)AL^{1-\alpha}-\theta\right]K(0), \tag{20}$$

given $K(0) > 0$.

Here, the following assumption is made.

Assumption 1 $0 \leq \theta < (1-\tau_W)\alpha AL^{1-\alpha} < (1-g)AL^{1-\alpha}$, where θ is defined in (18).

This assumption implies that $\theta < (1-g)AL^{1-\alpha}$, which guarantees that the economic variables (C, K) are positive on a BGP (see (20)). The first inequality $0 \leq \theta$ means that the economic growth rate θ is non-negative on the BGP, which is satisfied if ρ is sufficiently near 0. The second inequality $\theta < (1-\tau_W)\alpha AL^{1-\alpha} (= (1-\tau_W)r)$ guarantees that the NPG conditions (4) and (10) are satisfied on the BGP (see the BGP conditions (21) and (27)).

3.2 Balanced Growth Path

The conditions of the BGP are obtained from (17), (19) and (13) as

$$\begin{aligned} C(t) &= C(0)e^{\theta t}, \\ K(t) &= K(0)e^{\theta t}, \\ Y(t) &= AK(0)L^{1-\alpha}e^{\theta t}, \end{aligned} \tag{21}$$

with the consumption level (20) given $K(0) > 0$.

Using these conditions (21), I study the dynamic path of the public debt. Substituting (8) and (9) into (7) yields

$$\dot{B} = rB - (\tau_C C + \tau_L wL + \tau_W rW - gY).$$

It holds from (1), (13), (14) and (15) that

$$\begin{aligned}\dot{B} &= (1-\tau_W)\, rB - (\tau_C C + \tau_L wL + \tau_W rK - gAKL^{1-\alpha}) \\ &= (1-\tau_W)\,\alpha AL^{1-\alpha}B \\ &\quad - \left[\tau_C C + \tau_L (1-\alpha) AKL^{1-\alpha} + \tau_W \alpha AL^{1-\alpha}K - gAKL^{1-\alpha}\right].\end{aligned}$$

Thus,

$$\dot{B} = (1-\tau_W)\,\alpha AL^{1-\alpha}B - \left[\tau_C C + \{\tau_L (1-\alpha) + \tau_W \alpha - g\}\, AKL^{1-\alpha}\right]. \quad (22)$$

Substituting the BGP conditions with respect to $C(t)$ and $K(t)$ shown in (21) into (22) yields

$$\begin{aligned}\dot{B} &= (1-\tau_W)\,\alpha AL^{1-\alpha}B \\ &\quad - \left[\tau_C C(0) + \{(1-\alpha)\,\tau_L + \alpha\tau_W - g\}\, AL^{1-\alpha}K(0)\right] e^{\theta t}.\end{aligned}$$

It follows from (20) that the following holds:

$$\dot{B} = (1-\tau_W)\,\alpha AL^{1-\alpha}B - HK(0)\, e^{\theta t}, \quad (23)$$

where H is a variable defined by

$$H \equiv \tau_C \left[(1-g)\, AL^{1-\alpha} - \theta\right] + \left[(1-\alpha)\,\tau_L + \alpha\tau_W - g\right] AL^{1-\alpha}. \quad (24)$$

From (23), I obtain

$$\begin{aligned}B &= e^{(1-\tau_W)\alpha AL^{1-\alpha}t}\left[B(0) + \frac{H}{\theta - (1-\tau_W)\,\alpha AL^{1-\alpha}}K(0)\right] \\ &\quad - \frac{H}{\theta - (1-\tau_W)\,\alpha AL^{1-\alpha}}K(0)\, e^{\theta t}.\end{aligned}$$

From (18), it holds that

$$\begin{aligned}B &= e^{(1-\tau_W)\alpha AL^{1-\alpha}t}\left[B(0) - \frac{\sigma H}{\rho - (1-\tau_W)\,\alpha AL^{1-\alpha}(1-\sigma)}K(0)\right] \quad (25) \\ &\quad + \frac{\sigma H}{\rho - (1-\tau_W)\,\alpha AL^{1-\alpha}(1-\sigma)}K(0)\, e^{\theta t}.\end{aligned}$$

Thus, if and only if

$$B(0) = \frac{H}{\rho - (1-\tau_W)\,\alpha A L^{1-\alpha}(1-\sigma)} \sigma K(0), \tag{26}$$

then the public debt is on the BGP:

$$B(t) = B(0)\, e^{\theta t}. \tag{27}$$

The set of equations (21) and (27) represents the BGP in the present economy.

