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by

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Abstract. The LIBOR market (LM) model (Brace, Gatarek, and Musiela [8], Miltersen, Sandmann, Sondermann [21], and Jamshidian [18]) is a Heath-Jarrow-Morton model (Heath, Jarrow, and Morton [15]) specified to be an interest rate version of the celebrated Black-Scholes model of stock price, and is the most popular among practitioners and researchers. However, a statistical test (Kusuda [19]) rejected the LM model, and suggested that the deterministic volatility in the LIBOR market model should be replaced with a stochastic one and/or that a jump process should be introduced into the LM model. This paper presents a stochastic volatility jump-diffusion LM model using a general equilibrium security market model of Kusuda [19]. Approximate general equilibrium pricing formulas for caplet and swaption are derived exploiting the forward martingale measure approach (Jamshidian [17]) and a Fourier transform method (Heston [16], Bates [4], and Duffie, Pan, and Singleton [13]).

This paper is a revision and expansion of a part of Chapter 7 in my Ph.D. dissertation (Kusuda [19]) at the Department of Economics, University of Minnesota. I would like to thank my adviser Professor Jan Werner for his invaluable advice. I am grateful for comments of participants of presentations at Japanese Economic Association of Financial Econometrics and Engineering Winter 2003 Conference, University of Minnesota, Institute for Advanced Studies, Vienna, Hitotsubashi University, and Nagoya City University.

1. Introduction

In international financial markets, most interest rate related contracts refer to LIBOR (London InterBank Offered Rate[1]) rates, forward LIBOR rates, swap rates (a long term version of LIBOR rates), and forward swap rates. The two most frequently traded interest rate derivatives, *i.e.* a *caplet* and a *swaption*, are a European option on a forward LIBOR rate and on a forward swap rate, respectively. It seems reasonable to suppose that an ideal interest rate derivative pricing model should possess the following two properties: (i) Arbitrage-free pricing formulas for caplet and swaption are derived in the model. (ii) The model is statistically acceptable, *i.e.* the model can capture the dynamics of interest rates in real markets. The property (i) makes it possible to speedily calibrate the model using the formulas, and to price other interest rate derivatives exploiting the calibrated model. The property (ii) ensures that this pricing is accurate. The *LIBOR market model*, developed by Brace, Gatarek, and Musiela [8], Miltersen, Sandmann, Sondermann [21], and Jamshidian [18], is a Heath-Jarrow-Morton model (Heath, Jarrow, and Morton [15]) specified to be an interest rate version of the celebrated Black-Scholes model (Black and Scholes [7]) of stock price. In the Black-Scholes model, the change in stock price is subject to a lognormal distribution under the risk-neutral measure. In the LIBOR market (LM) model, the change in each forward LIBOR rate (resp. forward swap rate) is subject to a lognormal distribution (resp. an approximate lognormal distribution) under the associated equivalent martingale measure. Thus a Black-Scholes-like pricing formula (resp. approximate pricing formula) for each caplet (resp. swaption) is derived, which has made the LM model currently the most popular interest rate derivative pricing models among both practitioners and researchers. However, a statistical test conducted in Chapter 2 in Kusuda [19] rejected the LM model and showed that the distribution of the estimated discretized Wiener process, which is supposed to be a normal distribution, has much fatter tail than the normal distribution. This result suggests that the deterministic volatility in the LM model with a stochastic one and/or that a jump process should be introduced into the LM model. A stochastic volatility LM model (Andersen and Andreasen [2]) and jump-diffusion LM models (Glasserman and Kou [14], Kusuda [19]) have been proposed. However, these two extensions are not alternatives but complements in the sense a stochastic volatility term mainly affects the price of derivatives with long maturity while a jump term mainly affects that of derivatives short maturity. It has been, therefore, desired that a stochastic volatility jump-diffusion LM model which includes both a stochastic volatility LM model and a jump-diffusion LM model as special cases, is introdued and statistically tested. No stochastic volatility jump-diffusion LM model has been presented so far. The main purpose of this paper is to present a stochastic volatility jump-diffusion LM (SVJDLM, hereafter) model which is an extension of the LM model and possesses both the above properties (i) and (ii).

In the SVJDLM model, it is assumed like in most jump-diffusion option pricing models that the jump magnitude of forward LIBOR rate is a continuously distributed random variable at each jump occurring time. Under this assumption, the markets have uncountably infinite number of information sources, and no finite

[1]The LIBOR rate is the interest rate offered by banks on deposits from other banks in Eurocurrency markets and is frequently a reference rate of interest for loans in international financial markets. In the LIBOR market model, the dynamics of forward LIBOR rates are modeled. A representative real example of forward LIBOR rate is a Eurodollar future rate traded on the Chicago Mercantile Exchange. In the case of Eurodollar futures, the underlying instrument of Eurodollar future contracts is the 90-day LIBOR and future rates with 48 different times to maturity, *i.e.*, one month, two month, $\cdots$, one year, one year and three month, one year and six month, $\cdots$, ten years, are traded.

number of securities complete the markets.[2] In incomplete markets, the standard arbitrage-free pricing method cannot be applied because the payoff of an option is not guaranteed to be replicable. In many jump-diffusion option pricing models (Ahn and Thompson [1]; Bates [3] [4] Das and Foresi [12], Naik and Lee [22], *etc.*), general equilibrium (GE, hereafter) pricing method in incomplete markets is adopted. In these models, it is assumed that there is a representative agent with a CRRA utility, or that there are homogeneous agents with a common CRRA utility. In order to justify the assumption on the representative agent, it is required to show that there exists a security market equilibrium in which the representative agent admits a CRRA utility. However, it is very difficult to do so in incomplete markets. It is still difficult to show the existence of security market equilibria under the assumption of homogeneous agents.

Björk, Kabanov, and Runggaldier [5] assumed that every bond with any maturity date is traded in security markets with jump-diffusion information, and showed that under some regularity condition, the markets are *approximately complete* (see Björk, Di Masi, Kabanov, and Runggaldier [6]) in which any contingent claim can be approximately replicated with an arbitrary precision by an admissible portfolio of the bonds. Thus, arbitrage-free pricing method can be adopted in approximately complete bond markets. Glasserman and Kou [14] have presented a jump-diffusion LIBOR market model assuming approximately complete bond markets. In their model, the market price of risk is rather arbitrarily specified such that arbitrage-free pricing formulas for caplet and swaption can be derived. Here it must be noted that in the GE model, there is a functional relation among the market price of risk and the dynamics of aggregate consumption and commodity price in equilibrium. Thus, it is desired to verify that specification of the market price of risk in option pricing model can be consistent to the GE functional relation among the market price of risk and the dynamics of aggregate consumption and commodity price in some reasonable GE model. This should be verified, in particular, in the case when option prices depend on the market price of risk.

Recently, the author (Kusuda [19]) has introduced the notion of *approximate security market equilibrium* in which each agent is allowed to choose a consumption plan that is approximately financed with any prescribed precision by a budget feasible portfolio, and showed sufficient conditions for the existence and uniqueness of approximate security market equilibria in approximately complete bond markets. This paper presents an SVJDLM model assuming the GE approximately complete bond market model. First, since the nominal bond price processes can be exogenously given in the GE model, they are specified such that a compound Poisson jump process is introduced into the LM model, and the deteministic volatility in the LM model is replaced with a stochastic volatility which is subject to a square-root process. The GE dynamics of a forward LIBOR (resp. swap) rate is derived under the associated forward martingale measure because the pricing problem of a caplet (resp. swaption) is reduced to calculating the conditional expectation of the caplet's (resp. swaption's) payoff under the associated *forward martingale measure* (see Definition 4 and 5) introduced by Jamshidian [17]. The conditional distribution is analytically intractable to calculate the conditional expectation because of the presence of the stochastic volatility and jump terms. For a class of stochastic volatility jump-diffusion models of security price with the *affine jump-diffusion* (AJD) *structure* (see Duffie, Pan, and Singleton [13]) under the risk-neutral measure, Duffie, Pan, and Singleton [13] showed that an arbitrage-free pricing formula

[2] Merton [20] assumed that the market price of risk is zero in order to make the number of sources of the market information finite, and to complete the markets. However, an empirical analysis in Pan [23] showed that the market price of risk cannot be regarded as zero.

for European option can be derived using a Fourier transform method developed by Heston [16], Bates [4], and Duffie, Pan, and Singleton [13]. This paper exploits this idea, and approximates each forward LIBOR (resp. swap) rate and its volatility processes in order that the approximate system dynamics of each rate and its volatility admit an AJD structure under the associated forward martingale measure instead of the risk-neutral measure. Then an approximate GE pricing formula for its caplet (resp. swaption) is derived using the Fourier transform method.

Since the SVJDLM model admits an approximate AJD structure, an approximate characteristic function can be derived. Therefore, a characteristic function-based method developed by Carrasco and Florens [10], and Carrasco, Chernov, Florens, and Ghysels [11], can be employed to estimate and test the SVJDLM model.

The remainder of this paper is organized as follows. Section 2 reviews the GE model of security markets with jump-diffusion information. Section 3 specifies the SVJDLM model and derives the dynamics of forward LIBOR rates. Sections 4 and 5 derive approximate GE pricing formulas for a caplet and a swaption. Appendix A, B, and C introduce marked point process, Ito's Formula and Girsanov's Theorem, and basic concepts in bond markets, respectively. Appendix D shows proofs.

2. The GE Model of Security Markets with Jump-Diffusion Information

In this section, the GE model of security markets with jump-diffusion information is reviewed following Chapter 3 and 4 in Kusuda [19].

2.1. **Security Market Economy under Jump-Diffusion Uncertainty.** A continuous-time frictionless security market economy with time span $[0, T^\dagger]$ (abbreviated by $\mathbf{T}$, hereafter) for a fixed horizon time $T^\dagger > 0$ is considered. The agents' common subjective probability and information structure is modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F} := (\mathcal{F}_t)_{t\in\mathbf{T}}$ is the natural filtration generated by a d-dimensional Wiener process W and a *marked point process* $\nu(dt \times dz)$ (see Appendix A.1) on a Lusin space $(\mathbb{Z}, \mathcal{Z})$ ($\mathbb{Z} = \mathbb{R}^n$ in the SVJDLM model) with the P-intensity kernel $\lambda_t(dz)$. There is a single perishable consumption commodity. The commodity space is a Banach space $\mathbf{L}^\infty := \mathbf{L}^\infty(\Omega \times \mathbf{T}, \mathcal{P}, \mu)$ where $\mathcal{P}$ is the predictable σ-algebra on $\Omega \times \mathbf{T}$, and μ is the product measure of the probability measure P and the Lebesgue measure on $\mathbf{T}$. There are I agents. Each agent $i \in \{1, 2, \cdots, I\}$ (abbreviated by $\mathbf{I}$, hereafter) is represented by $(U, \bar{c}^i)$, where U is a common strictly increasing and continuous utility on the positive cone $\mathbf{L}^\infty_+$ of the consumption process and $\bar{c}^i \in \mathbf{L}^\infty_+$ is an endowment process, which is assumed to be nonzero. The economy mentioned above is described by a collection: $\mathbf{E} := ((\Omega, \mathcal{F}, \mathbb{F}, P), (U, \bar{c}^i)_{i\in\mathbf{I}})$. There are markets for the consumption commodity and securities at every date $t \in \mathbf{T}$. The traded securities are nominal-risk-free security (NOT the risk-free security) called the *money market account* and a continuum of zero-coupon bonds whose maturity times are $(0, T^\dagger]$, each of which pays one unit of cash (NOT one unit of the commodity) at its maturity time. Let p, B, and $(B^T)_{T\in(0,T^\dagger]}$ denote the processes of consumption commodity price, nominal money market account price, and nominal bond price, respectively. The collection $(B, (B^T)_{T\in(0,T^\dagger]})$ of security prices is abbreviated by $\mathbf{B}$, and called the *family of bond prices.*

Each agent is allowed to hold a portfolio consisting of the money market account and every bond with any maturity time $T \in [t, T^\dagger]$ at each time $t \in \mathbf{T}$.