Here, the following assumptions are made.

Assumption 2 $(1-\alpha)\,\tau_L + \alpha\tau_W - g > 0.$

Assumption 3 σ is sufficiently near 1.

Assumption 2 is met if the public spending-to-GDP ratio g is sufficiently small in relation to the income tax rates (τ_L, τ_W). Together with Assumption 1, Assumption 2 guarantees that the variable H is positive (see (24)). Assumptions 1–3 imply that $B(0)$ is positive (see (26)); i.e., the government is a borrower. This is a reasonable assumption given the high level of accumulated public debt in many advanced countries in the real world. Section 4 offers a numerical example that satisfies Assumptions 1–3.

The public debt-to-GDP ratio on the BGP is of great interest. It can be derived from (21), (27) and (26) as

$$\begin{aligned} \frac{B}{Y} &= \frac{\sigma H}{AL^{1-\alpha}\left[\rho - (1-\tau_W)\,\alpha A L^{1-\alpha}(1-\sigma)\right]} \\ &= \frac{\tau_C\left[(1-g)\,AL^{1-\alpha} - \theta\right] + \left[(1-\alpha)\,\tau_L + \alpha\tau_W - g\right]AL^{1-\alpha}}{AL^{1-\alpha}\left[\rho - (1-\tau_W)\,\alpha A L^{1-\alpha}(1-\sigma)\right]}\sigma \end{aligned} \tag{28}$$

The following proposition is established.

Proposition 1 *(Balanced Growth Path)*

In the present economy, the BGP is given by (21) *and* (27)*, with the initial level of the consumption* (20) *and the public debt* (26) *given* $K(0) > 0$. *The economic growth rate* θ *and the variable* H *are shown in* (18) *and* (24)*, respectively. The public debt-to-GDP ratio on the BGP is given by* (28)*.*

4 Sustainability of Public Debt

This section analyzes the initial level of public debt-to-GDP ratio that is compatible with the BGP. The section further examines how the level depends on the tax system (τ_C, τ_L, τ_W). A numerical example is given to illustrate the analyses.

4.1 Critical Level of Public Debt-to-GDP Ratio

It is clear from (25) that if $B(0) > \frac{H}{\rho-(1-\tau_W)\alpha AL^{1-\alpha}(1-\sigma)}\sigma K(0)$, then the public debt-to-capital ratio diverges to $+\infty$ as time passes. Further, from (21) and (25), the public debt-to-GDP ratio also diverges to $+\infty$, which contradicts the NPG condition (10). Thus, the initial level of public debt must be

$$B(0) \leq \frac{H}{\rho - (1-\tau_W)\alpha AL^{1-\alpha}(1-\sigma)}\sigma K(0).$$

Using $Y(0) = AL^{1-\alpha}K(0)$, the critical level of public debt-to-GDP ratio that is compatible with the BGP is given by

$$\frac{B(0)}{Y(0)} = \frac{H}{AL^{1-\alpha}\left[\rho - (1-\tau_W)\alpha AL^{1-\alpha}(1-\sigma)\right]}\sigma.$$

Substituting the definition of H (24) into the right hand side of the above inequality, I get a public debt-to-GDP ratio that is compatible with the BGP as follows:

$$\frac{B(0)}{Y(0)} = \frac{\begin{array}{c}\tau_C\left[(1-g)AL^{1-\alpha} - \left\{(1-\tau_W)\alpha AL^{1-\alpha} - \rho\right\}/\sigma\right] \\ +\left\{(1-\alpha)\tau_L + \alpha\tau_W - g\right\}AL^{1-\alpha}\end{array}}{AL^{1-\alpha}\left[\rho - (1-\tau_W)\alpha AL^{1-\alpha}(1-\sigma)\right]}\sigma \equiv \varphi. \quad (29)$$

Under Assumptions 1–3, it holds that $\varphi > 0$, where the variable φ is defined in (29). Equation (29) explicitly shows how the tax system (τ_C, τ_L, τ_W) effects the public debt sustainability.

I sum up this result as a theorem.