Definition 1. A *portfolio* is a stochastic process $\vartheta = (\vartheta^0, \vartheta^1(\cdot))$ that satisfies:

(1) The component ϑ^0 is a real-valued $\mathcal{P}$-measurable process.

(2) The component ϑ^1 is such that:
 (a) For every $(\omega, t) \in \Omega \times \mathbf{T}$, the set function $\vartheta^1_t(\omega, \cdot\,)$ is a signed finite Borel measure on $[t, T^\dagger]$.
 (b) For every Borel set $A \in [t, T^\dagger]$, the process $\vartheta^1_t(A)$ is $\mathcal{P}$-measurable.

An intuitive interpretation of this definition is that ϑ^0_t and $\vartheta^1_t(dT)$ are the number of units of the money market account and the number of bonds with maturity times $[T, T + dT]$ held at time t, respectively. The *value process* $\mathcal{V}_t(\vartheta^i_n)$ *of a portfolio* ϑ^i_n is given by

$$\mathcal{V}_t(\vartheta^i_n) = B_t \vartheta^{i0}_{nt} + \int_t^{T^\dagger} B^T_t \vartheta^{i1}_{nt}(dT) \qquad \forall t \in \mathbf{T}.$$

2.2. **Arbitrage-Free Approximately Complete Markets.** Let $n \in \mathbb{N}$. Let $\mathcal{L}^n$ denote the set of real-valued $\mathcal{P}$-measurable process X satisfying the integrability condition $\int_0^{T^\dagger} |X_s|^n \, ds < \infty$ P-a.s. Also let $\mathcal{L}^n(\lambda_t(dz) \times dt)$ denote the set of real-valued $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$-measurable process m satisfying the integrability condition $\int_0^{T^\dagger} \int_{-\infty}^{\infty} |m_s(z)|^n \, \lambda_s(dz) \, ds < \infty$ P-a.s. The notion of *implementable family of bond prices* is introduced.

Definition 2. A bond price family $\mathbf{B}$ is *implementable* if and only if the following conditions hold:

(1) (a) For every $T \in (0, T^\dagger]$, the dynamics of nominal bond price process B^T satisfies the following stochastic differential-difference equation (SDDE)

$$\frac{dB^T_t}{B^T_{t-}} = r^T_t \, dt + v^T_t \cdot dW_t + \int_{\mathbb{Z}} m^T_t(z) \, \{ \, \nu(dt \times dz) - \lambda_t(dz) \, dt \, \} \qquad \forall t \in [0, T)$$

with $B^T_T = 1$ where $r^T \in \mathcal{L}^1$, $v^T \in \prod_{j=1}^d \mathcal{L}^2$, and $m^T \in \mathcal{L}^1(\lambda_t(dz) \times dt)$ and satisfies $m^T_t(z) > -1$ P-a.s. for every $(t, z) \in \mathbf{T} \times \mathcal{Z}$. Moreover, the following hold:
 (i) For every $(\omega, t) \in \Omega \times \mathbf{T}$, $r^{\cdot}_t(\omega), v^{\cdot}_t(\omega) \in \mathbf{C}^1(\mathbf{T})$, and for every $(\omega, t, z) \in \Omega \times \mathbf{T} \times \mathbb{Z}$, $m^{\cdot}_t(\omega, z) \in \mathbf{C}^1(\mathbf{T})$.
 (ii) For every $T \in (0, T^\dagger]$, B^T is regular enough to allow for the differentiation under the integral sign and the interchange of integration order.
 (iii) For every $t \in \mathbf{T}$, bond price curves $B^{\cdot}_t$ are bounded P-a.e.
 (iv) The family of jump magnitude functions $m^{\cdot}_t(\,\cdot\,)$ is uniformly bounded μ-a.e.

 (b) The dynamics of nominal money market account price process satisfies $\frac{dB_t}{B_t} = r^B_t \, dt$ for every $t \in [0, T^\dagger)$ such that $B_0 = 1$ where r^B_t is given by $r^B_t = -\left.\frac{\partial \ln B^T_t}{\partial T}\right|_{T=t}$, and $r^B \geq 0$ μ-a.e.

(2) There exists a unique real-valued P-martingale $\Lambda^{\mathbf{B}}$ such that

$$\frac{d\Lambda^{\mathbf{B}}_t}{\Lambda^{\mathbf{B}}_{t-}} = -v^{\mathbf{B}}_t \cdot dW_t - \int_{\mathbb{Z}} m^{\mathbf{B}}_t(z) \, \{ \, \nu(dt \times dz) - \lambda_t(dz) \, dt \, \} \qquad \forall t \in [0, T^\dagger) \tag{2.1}$$

with $\Lambda^{\mathbf{B}}_0 = 1$ where $(v^{\mathbf{B}}, m^{\mathbf{B}}) \in \prod_{j=1}^{d_1} \mathcal{L}^2 \times \mathcal{L}^1(\lambda_t(dz) \times dt)$ satisfies

$$r^T_t = r^B_t + v^{\mathbf{B}}_t \cdot v^T_t + \int_{\mathbb{Z}} m^{\mathbf{B}}_t(z) m^T_t(z) \, \lambda_t(dz). \tag{2.2}$$

(3) The process $\frac{\Lambda^{\mathbf{B}}}{B}$ is bounded above and bounded away from zero μ-a.e.

The processes $v_t^{\mathbf{B}}$ and $m_t^{\mathbf{B}}(z)\lambda_t(dz)$ are called *market price of (nominal) diffusive risk* and *market price of (nominal) jump risk*, respectively. It follows from Ito's formula (see Appendix B.1) and Girsanov's Theorem (see Appendix B.2) that the existence of *risk-neutral measures*, which implies that markets are *arbitrage-free* (for definitions of risk-neutral measure and arbitrage-free, see Appendix C.2), is equivalent to the existence of process $\Lambda^{\mathbf{B}}$ satisfying the conditions (2.1) and (2.2). It should be noted that the process $\Lambda^{\mathbf{B}}$ satisfying conditions (2.1) and (2.2) is the Radon-Nikodym derivative of risk-neutral measure, and therefore the uniqueness of processes $\Lambda^{\mathbf{B}}$ satisfying conditions (2.1) and (2.2) is equivalent to the uniqueness of risk-neutral measures. Björk, Di Masi, Kabanov, and Runggaldier [6] showed that markets with any implementable bond price family are *approximately complete* in the sense that for any $T \in (0, t^\dagger]$ and any T*-contingent claim*, there exists a sequence of *admissible* self-financing portfolios such that the sequence of corresponding value processes converge to the claim at time T (for these definitions, see Appendices C.1, C.2, and C.3). Let $\bar{\mathcal{B}}$ and $\underline{\Theta}(\tilde{\mathbf{B}})$ denote the set of all implementable bond price families and the space of admissible portfolios, respectively.

2.3. **Approximate Security Market Equilibrium.** The notion of *approximate security market equilibrium* is introduced in which each agent is allowed to choose any consumption plan that is approximately financed with an arbitrary precision by a budget feasible portfolio.

Definition 3. A collection $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B}) \in \prod_{i\in\mathbf{I}} \mathbf{L}_+^\infty \times \mathbf{L}^\infty \times \bar{\mathcal{B}}$ constitutes an *approximate security market equilibrium for* $\mathbf{E}$ if and only if it follows that:

(1) For every $i \in \mathbf{I}$, $\hat{c}^i$ solves the problem $\max_{c^i \in \bar{\mathcal{C}}^i(p,\mathbf{B})} U^i(c^i)$ where

$$\bar{\mathcal{C}}^i(p,\mathbf{B}) = \{c^i \in \mathbf{L}_+^\infty \mid \exists (\vartheta_n^i)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \underline{\Theta}(\tilde{\mathbf{B}}) \quad \text{s.t.}$$

$$\mathcal{V}_t(\vartheta_n^i) = \int_0^t \vartheta_{ns}^{i0}\, dB_s + \int_0^t \int_s^{T^\dagger} \vartheta_{ns}^{i1}(dT)\, dB_s^T + \int_0^t p_s(\bar{c}_s^i - c_s^i)\, ds \qquad \forall (n,t) \in \mathbb{N}\times\mathbf{T},$$

$$\lim_{n\to\infty} \frac{\mathcal{V}_{T^\dagger}(\vartheta_n^i)}{B_{T^\dagger}} = 0 \quad \text{in } \mathbf{L}^2(\Omega, \mathcal{F}_{T^\dagger}, \tilde{P}^{\mathbf{B}})\,\}.$$

(2) The commodity market is cleared: $\sum_{i\in\mathbf{I}} \hat{c}^i = \sum_{i\in\mathbf{I}} \bar{c}^i$.

Hereafter, approximate security market equilibrium is abbreviated by ASM equilibrium. The following assumption is a sufficient condition for the existence of ASM equilibria.

Assumption 1. (1) *Every agent has a common CRRA utility U of the form $U(c) = E\left[\int_0^{T^\dagger} u(t, c_t)\, dt\right]$ where u is given by*

$$u(t,x) = e^{-\rho t}\frac{\gamma}{1-\gamma}\left(\left(\frac{x}{\gamma}\right)^{1-\gamma} - 1\right)$$

for some positive constants $\rho > 0$ and $\gamma > 0$.

(2) *The aggregate endowment is bounded away from zero μ-a.e.*

3. The Stochastic Volatility Jump-Diffusion LIBOR Market Model

In this section, the specification of SVJDLM (Stochastic Volatility Jump-Diffusion LIBOR Market) model is provided, and the GE (General Equilibrium) dynamics of a forward LIBOR rate is derived under the associated *forward martingale measure*, which is defined in the following.

Definition 4. Let $\mathbf{B} \in \bar{\mathcal{B}}$. For every $T \in (0, T^\dagger]$, a probability measure denoted by P^T on $(\Omega, \mathcal{F})$ is a *T-forward martingale measure at* $\mathbf{B}$ if and only if P^T is equivalent to P, and for every $T' \in (0, T^\dagger]$, $\frac{B^{T'}}{B^T}$ is a local P^T-martingale.

Let the common tenor of forward LIBOR rates be denoted by $\delta \in (0, 1]$. For every $T \in (0, T^\dagger - \delta]$, the *$T$-forward LIBOR rate process* L^T is defined by

$$L_t^T = \frac{1}{\delta}\left(\frac{B_t^T}{B^{T+\delta}} - 1\right) \qquad \forall t \in [0, T].$$

3.1. **The Stochastic Volatility Jump-Diffusion LIBOR Market Model.** For every $T \in (0, T^\dagger - \delta]$ and $t \in [0, T]$, the integer $\lceil \frac{T-t}{\delta} \rceil - 1$ is denoted by K_t^T, hereafter. The SVJDLM (Stochastic Volatility Jump-Diffusion LIBOR Market) model is specified by the set of Assumption 1 and the following two assumptions.