Theorem 1 *(Critical Level of Public Debt-to-GDP Ratio)*

The initial level of public debt-to-GDP ratio that is compatible with the BGP is given by (29).

The relationship in (29) is complicated; hence, I describe the case in which the utility function of the household is logarithmic. Substituting $\sigma = 1$ into (29) yields

$$\varphi = \frac{\begin{array}{c} \tau_C \left[(1-g) AL^{1-\alpha} - \left\{(1-\tau_W)\alpha AL^{1-\alpha} - \rho\right\}\right] \\ + \left\{(1-\alpha)\tau_L + \alpha\tau_W - g\right\} AL^{1-\alpha} \end{array}}{AL^{1-\alpha}\rho}. \tag{30}$$

This result is summarized as the following corollary.

Corollary 1 *(Critical Level of Public Debt-to-GDP Ratio for the Log-Utility Case)*

Assume that the utility function of the representative household is logarithmic. Then, the initial level of public debt-to-GDP ratio that is compatible with the BGP is given by (30)*.*

4.2 Marginal Rates of Substitution of the Tax Rates

The main purpose of this paper is to investigate how the critical level of the sustainable public debt-to-GDP ratio φ depends on the tax system (τ_C, τ_L, τ_W). To do this, I derive the marginal effects of the tax rates on φ. Some calculations from (29) yield

$$\begin{aligned}
\frac{\partial\varphi}{\partial\tau_C} &= \frac{(1-g) AL^{1-\alpha}\sigma - (1-\tau_W)\alpha AL^{1-\alpha} + \rho}{AL^{1-\alpha}\left[\rho - (1-\tau_W)\alpha AL^{1-\alpha}(1-\sigma)\right]}, \\
\frac{\partial\varphi}{\partial\tau_L} &= \frac{(1-\alpha)\sigma}{\rho - (1-\tau_W)\alpha AL^{1-\alpha}(1-\sigma)}, \\
\frac{\partial\varphi}{\partial\tau_W} &= \frac{\begin{array}{c} (\tau_C + \sigma)\left[\rho - (1-\tau_W)\alpha AL^{1-\alpha}(1-\sigma)\right] \\ -(1-\sigma) AL^{1-\alpha} \left[\begin{array}{c} \tau_C \left\{(1-g)\sigma - (1-\tau_W)\alpha + \frac{\rho}{AL^{1-\alpha}}\right\} \\ + \left\{(1-\alpha)\tau_L + \alpha\tau_W - g\right\}\sigma \end{array}\right] \end{array}}{\left[\rho - (1-\tau_W)\alpha AL^{1-\alpha}(1-\sigma)\right]^2}\alpha.
\end{aligned}$$

It is easily ascertained that under Assumptions 1–3, the sign of $\partial\varphi/\partial\tau_i$ is positive for any $i = C, L, W$.

By using the implicit function theorem, I can derive the marginal rate of policy substitutions.

$$-\frac{d\tau_L}{d\tau_C} = \frac{\partial\varphi/\partial\tau_C}{\partial\varphi/\partial\tau_L} = \frac{(1-g) AL^{1-\alpha}\sigma - (1-\tau_W)\alpha AL^{1-\alpha} + \rho}{(1-\alpha) AL^{1-\alpha}\sigma}, \tag{31}$$

$$
\begin{aligned}
-\frac{d\tau_W}{d\tau_C} &= \frac{\partial\varphi/\partial\tau_C}{\partial\varphi/\partial\tau_W} && (32)\\
&= \frac{(1-g)\,AL^{1-\alpha}\sigma - (1-\tau_W)\,\alpha AL^{1-\alpha} + \rho}{(\tau_C+\sigma)\left[\rho - (1-\tau_W)\,\alpha AL^{1-\alpha}(1-\sigma)\right] - \frac{(1-\sigma)\,AL^{1-\alpha}\left[\begin{array}{c}\tau_C\left\{\begin{array}{c}(1-g)\,\sigma\\ -(1-\tau_W)\,\alpha + \frac{\rho}{AL^{1-\alpha}}\end{array}\right\}\\ +\{(1-\alpha)\,\tau_L + \alpha\tau_W - g\}\,\sigma\end{array}\right]}{\rho-(1-\tau_W)\alpha AL^{1-\alpha}(1-\sigma)}\alpha AL^{1-\alpha}},
\end{aligned}
$$