Assumption 2. (1) *The Lusin space $(\mathbb{Z}, \mathcal{Z})$ is 1-dimensional Euclidean space where $d' \in \mathbb{N}$.*

(2) *The P-intensity kernel $\lambda_t(dz)$ is given by*

$$\lambda_t(dz) = \lambda_t \, \phi(z) \, dz \tag{3.1}$$

where λ and ϕ satisfy

$$\begin{aligned} \lambda_t &= \lambda_0 + \lambda_1 V_t \qquad \forall t \in \mathbf{T}, \\ \phi(z) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \qquad \forall z \in \mathbb{R} \end{aligned} \tag{3.2}$$

where $\lambda_0, \lambda_1 \in \mathbb{R}_+$, and V satisfies

$$dV_t = \kappa(\bar{V} - V_t)\, dt + \sqrt{V_t}\, \varsigma_V \cdot dW_t \qquad \forall t \in [0, T) \tag{3.3}$$

for some positive constants $\kappa, \bar{V}$ and some constant vector $\varsigma_V \in \mathbb{R}^d$.

(3) *The dynamics of the aggregate endowment process follows the SDDE*

$$\frac{d\bar{c}_t}{\bar{c}_{t-}} = r_t^{\bar{c}}\, dt + v_t^{\bar{c}} \cdot dW_t + \int_{\mathbb{R}} (e^{\sigma_{\bar{c}} z + \mu_{\bar{c}}} - 1) \{ \nu(dt \times dz) - \lambda_t(dz)\, dt \} \qquad \forall t \in [0, T^\dagger)$$

for some $r^{\bar{c}} \in \mathcal{L}^1$, $v^{\bar{c}} \in \prod_{j=1}^{d_1} \mathcal{L}^2$, and $(\sigma_{\bar{c}}, \mu_{\bar{c}}) \in \mathbb{R}^2$.

Assumption 3. *The family $\mathbf{B}$ of nominal bond price processes satisfies $\mathbf{B} \in \bar{\mathcal{B}}$ and the following conditions:*

(1) *The volatilities of bonds $\mathbf{B}$ are such that there exist processes $(\varsigma^T)_{T \in (0, T^\dagger - \delta]}$ and constant vectors $\varsigma_{\bar{c}}, \varsigma_p \in \mathbb{R}^d$ satisfying for every $T \in (0, T^\dagger - \delta]$ and $t \in [0, T)$,*

$$\begin{pmatrix} v_t^T \\ v_t^{\bar{c}} \\ v_t^p \end{pmatrix} = \sqrt{V_t} \begin{pmatrix} \varsigma_t^T \\ \varsigma_{\bar{c}} \\ \varsigma_p \end{pmatrix} \tag{3.4}$$

where ς^T satisfies

$$\varsigma_t^T \begin{cases} = \varsigma_t^{T - K_t^T \delta} - \sum_{k=1}^{K_t^T} \frac{\delta L_{t-}^{T-k\delta}}{1 + \delta L_{t-}^{T-k\delta}}\, \varsigma(T - k\delta - t) & \forall t \in [0, T - \delta) \\ \approx 0 & \forall t \in [T - \delta, T) \end{cases} \tag{3.5}$$

for some Borel measurable function $\varsigma\colon \mathbb{R}_+ \to \mathbb{R}^d$.

(2) *The jump magnitudes of bonds* $\mathbf{B}$ *are such that there exist Borel measurable functions* $\mu : \mathbb{R}_+ \to \mathbb{R}$ *and* $\sigma : \mathbb{R}_+ \to \mathbb{R}$ *such that for every* $T \in (0, T^\dagger - \delta]$,

$$m_t^T(z) \begin{cases} = \dfrac{1+m_t^{T-K_t^T\delta}(z)}{\prod_{k=1}^{K_t^T}\left(1+\frac{\delta L_{t-}^{T-k\delta}}{1+\delta L_{t-}^{T-k\delta}}\left(\exp[\sigma(T-k\delta-t)z+\mu(T-k\delta-t)]-1\right)\right)} - 1 & \forall (t,z) \in [0, T-\delta) \times \mathbb{R} \\ \approx 0 & \forall (t,z) \in [T-\delta, T) \times \mathbb{R}. \end{cases} \tag{3.6}$$

(3) *The Jump magnitude* $m^{\mathbf{B}}$ *of density process satisfies*

$$m_t^{\mathbf{B}}(z) = e^{\sigma_{\mathbf{B}} z + \mu_{\mathbf{B}}} \qquad \forall (t,z) \in \mathbf{T} \times \mathbb{R} \tag{3.7}$$

for some $(\sigma_{\mathbf{B}}, \mu_{\mathbf{B}}) \in \mathbb{R}^2$.

Remark 1. It shall be shown in Proposition 1 that $\sqrt{V_t}\,\varsigma(T-t)$ and $e^{\sigma(T-t)z+\mu(T-t)} - 1$ are the volatility and the jump magnitude of L_t^T, respectively.

3.2. **GE Dynamics of Forward LIBOR Rates.** It shall be shown in Section 4 that the pricing problem of a caplet on L^T is reduced to the calculation of expectation of the caplet's payoff under the $(T+\delta)$-forward martingale measure. Let $T^\delta = T + \delta$, hereafter. The following proposition presents that the bond price family $\mathbf{B}$ is supported as an ASM equilibrium, and the GE dynamics of T-forward LIBOR rate process under P^{T^δ} as well as under P in the SVJDLM model.

Proposition 1. *Under Assumption 1-3, the following hold:*

(1) *The collection* $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ *is an ASM equilibrium for* $\mathbf{E}$ *where* $p_t = \frac{\Lambda_t^{\mathbf{B}}}{B_t} u_c(t, \bar{c}_t)$ *In addition, if* $\gamma \le 1$, *then the ASM equilibrium is unique.*

(2) *The market prices of nominal diffusive risk and nominal jump risk satisfy for every* $t \in \mathbf{T}$,

$$v_t^{\mathbf{B}} = \gamma\, v_t^{\bar{c}} + v_t^p, \qquad m_t^{\mathbf{B}}(z)\,\lambda_t(dz) = (\lambda_0 + \lambda_1 V_t)(1 - e^{-(\gamma m_{\bar{c}} + m_p)\cdot z})\,dz, \tag{3.8}$$

in the equilibrium where v^p *and* $(e^{m_p \cdot z} - 1)$ *are the volatility and the jump magnitude of commodity price process, respectively.*

(3) *Let* $T \in (0, T^\dagger - \delta]$. *The dynamics of* T*-forward LIBOR rate process satisfy for every* $t \in [0, T)$,

$$\begin{aligned} \frac{dL_t^T}{L_{t-}^T} &= \Big\{\varsigma(T-t)\cdot(\gamma\,\varsigma_{\bar{c}} + \varsigma_p - \varsigma_t^{T^\delta})V_t \\ &- \int_{-\infty}^{\infty} (e^{\sigma(T-t)z+\mu(T-t)} - 1)(1 + m_t^{T^\delta}(z))e^{-(\gamma\sigma_{\bar{c}}+\sigma_p)z}\phi(z)\,dz\,(\lambda_0 + \lambda_1 V_t)\Big\}\,dt \\ &\qquad + \sqrt{V_t}\,\varsigma(T-t)\cdot dW_t + \int_{-\infty}^{\infty}(e^{\sigma(T-t)z+\mu(T-t)} - 1)\,\nu(dt\times dz), \end{aligned} \tag{3.9}$$

where $\varsigma_p = \gamma\varsigma_{\bar{c}} - \varsigma_{\mathbf{B}}$, $\sigma_p = -\big(\gamma\sigma_{\bar{c}} + \sigma_{\mathbf{B}}\big)$, *and* V *satisfies (3.3), or equivalently*

$$\begin{aligned} \frac{dL_t^T}{L_{t-}^T} &= \sqrt{V_t}\,\varsigma(T-t)\cdot dW_t^{T^\delta} + \int_{-\infty}^{\infty}(e^{\sigma(T-t)z+\mu(T-t)} - 1)\,\Big\{\nu(dt\times dz) - \lambda_t^{T^\delta}\phi_t^{T^\delta}(z)\,dz\,dt\Big\}, \\ dV_t &= \Big[\kappa\bar{V} - \Big\{\kappa + \varsigma_V\cdot(\gamma\varsigma_{\bar{c}} + \varsigma_p - \varsigma_t^{T^\delta})\Big\}\,V_t\Big]\,dt + \sqrt{V_t}\,\varsigma_V\cdot dW_t^{T^\delta}, \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} W_t^{T^\delta} &= W_t + \int_0^t (\gamma\,\varsigma_{\bar{c}} + \varsigma_p - \varsigma_s^{T^\delta})\sqrt{V_s}\,ds, \qquad \lambda_t^{T^\delta} = \iota_t^{T^\delta}(\lambda_0 + \lambda_1 V_t), \\ \phi_t^{T^\delta}(z) &= \frac{1}{\iota_t^{T^\delta}}(1 + m_t^{T^\delta}(z))e^{-(\gamma\sigma_{\bar{c}}+\sigma_p)z}\phi(z), \end{aligned} \tag{3.11}$$

where

$$\iota_t^{T^\delta} = \int_{-\infty}^{\infty} (1 + m_t^{T^\delta}(z)) e^{-(\gamma\sigma_{\bar{c}}+\sigma_p)z} \phi(z)\, dz, \tag{3.12}$$

and W^{T^δ} *and* $\lambda_t^{T^\delta} \phi_t^{T^\delta}(z)\, dz$ *is a* P^{T^δ}*-Wiener process and the* P^{T^δ}*-intensity kernel of* $\nu(dt \times dz)$*, respectively.*

Remark 2. As shown in (3.8), in equilibrium, the market price of diffusive risk is a function of the volatilities of the aggregate consumption and the commodity price, and the market price of jump risk is a function of the jump magnitudes of the aggregate consumption and the commodity price. In most conventional option pricing models, the market price of risk is rather arbitrarily specified. It is desired to verify that the option pricing model can be embedded in some reasonable GE model, or to put it more precisely that the specification of market price of risk can be consistent to the GE functional relation among the market price of risk and the dynamics of the aggregate consumption and the commodity price in some reasonable GE model. In particular, in the case when option prices depend on the market price of risk, it should be verified. As shown in Section 4 and 5, the GE prices of caplet and swaption depend on the market price of jump risk in the SVJDLM model.

Proof. See Appendix D.1. □

4. Approximate GE Pricing Formula for Caplet

In this section, an approximate GE pricing formula for a caplet, a European call option on a forward LIBOR rate, is derived in the SVJDLM model exploiting the forward martingale measure approach introduced by Jamshidian [17], the Fourier transform method developed by Heston [16], Bates [4], and Duffie, Pan, and Singleton [13], and approximations. Here a caplet is defined in the following.

Let $T \in (0, T^\dagger - \delta]$ and $K > 0$. A *caplet on* T*-forward LIBOR rate* L^T *with strike rate* K is a contingent claim with payoff $\delta\,(L_T^T - K)^+$ at time T^δ where $(L_T^T - K)^+ := \max\{L_T^T - K, 0\}$.

Let $\mathrm{Cpl}_t(L^T, K)$ denote the GE price of the caplet on T-forward LIBOR rate L^T with strike rate K at time t in an ASM equilibrium $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ for $\mathbf{E}$ in the SVJDLM model. Since the security markets are approximately complete in the SVJDLM model, there exists a sequence of replicable claims converging to the T^δ-contingent claims. Let $(\vartheta_n)_{n\in\mathbb{N}}$ denote the corresponding sequence of replicable claims. Since the value process of every replicating portfolio discounted by B^{T^δ} is a P^{T^δ}-martingale, the following holds:

$$\frac{\mathcal{V}_t(\vartheta_n)}{B_t^{T^\delta}} = E_t^{T^\delta}\left[\frac{\mathcal{V}_{T^\delta}(\vartheta_n)}{B_{T^\delta}^{T^\delta}}\right] = E_t^{T^\delta}[\mathcal{V}_{T^\delta}(\vartheta_n)] \tag{4.1}$$

where $E_t^{T^\delta}[\,\cdot\,] = E^{T^\delta}[\,\cdot\,|\,\mathcal{F}_t\,]$ and $E^{T^\delta}[\,\cdot\,]$ is the expectation operator under P^{T^δ}. Taking the limit of the both sides of (4.1) yields

$$\mathrm{Cpl}_t(L^T, K) = \delta B_t^{T^\delta} E_t^{T^\delta}[\,(L_T^T - K)^+\,]. \tag{4.2}$$

4.1. **Approximation to AJD Structure under Associated Forward Martingale Measure.** Let $Y^T = \ln L^T$ for every $t \in [0, T]$. Applying Ito's formula to (3.10) yields the GE dynamics of Y^T for every $s \in [t,T]$,

$$\begin{aligned} dY_s^T &= \left\{ -\iota_s^{T^\delta}\lambda_0 - \left(\iota_s^{T^\delta}\lambda_1 + \frac{1}{2}\|\varsigma(T-s)\|^2 \right) V_s \right\} ds \\ &\quad + \sqrt{V_s}\,\varsigma(T-s)\cdot dW_s^{T^\delta} + \int_{-\infty}^{\infty} \big(\sigma(T-s)z + \mu(T-s)\big)\,\nu_s^{T^\delta}(ds\times dz), \\ dV_s &= \left[\kappa\bar{V} - \left\{ \kappa + \varsigma_V \cdot \left(\gamma\varsigma_{\bar{c}} + \varsigma_p - \varsigma_s^{T^\delta} \right) \right\} V_s \right] ds + \sqrt{V_s}\,\varsigma_V \cdot dW_s^{T^\delta}, \end{aligned} \tag{4.3}$$

The distribution Y^T conditional on $\mathcal{F}_t$ is analytically intractable because of the stochastic volatility and jump terms. For a class of stochastic volatility jump-diffusion models of security price with the *affine jump-diffusion* (AJD) *structure* (see Duffie, Pan, and Singleton [13]) under the risk-neutral measure, Duffie, Pan, and Singleton [13] showed that an arbitrage-free pricing formula for European option can be derived using a Fourier transform method developed by Heston [16], Bates [4], and Duffie, Pan, and Singleton [13]. This paper exploits this idea, and approximates (Y_s^T, V_s) in order that the approximate system dynamics of (Y_s^T, V_s) admit an AJD structure under P^{T^δ}. In (4.3), each of $\varsigma(T-s), \mu(T-s), \sigma(T-s), \varsigma_s^{T^\delta}, m_s^{T^\delta}(z), \iota_s^{T^\delta}, \lambda_s^{T^\delta}$, and $\phi_s^{T^\delta}$ includes a stochastic process or a time variable, and therefore they are approximated as constants during each time interval defined as follows. Let $I_n = [(n-1)\delta, n\delta]$ and $\tau_n = n - \frac{1}{2}\delta$ for every $n \in \mathbb{N}$.[3] First, $\varsigma(T-s), \mu(T-s)$ and $\sigma(T-s)$ sre approximated to $\tilde{\varsigma}, \tilde{\mu}$, and $\tilde{\sigma}$, resectively, which are defined by

$$\begin{pmatrix} \tilde{\varsigma}(T-s) \\ \tilde{\mu}(T-s) \\ \tilde{\sigma}(T-s) \end{pmatrix} = \begin{pmatrix} \varsigma(\tau_n) \\ \mu(\tau_n) \\ \sigma(\tau_n) \end{pmatrix} \quad \text{if } T-s \in I_n. \tag{4.4}$$

Next, $\varsigma_s^{T^\delta}$ and $m_s^{T^\delta}(z)$ are approximated to

$$\varsigma_s^{T^\delta} = \varsigma_s^{T-K_s^T\delta} - \sum_{k=1}^{K_s^T} \frac{\delta L_{s-}^{T-k\delta}}{1+\delta L_{s-}^{T-k\delta}}\,\varsigma(T-k\delta-s) \approx -\sum_{k=1}^{K_s^T} \frac{\delta L_{t-}^{T-k\delta}}{1+\delta L_{t-}^{T-k\delta}}\,\tilde{\varsigma}(T-k\delta-s)$$

and

$$\begin{aligned} m_s^{T^\delta}(z) &= \frac{1+m_s^{T^\delta-K_s^{T^\delta}\delta}(z)}{\prod_{k=1}^{K_s^{T^\delta}}\left(1+\frac{\delta L_{s-}^{T^\delta-k\delta}}{1+\delta L_{s-}^{T^\delta-k\delta}}\left(e^{\sigma(T^\delta-k\delta-s)z+\mu(T^\delta-k\delta-s)}-1\right)\right)} - 1 \\ &\approx \exp\left[-\sum_{k=1}^{K_s^{T^\delta}} \ln\left[1+\frac{\delta L_{t-}^{T^\delta-k\delta}}{1+\delta L_{t-}^{T^\delta-k\delta}}\left(e^{\tilde{\sigma}(T^\delta-k\delta-s)z+\tilde{\mu}(T^\delta-k\delta-s)}-1\right)\right]\right] - 1 \\ &\approx \exp\left[-\sum_{k=1}^{K_s^{T^\delta}} \ln\left[1+\frac{\delta L_{t-}^{T^\delta-k\delta}}{1+\delta L_{t-}^{T^\delta-k\delta}}\left\{e^{\tilde{\mu}(T^\delta-k\delta-s)}\left(1+\tilde{\sigma}(T^\delta-k\delta-s)z\right)-1\right\}\right]\right] - 1 \\ &\approx \exp\left[-\sum_{k=1}^{K_s^{T^\delta}} \frac{\delta L_{t-}^{T^\delta-k\delta}}{1+\delta L_{t-}^{T^\delta-k\delta}}\left\{e^{\tilde{\mu}(T^\delta-k\delta-s)}\tilde{\sigma}(T^\delta-k\delta-s)z + e^{\tilde{\mu}(T^\delta-k\delta-s)}-1\right\}\right] - 1, \end{aligned}$$

respectively. Here the approximations $\varsigma_s^{T-K_s^T\delta}, \ln(1+m_s^{T^\delta-K_s^{T^\delta}\delta}(z)) \approx 0$, $L_s^{T^\delta-k\delta} \approx L_t^{T^\delta-k\delta}, e^{\tilde{\sigma}(T^\delta-k\delta-t)z} \approx 1+\tilde{\sigma}(T^\delta-k\delta-t)z$, and $\ln[1+Z_s] \approx Z_s$ were used where

[3] If this approximation is computationally infeasible, then one can set $I_0 = [0,\delta], \tau_0 = \frac{\delta}{2}$, $I_n = [2^{n-1}\delta, 2^n\delta]$, $\tau_n = \frac{2^n\delta+2^{n-1}\delta}{2}$ for every $n \in \mathbb{N}$, and replace K_s^T with $\tilde{K}_s^T = \lceil \frac{\tau_n}{\delta} \rceil - 1$ if $T-s \in I_n$.

$Z_s = \frac{\delta L^{T^\delta - k\delta}_{t-}}{1+\delta L^{T^\delta - k\delta}_{t-}} \{e^{\tilde{\mu}(T^\delta - k\delta - s)}\tilde{\sigma}(T^\delta - k\delta - s)z + e^{\tilde{\mu}(T^\delta - k\delta - s)} - 1\}$. Let $\tilde{\varsigma}^{T^\delta}_s$, $\tilde{m}^{T^\delta}_s(z)$, and $\tilde{\iota}^{T^\delta}_s$ denote the approximations of $\varsigma^{T^\delta}_s$, $m^{T^\delta}_s(z)$, and $\iota^{T^\delta}_s$, respectively, *i.e.* they are given by

$$\begin{aligned}
\tilde{\varsigma}^{T^\delta}_s &= -\sum_{k=1}^{K^T_s} \frac{\delta L^{T-k\delta}_{t-}}{1+\delta L^{T-k\delta}_{t-}} \tilde{\varsigma}(T-k\delta-s), \\
\tilde{m}^{T^\delta}_s(z) &= \exp\left[-\sum_{k=1}^{K^{T^\delta}_s} \frac{\delta L^{T^\delta-k\delta}_{t-}}{1+\delta L^{T^\delta-k\delta}_{t-}} \left\{e^{\tilde{\mu}(T^\delta-k\delta-s)}\tilde{\sigma}(T^\delta-k\delta-s)z + e^{\tilde{\mu}(T^\delta-k\delta-s)} - 1\right\}\right] - 1, \\
\tilde{\iota}^{T^\delta}_s &= \int_{-\infty}^{\infty} (1+\tilde{m}^{T^\delta}_s(z))e^{-(\gamma\sigma_{\bar{c}}+\sigma_p)z}\phi(z)\,dz \\
&= \exp\left[\frac{1}{2}(\bar{z}^{T^\delta}_s)^2 - \sum_{k=1}^{K^{T^\delta}_s} \frac{\delta L^{T-k\delta}_{t-}}{1+\delta L^{T-k\delta}_{t-}} \left(e^{\tilde{\mu}(T^\delta-k\delta-s)} - 1\right)\right]
\end{aligned} \tag{4.5}$$

where

$$\bar{z}^{T^\delta}_s = \sum_{k=1}^{K^{T^\delta}_s} \frac{\delta L^{T-k\delta}_{t-}}{1+\delta L^{T-k\delta}_{t-}} e^{\tilde{\mu}(T^\delta-k\delta-s)}\tilde{\sigma}(T^\delta-k\delta-s) + \gamma\sigma_{\bar{c}} + \sigma_p \tag{4.6}$$

Finally, $\lambda^{T^\delta}_s$ and $\phi^{T^\delta}_s(z)$ are approximated to $\tilde{\lambda}^{T^\delta}_s$ and $\tilde{\phi}^{T^\delta}_s(z)$, respectively, which are given by

$$\begin{aligned}
\tilde{\lambda}^{T^\delta}_s &= \tilde{\iota}^{T^\delta}_s(\lambda_0 + \lambda_1 V_s), \\
\tilde{\phi}^{T^\delta}_s(z) &= \frac{1}{\tilde{\iota}^{T^\delta}_s}(1+\tilde{m}^{T^\delta}_s(z))e^{-(\gamma\sigma_{\bar{c}}+\sigma_p)z}\phi(z) \\
&= \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(z-\bar{z}^{T^\delta}_s\right)^2\right].
\end{aligned} \tag{4.7}$$

Using the approximations (4.4)-(4.7), define the *approximate GE caplet price* $\widetilde{\mathrm{Cpl}}_t(L^T, K)$ by

$$\widetilde{\mathrm{Cpl}}_t(L^T, K) = \delta B^{T^\delta}_t E^{T^\delta}_t\left[(\tilde{L}^T_T - K)^+\right] \tag{4.8}$$

where $\tilde{L}^T_T = e^{\tilde{Y}^T_T}$ and $\tilde{Y}^T_T$ is given by

$$\begin{aligned}
d\tilde{Y}^T_s &= \left\{-\tilde{\iota}^{T^\delta}_s\lambda_0 - \left(\tilde{\iota}^{T^\delta}_s\lambda_1 + \frac{1}{2}\|\tilde{\varsigma}(T-s)\|^2\right)\tilde{V}_s\right\}ds \\
&\quad + \sqrt{\tilde{V}_s}\,\tilde{\varsigma}(T-s)\cdot dW^{T^\delta}_s + \int_{-\infty}^{\infty}\left(\tilde{\sigma}(T-s)z + \tilde{\mu}(T-s)\right)\tilde{\nu}^{T^\delta}(ds\times dz), \qquad (4.9) \\
d\tilde{V}_s &= \left[\kappa\bar{V} - \left\{\kappa + \varsigma_V\cdot\left(\gamma\varsigma_{\bar{c}} + \varsigma_p - \tilde{\varsigma}^{T^\delta}_s\right)\right\}\tilde{V}_s\right]ds + \sqrt{\tilde{V}_s}\,\varsigma_V\cdot dW^{T^\delta}_s,
\end{aligned}$$

with $(\tilde{Y}^T_t, \tilde{V}_t)' = (Y^T_t, V_t)'$ where $\tilde{\nu}^{T^\delta}(ds\times dz)$ is the marked point process with P^{T^δ}-intensity kernel $\tilde{\lambda}^{T^\delta}_s\,\tilde{\phi}^{T^\delta}_s(z)\,dz$.