$$
\begin{aligned}
-\frac{d\tau_W}{d\tau_L} &= \frac{\partial\varphi/\partial\tau_L}{\partial\varphi/\partial\tau_W} && (33)\\
&= \frac{(1-\alpha)\,\sigma}{(\tau_C+\sigma)\left[\rho - (1-\tau_W)\,\alpha AL^{1-\alpha}(1-\sigma)\right] - \frac{(1-\sigma)\,AL^{1-\alpha}\left[\begin{array}{c}\tau_C\left\{\begin{array}{c}(1-g)\,\sigma\\ -(1-\tau_W)\,\alpha + \frac{\rho}{AL^{1-\alpha}}\end{array}\right\}\\ +\{(1-\alpha)\,\tau_L + \alpha\tau_W - g\}\,\sigma\end{array}\right]}{\rho-(1-\tau_W)\alpha AL^{1-\alpha}(1-\sigma)}\alpha}.
\end{aligned}
$$

How are results (31)–(33) interpreted from an economic standpoint? First, see (31). Think of a situation in which the economy is on the BGP. Then, the debt-to-GDP ratio remains constant over time at the level shown in (29). Suppose that the government plans to increase the consumption tax rate τ_C at the marginal level. In such a situation,

Q. To what extent can the government cut the labor tax rate τ_L while sustaining the public debt-to-GDP ratio at the constant level (29)?

A. The answer is given by (31).

If the government changes τ_C and τ_L, then the behavior of the households is influenced, and thus the equilibrium path is affected. That effect is captured by the conditions of BGP (20), (21), (26) and (27). Accordingly, the critical level of the public debt-to-GDP ratio is affected by the policy change (see (29)). The equation (31) explicitly demonstrates that the government can cut the labor tax rate by the level shown in (31). The relationships (32) and (33) can be interpreted in much the same way.

The following theorem is established:

Theorem 2 *(Marginal Rates of Substitution of the Tax Rates)*

Assume that the economy is on the BGP, and that the government marginally increases the consumption tax rate τ_C*. Then, the government can cut the labor tax rate* τ_L *by the level shown in* (31)*, or cut the interest tax rate* τ_W *by the level shown in* (32) *while maintaining the initial level of the public debt-to-GDP ratio at a constant level* (29)*. In much the same way, if the government marginally increases the labor tax rate, then it can cut the interest tax rate by the level shown in* (33) *while maintaining the debt-to-GDP ratio at a constant level* (29)*.*

The relationships in (31)–(33) are complicated; hence I describe the case in which the utility functions of the households are logarithmic. Substituting $\sigma = 1$ into (31)-(33) yields

$$-\frac{d\tau_L}{d\tau_C} = \frac{\partial\varphi/\partial\tau_C}{\partial\varphi/\partial\tau_L} = \frac{(1-g)\,AL^{1-\alpha} - (1-\tau_W)\,\alpha AL^{1-\alpha} + \rho}{(1-\alpha)\,AL^{1-\alpha}}, \tag{34}$$

$$-\frac{d\tau_W}{d\tau_C} = \frac{\partial\varphi/\partial\tau_C}{\partial\varphi/\partial\tau_W} = \frac{(1-g)\,AL^{1-\alpha} - (1-\tau_W)\,\alpha AL^{1-\alpha} + \rho}{(\tau_C+1)\,\alpha AL^{1-\alpha}}, \tag{35}$$

$$-\frac{d\tau_W}{d\tau_L} = \frac{\partial\varphi/\partial\tau_L}{\partial\varphi/\partial\tau_W} = \frac{1-\alpha}{(\tau_C+1)\,\alpha}. \tag{36}$$

These results are summarized as the following corollary.

Corollary 2 *(Marginal Rates of Substitution of the Tax Rates for the Log-Utiliy Case)*

Let the utility function of the representative household be logarithmic. Assume that the economy is on the BGP, and that the government marginally increases the consumption tax rate τ_C*. The government can cut the labor tax rate* τ_L *by the level shown in* (34) *or cut the interest tax rate* τ_W *by the level shown in* (35) *while maintaining the initial level of the public debt-to-GDP ratio* φ *at a constant level* (30)*. In much the same way, if the government marginally increases the labor tax rate* τ_L*, then it can cut the interest tax rate* τ_W *by the level shown in* (36) *while maintaining the debt-to-GDP ratio* φ *at a constant level* (30)*.*

4.3 Required Tax Rates for the Sustainability of the Public Debt

This subsection derives the tax rates that are compatible with the sustainable public debt, given the initial level of the public debt-to-GDP ratio $B(0)/Y(0)$. In other words, the equation (29) is solved for each tax rate, e.g., τ_C, given $B(0)/Y(0)$ and other tax rates, e.g., (τ_L, τ_W).