Remark 3. The approximation $\tilde{m}^{T^\delta}_t$ for $m^{T^\delta}_s$ looks quite rough at a glance. However, it seems reasonable to suppose that $m^{T^\delta}_s$ is fairly close to zero under forecast values of parameters. Therefore, $\tilde{m}^{T^\delta}_t$ can be regarded as a fairly good approximation for $m^{T^\delta}_s$ from the viewpoint of deriving an approximate price of $\mathrm{Cpl}_t(L^T, K)$.

4.2. **The Fourier Transform Method and Pricing Formula for Caplet.** Since the system dynamics of $\tilde{X} = (\tilde{Y}_t^T, \tilde{V}_t)'$ possesses an AJD structure under P^{T^δ}, the Fourier transform method shown in Duffie, Pan, and Singleton [13] can be applied to $\tilde{X}^T$. Define the function $\psi^{T^\delta}_{t,\tilde{X}^T}\colon \mathbb{C}^2 \times \mathbb{R} \times \mathbb{R}_+ \times [t,T] \to \mathbb{C}$ by

$$\psi^{T^\delta}_{t,\tilde{X}^T}(\xi, \tilde{X}_s^T, s) = E_s^{T^\delta}\left[e^{\xi\cdot \tilde{X}_T^T}\right].$$

The function $\psi^{T^\delta}_{t,\tilde{X}^T}$ is derived in analytic form as shown in the following lemma.

Lemma 1. *It follows that*

$$\psi^{T^\delta}_{t,\tilde{X}^T}((\xi_1,0)', (\tilde{Y}_s^T, \tilde{V}_s)', s) = e^{\alpha(\tau;\xi_1)+\xi_1\tilde{Y}_s^T+\beta(\tau;\xi_1)\tilde{V}_s} \tag{4.10}$$

where $\tau = T - s$, *and*

$$\begin{aligned}
\alpha(\tau;\xi_1) &= \begin{cases} \alpha_0(\tau;\xi_1,\tau_0) & \text{if } \tau \in I_0 \\ \alpha(\tau_{n-1};\xi_1) + \alpha_0(\tau-\tau_{n-1};\xi_1,\tau_n) & \text{if } \tau \in I_n, \end{cases} \\
\beta(\tau;\xi_1) &= \begin{cases} \beta_0(\tau;\xi_1,\tau_0) & \text{if } \tau \in I_0 \\ \beta(\tau_{n-1};\xi_1) + \beta_0(\tau-\tau_{n-1};\xi_1,\tau_n) & \text{if } \tau \in I_n, \end{cases}
\end{aligned} \tag{4.11}$$

with

$$\begin{aligned}
\alpha_0(\tau;\xi_1,\tau_n) &= \tilde{\iota}_s^{T^\delta}\lambda_0\left[\exp\left[\frac{1}{2}\tilde{\sigma}(\tau)\xi_1^2 + \left(\tilde{\mu}(\tau) + \bar{z}_s^{T^\delta}\tilde{\sigma}(\tau)\right)\xi_1\right] - \xi_1 - 1\right]\tau \\
&\quad - \kappa\bar{V}\left[\frac{b_n+d_n}{a}\tau + \frac{2}{a}\ln\left[1 - \frac{b_n+d_n}{2d_n}(1-e^{-d_n\tau})\right]\right], \\
\beta_0(\tau;\xi_1,\tau_n) &= \frac{c_n(1-e^{-d_n\tau})}{2d_n - (b_n+d_n)(1-e^{-d_n\tau})},
\end{aligned}$$

where

$$\begin{aligned}
a &= \|\varsigma_V\|^2, \qquad b_n = \varsigma(\tau_n)\cdot\varsigma_V\xi_1 - \left\{\kappa + \varsigma_V\cdot\left(\gamma\varsigma_{\bar{c}} + \varsigma_p - \tilde{\varsigma}_s^{T^\delta}\right)\right\}, \\
c_n &= 2\tilde{\iota}_s^{T^\delta}\lambda_1\exp\left[\frac{1}{2}\tilde{\sigma}(\tau)\xi_1^2 + \left(\tilde{\mu}(\tau) + \bar{z}_s^{T^\delta}\tilde{\sigma}(\tau)\right)\xi_1\right] \\
&\quad + \|\varsigma(\tau_n)\|^2\xi_1^2 - \left(2\tilde{\iota}_s^{T^\delta}\lambda_1 + \|\varsigma(\tau_n)\|^2\right)\xi_1 - 2\tilde{\iota}_s^{T^\delta}\lambda_1, \\
d_n &= \sqrt{|b_n^2 - a\,c_n|}\,\exp\left[\frac{i\arg(b_n^2 - a\,c_n)}{2}\right].
\end{aligned}$$

Proof. See Appendix D.2. □

Using Lemma 1, the approximate GE caplet price is derived as shown in the following proposition.

Proposition 2. *Let* $T \in (0, T^\dagger - \delta]$ *and* $K \in \mathbb{R}_{++}$. *Under Assumption 1-3, it follows that for a given ASM equilibrium* $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ *for* $\mathbf{E}$ *where* $p_t = \frac{B}{\Lambda^{\mathbf{B}}}u_c^{\hat{\alpha}}(t,\bar{c}_t)$, *the approximate GE caplet price* $\widetilde{\mathrm{Cpl}}_t(L^T,K)$ *satisfies the formula*

$$\begin{aligned}
\widetilde{\mathrm{Cpl}}_t(L^T,K) = \delta B_t^{T^\delta}\Bigg[&\frac{1}{2}e^{\alpha(T-t;1)+Y_t^T+\beta(\tau;1)V_t} \\
&- \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\left[e^{\alpha(T-t;1-iv)+(1-iv)Y_t^T+\beta(\tau;1-iv)V_t}K^{iv}\right]}{v}dv \\
&+ K\left(\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\left[e^{\alpha(T-t;-iv)-ivY_t^T+\beta(\tau;-iv)V_t}K^{iv}\right]}{v}dv\right)\Bigg]
\end{aligned} \tag{4.12}$$

where $Y_t^T = \ln L_t^T$*, and* $(\alpha, \beta)'$ *is given in (4.11).*

Proof. For every $y \in \mathbb{R}$ and $a, b \in \mathbb{R}^n$, let $G_{a,b}(y; \tilde{X}_t^T)$ denote the price of a security at time t, which pays $e^{a\cdot \tilde{X}_T^T}$ at time T in the event such that $b \cdot \tilde{X}_T^T \leq y$. Then the approximate caplet price $\widetilde{\mathrm{Cpl}}_t(L^T, K)$ is

$$\widetilde{\mathrm{Cpl}}_t(L^T, K) = \delta\Big\{G_{(1,0),(-1,0)}(-\ln K; \tilde{X}_t^T) - KG_{(0,0),(-1,0)}(-\ln K; \tilde{X}_t^T)\Big\}. \quad (4.13)$$

On the other hand, the following holds:

$$G_{a,b}(y; \tilde{X}_t^T) = B_t^{T^\delta} E_t^{T^\delta}\Big[e^{a\cdot \tilde{X}_T^T} 1_{b\cdot \tilde{X}_T \leq y}\Big]. \quad (4.14)$$

Thus, the Fourier-Stieltjes transform of $G_{a,b}(\,\cdot\,; \tilde{X}_t^T)$, if well defined, is given by

$$\int_0^\infty e^{ivy} dG_{a,b}(y; \tilde{X}_t^T) = \psi_{t,\tilde{X}^T}^{T^\delta}(a + ivb, \tilde{X}_t^T, t).$$

It follows from the extended Lévy inversion formula given in Duffie, Pan, and Singleton [13] that

$$G_{a,b}(y; \tilde{X}_t^T) = \frac{1}{2}\psi_{t,\tilde{X}^T}^{T^\delta}(a, \tilde{X}_t^T, t) - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\left[\psi_{t,\tilde{X}^T}^{T^\delta}(a + ivb, \tilde{X}_t^T, t)e^{-ivy}\right]}{v} dv. \quad (4.15)$$

Therefore, (4.12) follows from (4.13)-(4.15), and (4.10). □

5. Approximate GE Pricing Formula for Swaption

Next, an approximate GE pricing formula for a swaption is derived in the SVJDLM model. Since a swaption is a European call option on a forward swap rate, the approximate GE pricing formula for a swaption can be derived in a similar procedure as shown in Section 3 and 4. First, the GE dynamics of forward swap rate process is derived under the associated forward martingale measure. Then the forward swap rate and its volatility are approximated in order that the system dynamics of the approximate processes possesses an AJD structure under the associated forward martingale measure. Finally, the Fourier transform method is applied to the approximate processes to derive the approximate GE pricing formula for its swaption.

Let $N \in \mathbb{N}$ and $T \in (0, T^\dagger - N\delta]$. Then a N-period T*-forward swap rate process* $L^{T,N}$ is defined by

$$L_t^{T,N} = \frac{1}{\delta}\left(\frac{B_t^{T,N}}{B_t^{T^\delta,N}} - 1\right) \qquad \forall t \in [0, T^\delta) \quad (5.1)$$

where $B_t^{T,N} = \sum_{j=1}^N B_t^{T+(j-1)\delta}$. We call $L^{T,N}$ (T,N)*-forward swap rate process*, hereafter. A *payer swaption on* (T,N)*-forward swap rate* $L^{T,N}$ *with strike rate* $K \in \mathbb{R}_{++}$ is a contingent claim with fixed payoffs $\delta(L_T^{T,N} - K)^+$ at time $T + \delta, T + 2\delta, ..., T + N\delta$. We call the swaption (T,N)-forward swaption.

Let $\mathrm{PS}_t(L^{T,N}, K)$ denote the GE price of the (T,N)-swaption with strike rate K at time t in an ASM equilibrium $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ for $\mathbf{E}$ in the SVJDLM Model. In order to derive the GE price of the swaption, a forward martingale measure called (T^δ, N)*-forward martingale measure* is exploited.

Definition 5. Let $\mathbf{B} \in \bar{\mathcal{B}}$. For every $N \in \mathbb{N}$ and $T \in (0, T^\dagger - N\delta]$, a probability measure denoted by $P^{T,N}$ on $(\Omega, \mathcal{F})$ is a (T,N)*-forward martingale measure at* $\mathbf{B}$ if and only if $P^{T,N}$ is equivalent to P, and $\frac{\mathbf{B}}{B^{T,N}}$ is a local $P^{T,N}$-martingale.

The GE price of the (T,N)-forward swaption at time $t \in [0,T)$ satisfies

$$\mathrm{PS}_t(L^{T,N}, K) = \delta B_t^{T^\delta,N} E_t^{T^\delta,N}\left[\,(L_T^{T,N} - K)^+\,\right] \tag{5.2}$$

where $E^{T^\delta,N}$ is the expectation operator under the (T,N)-forward martingale measure $P^{T^\delta,N}$.

5.1. **Dynamics of Forward Swap Rate under Associated Forward Martingale Measure.** First, it is straightforward to see that the GE dynamics of $B^{T,N}$ satisfies the following SDDE:

$$\frac{dB_t^{T,N}}{B_{t-}^{T,N}} = r^{T,N}\,dt + \sqrt{V_t}\,\varsigma_t^{T,N}\cdot dW_t + \int_{-\infty}^{\infty} m_t^{T,N}(z)\,\{\nu(dt\times dz) - \lambda_t\,\phi(z)\,dz\,dt\,\} \tag{5.3}$$

where

$$\begin{aligned} r^{T,N} &= r_t^B + v_t^{\mathbf{B}}\cdot v_t^{T,N} + \int_{-\infty}^{\infty} m_t^{\mathbf{B}}(z) m_t^{T,N}(z)\phi(z)\,dz\lambda_t, \\ \varsigma_t^{T,N} &= \sum_{n=1}^{N} \frac{B_t^{T+(n-1)\delta}}{B_t^{T,N}}\varsigma_t^{T+(n-1)\delta}, \qquad m_t^{T,N}(z) = \sum_{n=1}^{N}\frac{B_t^{T+(n-1)\delta}}{B_t^{T,N}} m_t^{T+(n-1)\delta}(z). \end{aligned} \tag{5.4}$$

Then the following proposition is obtained in the same way as shown in Proposition 1.

Proposition 3. *Let $N \in \mathbb{N}$ and $T \in (0, T^\dagger - (N+1)\delta]$. Under Assumption 1-3, the dynamics of (T,N)-forward swap rate process satisfies for every $t \in [0,T)$,*

$$\begin{aligned} \frac{dL_t^{T,N}}{L_{t-}^{T,N}} &= \varsigma_t^N(T)\cdot dW_t^{T^\delta,N} + \int_{-\infty}^{\infty} \eta_t^{T,N}(z)\,\{\nu(dt\times dz) - \lambda_t^{T^\delta,N}\,\phi_t^{T^\delta,N}(z)\,dz\,dt\,\}, \\ dV_t &= \left[\kappa\bar{V} - \left\{\kappa + \varsigma_V\cdot(\gamma\varsigma_{\bar{c}} + \varsigma_p - \varsigma_t^{T^\delta,N})\right\} V_t\right] dt + \sqrt{V_t}\,\varsigma_V\cdot dW_t^{T^\delta,N}, \end{aligned} \tag{5.5}$$

where

$$\varsigma_t^N(T) = \frac{1+\delta L_{t-}^{T,N}}{\delta L_{t-}^{T,N}}(\varsigma_t^{T,N} - \varsigma_t^{T^\delta,N}), \qquad \eta_t^{T,N}(z) = \frac{1+\delta L_{t-}^{T,N}}{\delta L_{t-}^{T,N}}\left(\frac{1+m_t^{T,N}(z)}{1+m_t^{T^\delta,N}(z)} - 1\right),$$

$$W_t^{T^\delta,N} = W_t + \int_0^t (\gamma\, v_s^{\bar{c}} + v_s^p - v_s^{T^\delta,N})\,ds, \quad \lambda_t^{T^\delta,N} = \iota_t^{T^\delta,N}(\lambda_0 + \lambda_1 V_t),$$

$$\phi_t^{T^\delta,N}(z) = \frac{1}{\iota_t^{T^\delta,N}}(1 + m_t^{T^\delta,N}(z))e^{-(\gamma\sigma_{\bar{c}}+\sigma_p)\cdot z}$$

where

$$\iota_t^{T^\delta,N} = \int_{-\infty}^{\infty}(1+m_t^{T^\delta,N}(z))e^{-(\gamma\sigma_{\bar{c}}+\sigma_p)z}\phi(z)\,dz,$$

and $W_t^{T^\delta}$ and $\lambda_t^{T^\delta}\,\phi_t^{T^\delta}(z)\,dz$ are a $P^{T^\delta,N}$-Wiener process and the $P^{T^\delta,N}$-intensity kernel of $\nu(dt\times dz)$, respectively.

5.2. **Approximation to AJD Structure under Associated Forward Martingale Measure.** Let $Y^{T,N} = \ln L^{T,N}$. As conducted in the previous section, the system dynamics of $(Y^{T,N}, V)'$ is approximated in order to admit an AJD structure under $\mathbb{P}^{\mathbb{T}^\delta,\mathbb{N}}$. First, $\varsigma_s^N(T)$ is approximated as follows:

$$
\begin{aligned}
\varsigma_s^N(T) &= \frac{1+\delta L_{s-}^{T,N}}{\delta L_{s-}^{T,N}} \sum_{n=1}^{N} \left(\frac{B_s^{T+(n-1)\delta}}{B_s^{T,N}} \varsigma_s^{T+(n-1)\delta} - \frac{B_s^{T^\delta+(n-1)\delta}}{B_s^{T^\delta,N}} \varsigma_s^{T+n\delta} \right) \\
&\approx \frac{1+\delta L_{s-}^{T,N}}{\delta L_{s-}^{T,N}} \sum_{n=1}^{N} \frac{B_s^{T+(n-1)\delta}}{B_s^{T,N}} (\varsigma_s^{T+(n-1)\delta} - \varsigma_s^{T+n\delta}) \\
&= \frac{1+\delta L_{s-}^{T,N}}{\delta L_{s-}^{T,N}} \sum_{n=1}^{N} \frac{B_s^{T+(n-1)\delta}}{B_s^{T,N}} \frac{\delta L_{s-}^{T+(n-1)\delta}}{1+\delta L_{s-}^{T+(n-1)\delta}} \varsigma(T+(n-1)\delta - s) \\
&\approx \frac{1+\delta L_{s-}^{T,N}}{\delta L_{s-}^{T,N}} \sum_{n=1}^{N} \frac{B_s^{T+(n-1)\delta}}{B_s^{T,N}} \frac{\delta L_{s-}^{T,N}}{1+\delta L_{s-}^{T,N}} \varsigma(T+(n-1)\delta - s) \\
&\approx \sum_{n=1}^{N} \frac{B_t^{T+(n-1)\delta}}{B_t^{T,N}} \tilde{\varsigma}(T+(n-1)\delta - s).
\end{aligned}
$$

Here the following approximations were used.

$$
\frac{B_s^{T+(n-1)\delta}}{B_s^{T,N}} \approx \frac{B_s^{T^\delta+(n-1)\delta}}{B_s^{T^\delta,N}}, \qquad \frac{\delta L_{s-}^{T+(n-1)\delta}}{1+\delta L_{s-}^{T+(n-1)\delta}} \approx \frac{\delta L_{s-}^{T,N}}{1+\delta L_{s-}^{T,N}}, \qquad \frac{B_s^{T+(n-1)\delta}}{B_s^{T,N}} \approx \frac{B_t^{T+(n-1)\delta}}{B_t^{T,N}}.
$$

Next, $m_s^{T^\delta,N}(z)$ is approximated to

$$
\begin{aligned}
& m_s^{T^\delta,N}(z) = \sum_{n=1}^{N} \frac{B_s^{T^\delta+(n-1)\delta}}{B_s^{T^\delta,N}} m_s^{T^\delta+(n-1)\delta}(z) \\
&\approx \sum_{n=1}^{N} \frac{B_s^{T^\delta+(n-1)\delta}}{B_s^{T^\delta,N}} \left[\exp\left[-\sum_{k=1}^{K_s^{T^\delta+(n-1)\delta}} \frac{\delta L_{t-}^{T^\delta+(n-1)\delta-k\delta}}{1+\delta L_{t-}^{T^\delta+(n-1)\delta-k\delta}} \right.\right. \\
&\quad \left.\left. \times \left\{ e^{\tilde{\mu}(T^\delta+(n-1)\delta-k\delta-s)} \Big(1+\tilde{\sigma}(T^\delta+(n-1)\delta-k\delta-s)z\Big) - 1 \right\} \right] - 1 \right] \\
&\approx -\sum_{n=1}^{N} \sum_{k=1}^{K_s^{T+n\delta}} \frac{B_t^{T^\delta+(n-1)\delta}}{B_t^{T^\delta,N}} \frac{\delta L_{t-}^{T^\delta+(n-1-k)\delta}}{1+\delta L_{t-}^{T^\delta+(n-1-k)\delta}} \\
&\quad \times \left\{ e^{\tilde{\mu}(T^\delta+(n-1-k)\delta-s)} \tilde{\sigma}(T^\delta+(n-1-k)\delta-s)z + e^{\tilde{\mu}(T^\delta+(n-1-k)\delta-s)} - 1 \right\} \\
&\approx \exp\left[-\sum_{n=1}^{N} \sum_{k=1}^{K_s^{T+n\delta}} \frac{B_t^{T^\delta+(n-1)\delta}}{B_t^{T^\delta,N}} \frac{\delta L_{t-}^{T^\delta+(n-1-k)\delta}}{1+\delta L_{t-}^{T^\delta+(n-1-k)\delta}} \right. \\
&\quad \left. \times \left\{ e^{\tilde{\mu}(T^\delta+(n-1-k)\delta-s)} \tilde{\sigma}(T^\delta+(n-1-k)\delta-s)z + e^{\tilde{\mu}(T^\delta+(n-1-k)\delta-s)} - 1 \right\} \right] - 1.
\end{aligned}
$$

Let $\tilde{\varsigma}^N(T-s)$ and $\tilde{m}_s^{T^\delta,N}(z)$ denote the approximations of $\varsigma^N(T-s)$ and $m_s^{T^\delta,N}(z)$, *i.e.* they are given by

$$\tilde{\varsigma}^N(T-s) = \sum_{n=1}^{N} \frac{B_t^{T+(n-1)\delta}}{B_t^{T,N}} \tilde{\varsigma}(T+(n-1)\delta - s),$$

$$\begin{aligned}\tilde{m}_s^{T^\delta,N}(z) \approx \exp\Bigg[&- \sum_{n=1}^{N} \sum_{k=1}^{K_s^{T+n\delta}} \frac{B_t^{T^\delta+(n-1)\delta}}{B_t^{T^\delta,N}} \frac{\delta L_{t-}^{T^\delta+(n-1-k)\delta}}{1+\delta L_{t-}^{T^\delta+(n-1-k)\delta}} \\ &\times \Big\{ e^{\tilde{\mu}(T^\delta+(n-1-k)\delta-s)} \tilde{\sigma}(T^\delta+(n-1-k)\delta-s) z + e^{\tilde{\mu}(T^\delta+(n-1-k)\delta-s)} - 1 \Big\} \Bigg] - 1,\end{aligned} \tag{5.6}$$

respectively. Therefore, $\eta_s^{T,N}(z)$ is approximated to

$$\begin{aligned}\eta_s^{T,N}(z) &= \frac{1+\delta L_{s-}^{T,N}}{\delta L_{s-}^{T,N}} \left(\frac{1+m_s^{T,N}(z)}{1+m_s^{T^\delta,N}(z)} - 1 \right) \\ &\approx \frac{1+\delta L_{s-}^{T,N}}{\delta L_{s-}^{T,N}} \left(\frac{1+\tilde{m}_s^{T,N}(z)}{1+\tilde{m}_s^{T^\delta,N}(z)} - 1 \right) \\ &\approx \exp\big[e^{\tilde{\mu}(T^\delta+(n-1-k)\delta-s)} \tilde{\sigma}^N(T-s) z + e^{\tilde{\mu}(T^\delta+(n-1-k)\delta-s)} - 1 \big] - 1.\end{aligned} \tag{5.7}$$