Solving the equation (29) yields

$$\tau_C = \frac{\begin{array}{c} AL^{1-\alpha}\left[\rho - (1-\tau_W)\,\alpha AL^{1-\alpha}(1-\sigma)\right]\frac{B(0)}{Y(0)} \\ -\left[(1-\alpha)\,\tau_L + \alpha\tau_W - g\right]AL^{1-\alpha}\sigma \end{array}}{(1-g)\,AL^{1-\alpha}\sigma - \left[(1-\tau_W)\,\alpha AL^{1-\alpha} - \rho\right]}. \tag{37}$$

In much the same way, it holds that

$$\tau_L = \frac{\begin{array}{c} AL^{1-\alpha}\left[\rho - (1-\tau_W)\,\alpha AL^{1-\alpha}(1-\sigma)\right]\frac{B(0)}{Y(0)} \\ -\tau_C\left[(1-g)\,AL^{1-\alpha}\sigma - (1-\tau_W)\,\alpha AL^{1-\alpha} + \rho\right] \\ -(\alpha\tau_W - g)\,AL^{1-\alpha}\sigma \end{array}}{(1-\alpha)\,AL^{1-\alpha}\sigma}, \tag{38}$$

$$\tau_W = \frac{\begin{array}{c} \tau_C\left[(1-g)\,\sigma - \alpha + \rho\frac{1}{AL^{1-\alpha}}\right] + \left[(1-\alpha)\,\tau_L - g\right]\sigma \\ -\left[\rho - \alpha AL^{1-\alpha}(1-\sigma)\right]\frac{B(0)}{Y(0)} \end{array}}{\alpha\left[AL^{1-\alpha}(1-\sigma)\frac{B(0)}{Y(0)} - \tau_C - \sigma\right]}. \tag{39}$$

Note that under Assumptions 1–3,

$$\frac{\partial\tau_i}{\partial\left(B(0)/Y(0)\right)} > 0,$$

for any $i = C, L, W$. This means that when the initial level of public debt-to-GDP ratio is high, the tax rate τ_i must be set at a high level given other parameters, e.g., τ_j $(j \neq i)$.

The following theorem is established.

Theorem 3 *(Required Levels of Tax Rates)*

The tax rates for consumption, labor income and interest income that are required for the economy to be on the BGP are presented by (37)*,* (38) *and* (39)*, respectively.*

For simplicity, consider the log-utility case. Then, substituting $\sigma = 1$ into (37)-(39) I obtain that

$$\tau_C = \frac{AL^{1-\alpha}\rho\frac{B(0)}{Y(0)} - [(1-\alpha)\tau_L + \alpha\tau_W - g] AL^{1-\alpha}}{(1-g) AL^{1-\alpha} - [(1-\tau_W)\alpha AL^{1-\alpha} - \rho]}, \tag{40}$$

$$\tau_L = \frac{\begin{array}{c} AL^{1-\alpha}\rho\frac{B(0)}{Y(0)} - \tau_C [\{(1-g) - (1-\tau_W)\alpha\} AL^{1-\alpha} + \rho] \\ - (\alpha\tau_W - g) AL^{1-\alpha} \end{array}}{(1-\alpha) AL^{1-\alpha}}, \tag{41}$$

$$\tau_W = \frac{\rho\frac{B(0)}{Y(0)} - \tau_C \left[(1-g) - \alpha + \rho\frac{1}{AL^{1-\alpha}}\right] - (1-\alpha)\tau_L + g}{\alpha(\tau_C + 1)}. \tag{42}$$

These results are summarized as the following corollary.