Moreover, $\iota_s^{T^\delta,N}$ is approximated to $\tilde{\iota}_s^{T^\delta,N}$ which is given by

$$\begin{aligned}\tilde{\iota}_s^{T^\delta,N} = &\int_{-\infty}^{\infty} (1+\tilde{m}_s^{T^\delta,N}(z)) e^{-(\gamma\sigma_{\bar{c}}+\sigma_p)z} \phi(z)\, dz = \exp\Bigg[\frac{1}{2} (\bar{z}_s^{T^\delta,N})^2 \\ &- \sum_{n=1}^{N} \sum_{k=1}^{K_s^{T+n\delta}} \frac{B_t^{T^\delta+(n-1)\delta}}{B_t^{T^\delta,N}} \frac{\delta L_{t-}^{T^\delta+(n-1-k)\delta}}{1+\delta L_{t-}^{T^\delta+(n-1-k)\delta}} \left(e^{\tilde{\mu}(T^\delta+(n-1-k)\delta-s)} - 1 \right) \Bigg].\end{aligned} \tag{5.8}$$

where

$$\begin{aligned}\bar{z}_s^{T^\delta,N} = &\sum_{n=1}^{N} \sum_{k=1}^{K_s^{T+n\delta}} \frac{B_t^{T^\delta+(n-1)\delta}}{B_t^{T^\delta,N}} \frac{\delta L_{t-}^{T^\delta+(n-1-k)\delta}}{1+\delta L_{t-}^{T^\delta+(n-1-k)\delta}} \\ &\times e^{\tilde{\mu}(T^\delta+(n-1-k)\delta-s)} \tilde{\sigma}(T^\delta+(n-1-k)\delta-s) + \gamma\sigma_{\bar{c}} + \sigma_p.\end{aligned} \tag{5.9}$$

Finally, $\lambda_s^{T^\delta,N}$ and $\phi_s^{T^\delta,N}(z)$ are approximated to $\tilde{\lambda}_s^{T^\delta,N}$ and $\tilde{\phi}_s^{T^\delta,N}(z)$, respctively, which are defined by

$$\begin{aligned}\tilde{\lambda}_s^{T^\delta,N} &= \tilde{\iota}_s^{T^\delta,N} (\lambda_0 + \lambda_1 V_s), \\ \tilde{\phi}_s^{T^\delta,N}(z) &= \frac{1}{\tilde{\iota}_s^{T^\delta,N}} (1+\tilde{m}_s^{T^\delta,N}(z)) e^{-(\gamma\sigma_{\bar{c}}+\sigma_p)z} \phi(z) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(z - \bar{z}_s^{T^\delta,N} \right)^2 \right],\end{aligned} \tag{5.10}$$

respectively.

Using the approximations (5.2)-(5.10), define the *approximate GE swaption price* $\widetilde{\mathrm{PS}}_t(L^{T,N},K)$ by

$$\widetilde{\mathrm{PS}}_t(L^{T,N},K) = \delta B_t^{T^\delta,N} E_t^{T^\delta,N} \big[(\tilde{L}_T^{T,N} - K)^+ \big] \tag{5.11}$$

where $\tilde{L}_T^{T,N} = e^{\tilde{Y}_T^{T,N}}$ and $\tilde{Y}_T^{T,N}$ are given by

$$
\begin{aligned}
d\tilde{Y}_s^{T,N} &= \left\{ -\tilde{\iota}_s^{T^\delta,N}\lambda_0 - \left(\tilde{\iota}_s^{T^\delta,N}\lambda_1 + \frac{1}{2}\|\tilde{\varsigma}^N(T-s)\|^2 \right) \tilde{V}_s \right\} ds \\
&\quad + \sqrt{\tilde{V}_s}\, \tilde{\varsigma}^N(T-s)\cdot dW_s^{T^\delta,N} + \int_{-\infty}^{\infty} (\tilde{\sigma}^N(T-s)z + \tilde{\mu}^N(T-s))\, \tilde{\nu}^N(ds\times dz), \\
d\tilde{V}_s &= \left[\kappa\bar{V} - \left\{ \kappa + \varsigma_V \cdot \left(\gamma\varsigma_{\bar{c}} + \varsigma_p - \tilde{\varsigma}_s^{T^\delta,N} \right) \right\} \tilde{V}_s \right] ds + \sqrt{\tilde{V}_s}\, \varsigma_V \cdot dW_s^{T^\delta,N},
\end{aligned}
\tag{5.12}
$$

with $(\tilde{Y}_t^{T,N}, \tilde{V}_t)' = (Y_t^{T,N}, V_t)'$ where $\tilde{\nu}^N(ds\times dz)$ is the marked point process with P^{T^δ}-intensity kernel $\tilde{\lambda}_s^{T^\delta,N}\, \tilde{\phi}_s^{T^\delta,N}(z)\, dz$.

5.3. **Fourier Transform Method and GE Pricing Formula for Swaption.** Since the system dynamics of $\tilde{X}^{T,N} = (\tilde{Y}^{T,N}, \tilde{V})'$ admits an AJD structure under $P^{T^\delta,N}$, the Fourier transform method can be applied to $\tilde{X}^{T,N}$. Define the function $\psi_{t,\tilde{X}^{T,N}}^{T^\delta,N}\colon \mathbb{C}^2\times\mathbb{R}\times\mathbb{R}_+\times[t,T]\to\mathbb{C}$ by

$$
\psi_{t,\tilde{X}^{T,N}}^{T^\delta}(\xi, \tilde{X}_s^{T,N}, s) = E_s^{T^\delta}\left[e^{\xi\cdot\tilde{X}_T^{T,N}} \right].
$$

The function $\psi_{t,\tilde{X}^{T,N}}^{T^\delta,N}$ is derived in analytic form in a similar way as shown in Lemma 1.

Lemma 2. *It follows that*

$$
\psi_{t,\tilde{X}^{T,N}}^{T^\delta}((\xi_1,0)', (\tilde{Y}_s^{T,N}, \tilde{V}_s)', s) = e^{\alpha^N(\tau;\xi_1)+\xi_1\tilde{Y}_s^{T,N}+\beta^N(\tau;\xi_1)\tilde{V}_s}
\tag{5.13}
$$

where $\tau = T-s$, *and*

$$
\begin{aligned}
\alpha^N(\tau;\xi_1) &= \begin{cases} \alpha_0^N(\tau;\xi_1,\tau_0) & \text{if } \tau\in I_0 \\ \alpha^N(\tau_{n-1};\xi_1) + \alpha_0^N(\tau-\tau_{n-1};\xi_1,\tau_n) & \text{if } \tau\in I_n, \end{cases} \\
\beta^N(\tau;\xi_1) &= \begin{cases} \beta_0^N(\tau;\xi_1,\tau_0) & \text{if } \tau\in I_0 \\ \beta^N(\tau_{n-1};\xi_1) + \beta_0^N(\tau-\tau_{n-1};\xi_1,\tau_n) & \text{if } \tau\in I_n, \end{cases}
\end{aligned}
\tag{5.14}
$$

with

$$
\begin{aligned}
\alpha_0^N(\tau;\xi_1,\tau_n) &= \tilde{\iota}_s^{T^\delta,N}\lambda_0 \left[\exp\left[\frac{1}{2}\tilde{\sigma}^N(\tau)\xi_1^2 + (\tilde{\mu}^N(\tau) + \bar{z}_s^{T^\delta,N}\tilde{\sigma}^N(\tau))\xi_1 \right] - \xi_1 - 1 \right]\tau \\
&\qquad - \kappa\bar{V}\left[\frac{b_n^N + d_n^N}{a}\tau + \frac{2}{a}\ln\left[1 - \frac{b_n^N + d_n^N}{2d_n^N}(1-e^{-d_n^N\tau}) \right] \right], \\
\beta_0^N(\tau;\xi_1,\tau_n) &= \frac{\iota_n^N(1-e^{-d_n^N\tau})}{2d_n^N - (b_n^N+d_n^N)(1-e^{-d_n^N\tau})},
\end{aligned}
$$

where

$$
\begin{aligned}
a &= \|\varsigma_V\|^2, \qquad b_n^N = \tilde{\varsigma}^N(\tau)\cdot\varsigma_V\xi_1 - \{\kappa + \varsigma_V\cdot(\gamma\varsigma_{\bar{c}} + \varsigma_p - \tilde{\varsigma}_s^{T^\delta})\}, \\
\iota_n^N &= 2\tilde{\iota}_s^{T^\delta,N}\lambda_1 \exp\left[\frac{1}{2}\tilde{\sigma}^N(\tau)\xi_1^2 + (\tilde{\mu}^N(\tau) + \bar{z}_s^{T^\delta,N}\tilde{\sigma}^N(\tau))\xi_1 \right] \\
&\qquad + \|\tilde{\varsigma}^N(\tau)\|^2\xi_1^2 - \left(2\tilde{\iota}_s^{T^\delta,N}\lambda_1 + \|\tilde{\varsigma}^N(\tau)\|^2 \right)\xi_1 - 2\tilde{\iota}_s^{T^\delta,N}\lambda_1, \\
d_n &= \sqrt{|(b_n^N)^2 - a\,\iota_n^N|}\,\exp\left[\frac{i\arg((b_n^N)^2 - a\,\iota_n^N)}{2} \right].
\end{aligned}
$$

Then the approximate GE swaption price is derived in the same way as in Proposition 2.

Proposition 4. *Let $N \in \mathbb{N}$, $T \in (0, T^\dagger - (N+1)\delta]$ and $K \in \mathbb{R}_{++}$. Under Assumption 1-3, it follows that for a given ASM equilibrium $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ for $\mathbf{E}$ where $p_t = \frac{B}{\Lambda^{\mathbf{B}}} u_c^{\hat{\alpha}}(t, \bar{c}_t)$, the approximate GE swaption price $\widetilde{\mathrm{SP}}_t(L^T, K)$ satisfies the formula*

$$\begin{aligned}\widetilde{\mathrm{SP}}_t(L^T, K) = \delta B_t^{T^\delta, N} \Bigg[& \frac{1}{2} e^{\alpha^N(T-t;1) + Y_t^{T,N} + \beta^N(T-t;1)V_t} \\ & - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\left[e^{\alpha^N(T-t;1-iv) + (1-iv)Y_t^{T,N} + \beta^N(T-t;1-iv)V_t} K^{iv} \right]}{v} dv \\ & + K\left(\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\left[e^{\alpha^N(T-t;-iv) - ivY_t^{T,N} + \beta^N(T-t;-iv)V_t} K^{iv} \right]}{v} dv \right)\Bigg] \end{aligned} \tag{5.15}$$

where $Y_t^{T,N} = \ln L_t^{T,N}$, and $(\alpha^N, \beta^N)'$ are given in (5.14).