Corollary 3 *(Required Levels of Tax Rates for the Log-Utility Case)*
Assume that the utility function of the representative household is logarithmic. Then, the tax rates for consumption, labor income and interest income that are required for the economy to be on the BGP are presented by (40)*,* (41) *and* (42)*, respectively.*

4.4 Numerical Example

This subsection offers a numerical example, which illustrates the analyses presented above. I specify the parameters, except for (τ_C, τ_L, τ_W) and $B(0)/Y(0)$, as $A = L = \sigma = 1$, $\alpha = 0.25$, $\rho = 0.1$, $g = 0.15$. In what follows, I often specify $\tau_L = \tau_W (= \tau_C) = 0.2$, and it is easily ascertained that this parameter constellation satisfies Assumptions 1–3.

From (30), it holds that

$$\varphi = 7\tau_C + \frac{15}{2}\tau_L + \frac{5}{2}\tau_W + \frac{5}{2}\tau_C\tau_W - \frac{3}{2}, \tag{43}$$

which shows how the sustainability of public debt depends on the tax system (τ_C, τ_L, τ_W). If $\tau_C = \tau_L = \tau_W = 0.2$, then it is obtained from (43) that $\varphi = 2$. That is, if the tax rates are all set at 20%, then 200% of the public debt-to-GDP ratio is allowed for on-going economic growth.

The marginal rates of substitution of the tax rates are calculated from (34)–(36) as

$$-\frac{d\tau_L}{d\tau_C} = \frac{1}{3}\tau_W + \frac{14}{15}, \tag{44}$$

$$-\frac{d\tau_W}{d\tau_C} = \frac{14+5\tau_W}{5(\tau_C+1)}, \tag{45}$$

$$-\frac{d\tau_W}{d\tau_L} = \frac{3}{\tau_C+1}. \tag{46}$$

See (44). Since $\tau_W \in [0,1)$, it holds that $-d\tau_L/d\tau_C \in \left[\frac{14}{15}, \frac{19}{15}\right)$. Especially, if $\tau_W = 0.2$, then $-d\tau_L/d\tau_C = 1$. In other words, assume that the tax rate on interest income is 20%, and that it is fixed. In such a situation, if a government increases the consumption tax rate by 1%, then it can reduce the tax rate for labor income by 1% while maintaining a sustainable debt policy.

The required tax rates for the sustainability of public debt are derived from (40)–(42) as

$$\tau_C = \frac{2\frac{B(0)}{Y(0)} - 15\tau_L - 5\tau_W + 3}{5\tau_W + 14}, \tag{47}$$

$$\tau_L = \frac{2}{15}\frac{B(0)}{Y(0)} - \frac{14}{15}\tau_C - \frac{1}{3}\tau_W - \frac{1}{3}\tau_C\tau_W + \frac{1}{5}, \tag{48}$$

$$\tau_W = \frac{2\frac{B(0)}{Y(0)} - 14\tau_C - 15\tau_L + 3}{5(\tau_C+1)}. \tag{49}$$

Assume that $\tau_L = \tau_W = 0.2$, and see (47). Then, the relationship between the initial level of the public debt-to-GDP ratio and the consumption tax rate that is required for the sustainability of public debt is given by

$$\tau_C = \frac{2}{15}\frac{B(0)}{Y(0)} - \frac{1}{15}.$$

When $B(0)/Y(0) = 2.30$, the required consumption tax rate is $\tau_C = 0.24$. If $B(0)/Y(0) = 2.40$, then the required consumption tax rate is $\tau_C \doteqdot 0.253$. Thus, if the public debt-to-GDP ratio increases from 2.30 to 2.40, then the consumption tax rate must be increased by approximately 1.3% to maintain fiscal sustainability.

5 Conclusion

This paper explicitly derives the critical level of public debt that is compatible with a balanced economic growth. The model considers a complex tax system—a combination of tax rates for consumption, labor income and interest income. Analyzing the critical level reveals the marginal rates of substitution of tax rates. That is, by how much a government can reduce one tax rate when another one is slightly increased while maintaining a sustainable debt policy. Further, the required tax rates to sustain a given level of public debt-to-GDP ratio are computed.

Although the results obtained in this paper are relevant to economists and to policy makers in countries that are suffering from a highly accumulated public debt, the model used is somewhat simple. Some extensions should be implemented in possible future research. First, this paper assumes that the population is constant over time. Given the declining birth rates observed in many advanced countries, it is desirable to take a population decline into account. Second, progressive income tax rates should be included. Finally, it is important to include the productive expenditure of the government.

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