Appendix A. Marked Point Process

A.1. **Definition.** A double sequence $(s_n, Z_n)_{n\in\mathbb{N}}$ is considered where s_n is the occurrence time of nth jump and Z_n is a random variable taking its values on a measurable space $(\mathbb{Z}, \mathcal{Z})$ at time s_n. Define a random counting measure $\nu(dt \times dz)$ by

$$\nu([0,t] \times A) = \sum_{n\in\mathbb{N}} 1_{\{s_n \le t,\, Z_n \in A\}} \qquad \forall (t, A) \in \mathbf{T} \times \mathcal{Z}.$$

This counting measure $\nu(dt \times dz)$ is called the $\mathbb{Z}$-*marked point process*.

Let λ be such that

(1) For every $(\omega, t) \in \Omega \times (0, T^\dagger]$, the set function $\lambda_t(\omega, \cdot)$ is a finite Borel measure on $\mathbb{Z}$.
(2) For every $A \in \mathcal{Z}$, the process $\lambda(A)$ is $\mathcal{P}$-measurable and satisfies $\lambda(A) \in \mathcal{L}^1$.

If the equation $E\left[\int_0^{T^\dagger} Y_s\, \nu(ds \times A)\right] = E\left[\int_0^{T^\dagger} Y_s \lambda_s(A)\, ds\right]$ holds for every $A \in \mathcal{Z}$ for any nonnegative $\mathcal{P}$-measurable process Y, then it is said that the marked point process $\nu(dt \times dz)$ has the P-*intensity kernel* $\lambda_t(dz)$.

A.2. **Integration Theorem.** Let $\nu(dt \times dz)$ be a $\mathbb{Z}$-marked point process with the P-intensity kernel $\lambda_t(dz)$. Let m be a $\mathcal{P} \otimes \mathcal{Z}$-measurable process. It follows that:

(1) If the integrability condition $E\left[\int_0^{T^\dagger}\int_{\mathbb{Z}} |m_s(z)| \lambda_s(z)\, ds\right] < \infty$ holds, then the process $\int_0^t \int_{\mathbb{Z}} m_s(z) \{\, \nu(ds \times dz) - \lambda_s(dz)\, ds \,\}$ is a P-martingale.
(2) If $m \in \mathcal{L}(\lambda_t(dz))$, then the process $\int_0^t \int_{\mathbb{Z}} m_s(z) \{\, \nu(ds \times dz) - \lambda_s(dz)\, ds \,\}$ is a local P-martingale.

Proof. See p.235 in Brémaud [9]. □

Appendix B. Ito's Formula and Girsanov's Theorem

B.1. **Ito's Formula.** Let $X = (X^1, ..., X^n)'$ be a n-dimensional semimartingales, and g be a real-valued $\mathbf{C}^2$-function on $\mathbb{R}^n$. Then $g(X)$ is a semimartingale of the form

$$\begin{aligned} g(X_t) = g(X_0) + \sum_{i=1}^n \int_0^t \frac{\partial}{\partial x_i} g(X_{s-})\, dX_s^i + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} g(X_{s-})\, d\langle X_s^{ic}, X_s^{jc}\rangle \\ + \sum_{0\le s\le t} \left\{ g(X_s) - g(X_{s-}) + \sum_{i=1}^n \frac{\partial}{\partial x_i} g(X_{s-})\, \Delta X_s^i \right\} \end{aligned}$$

where X^{ic} is the continuous part of X^{ic} and $\langle X^{ic}, X^{jc}\rangle$ is the quadratic covariation of X^{ic} and X^{jc}.

B.2. **Girsanov's Theorem.**

(1) Let $v \in \prod_{j=1}^{d} \mathcal{L}^2$ and $m \in \mathcal{L}^1(\lambda_t(dz) \times dt)$. Define a real-valued process Λ by

$$\frac{d\Lambda_t}{\Lambda_{t-}} = -v_t \cdot dW_t - \int_{\mathbb{Z}} m_t(z) \{ \nu(dt \times dz) - \lambda_t(dz)\, dt\} \qquad \forall t \in [0, T^\dagger)$$

with $\Lambda_0 = 1$. If $E[\Lambda_{T^\dagger}] = 1$, then there exists a probability measure $\tilde{P}$ on $(\Omega, \mathcal{F}, \mathbb{F})$ given by the Radon-Nikodym derivative $d\tilde{P} = \Lambda_{T^\dagger}\, dP$ such that:
   (a) The measure $\tilde{P}$ is equivalent to P.
   (b) The process given by $\tilde{W} = W_t + \int_0^t v_s\, ds$ for every $t \in \mathbf{T}$ is a $\tilde{P}$-Wiener process.
   (c) The marked point process $\nu(dt \times dz)$ has the $\tilde{P}$-intensity kernel $\tilde{\lambda}_t(dz)$ such that $\tilde{\lambda}_t(dz) = (1 - m_t(z))\lambda_t(dz)$ for every $(t, z) \in \mathbf{T} \times \mathbb{Z}$.

(2) Every probability measure equivalent to P has the structure above.

Appendix C. Basic Concepts in Bond Markets

C.1. **Feasible, Self-Financing, and Admissible Portfolios.** Let X denote a real-valued $\mathcal{P}$-measurable process. The *discounted process of* X is defined by $\tilde{X}^{\mathbf{B}} = \frac{X}{B}$. Let $\tilde{\mathbf{B}}$ denote the discounted bond price family $(1, (\tilde{B}^{T\mathbf{B}})_{T\in\mathbf{T}})$. In bond markets with jump-diffusion information, notions of *feasible*, *self-financing*, and *admissible portfolios* are defined as follows.

Definition 6. Let $\mathbf{B} \in \mathcal{B}$.

(1) A portfolio ϑ is a *feasible portfolio at* $\mathbf{B}$ if and only if it follows that:

$$B_t r_t^B \vartheta_t^0 \in \mathcal{L}^1, \quad \int_t^{T^\dagger} |B_t^T r_t^T|\, |\vartheta_t^1(dT)| \in \mathcal{L}^1$$

$$\int_t^{T^\dagger} |B_t^T v_t^T |\vartheta_t^1(dT)| \in \mathcal{L}^2, \quad \int_t^{T^\dagger} |B_t^T m_t^T(z)|\, |\vartheta_t^1(dT)| \in \mathcal{L}^1(\lambda_t(dz) \times dt).$$

Let $\Theta(\mathbf{B})$ denote the space of feasible portfolios at $\mathbf{B}$.

(2) A feasible portfolio $\vartheta \in \Theta(\mathbf{B})$ at $\mathbf{B}$ is a *self-financing portfolio at* $\mathbf{B}$ if and only if its value process satisfies

$$\mathcal{V}_t(\vartheta) = \mathcal{V}_0(\vartheta) + \int_0^t \vartheta_s^0\, dB_s + \int_0^t \int_s^{T^\dagger} \vartheta_s^1(dT)\, dB_s^T \qquad \forall t \in \mathbf{T}.$$

(3) A feasible portfolio $\vartheta \in \Theta(\mathbf{B})$ at $\mathbf{B}$ is an *admissible portfolio at* $\mathbf{B}$ if and only if there exists a nonnegative number b such that $\tilde{\mathcal{V}}_t^{\mathbf{B}}(\vartheta) := \frac{\mathcal{V}_t(\vartheta)}{B_t} \geq -b$ P-a.s.

C.2. **Arbitrage-Free Markets and Risk-Neutral Measures.** In bond markets with jump-diffusion information, definitions of *arbitrage portfolio*, *arbitrage-free*, and *risk-neutral measure* are given in the following.

Definition 7. Let $\mathbf{B} \in \mathcal{B}$.

(1) A self-financing portfolio $\vartheta \in \Theta(\mathbf{B})$ at $\mathbf{B}$ is an *arbitrage portfolio at* $\mathbf{B}$ if and only if there exist $0 \leq t < T \leq T^\dagger$ such that $\vartheta_s = 0$ for every $s \in [0, t)$ and either of the following:
   (a) $\mathcal{V}_t(\vartheta) \leq 0$ P-a.s., and $\mathcal{V}_T(\vartheta) > 0$, *i.e.* $\mathcal{V}_T(\vartheta) \geq 0$ P-a.s. and $P(\{\mathcal{V}_T(\vartheta) > 0\}) > 0$.
   (b) $\mathcal{V}_t(\vartheta) < 0$, and $\mathcal{V}_T(\vartheta) \geq 0$ P-a.s.

(2) Markets are *arbitrage-free at* $\mathbf{B}$ if and only if there exists no arbitrage portfolio in the space of admissible portfolios.
(3) A probability measure $\tilde{P}^{\mathbf{B}}$ on $(\Omega, \mathcal{F})$ is a *risk-neutral measure at* $\mathbf{B}$ if and only if $\tilde{P}^{\mathbf{B}}$ is equivalent to P, and $\tilde{\mathbf{B}}$ are local $\tilde{P}^{\mathbf{B}}$-martingales.

C.3. **Approximately Complete Markets.** Definitions of *contingent claim*, *replicable claim*, and *approximately complete* are given as follows.

Definition 8. Let $\mathbf{B} \in \mathcal{B}$.

(1) For every $T \in (0, T^{\dagger}]$, a *contingent* T*-claim at* $\mathbf{B}$ is a $\mathcal{F}_T$-measurable random variable X such that $\frac{X}{B_T} \in \mathbf{L}^{\infty}_{+}(\Omega, \mathcal{F}_T)$ where $\mathbf{L}^{\infty}(\Omega, \mathcal{F}_T)$ is the space of almost surely bounded $\mathcal{F}_T$-measurable random variables.
(2) A contingent T-claim X is *replicable at* $\mathbf{B}$ if and only if there exists an admissible self-financing portfolio $\vartheta \in \underline{\Theta}(\tilde{\mathbf{B}})$ such that its value process satisfies $\mathcal{V}_T(\vartheta) = X$.
(3) Markets are *approximately complete at* $\mathbf{B}$ if and only if for any $T \in (0, T^{\dagger}]$ and any T-contingent claim X there exists a sequence of replicable claims $(X_n)_{n \in \mathbb{N}}$ such that $\lim_{n\to\infty} \frac{X_n}{B_T} = \frac{X}{B_T}$ in $\mathbf{L}^2(\Omega, \mathcal{F}_T, \tilde{P}^{\mathbf{B}})$.

Appendix D. Proofs

D.1. **Proof of Proposition 1.** For proofs of 1 and 2, see Kusuda [19]. It follows from (3.5) and (3.6) in Assumption 3 that

$$\begin{aligned} \varsigma(T-t) &= \frac{1+\delta L^T_{t-}}{\delta L^T_{t-}} (\varsigma^T_t - \varsigma^{T^\delta}_t), \\ e^{\sigma(T-t)z + \mu(T-t)} - 1 &= \frac{1+\delta L^T_{t-}}{\delta L^T_{t-}} \left(\frac{1+m^T_t(z)}{1+m^{T^\delta}_t(z)} - 1 \right). \end{aligned} \tag{D.1}$$

The dynamics of GE bond price process is

$$\frac{dB^T_t}{B^T_{t-}} = r^T\, dt + v^T_t \cdot dW_t + \int_{-\infty}^{\infty} m^T_t(z) \left\{ \nu(dt \times dz) - \lambda_t\, \phi(z)\, dz\, dt \right\} \qquad \forall t \in [0, T)$$

where $r^T = r^B_t + v^{\mathbf{B}}_t \cdot v^T_t + \lambda_t \int_{-\infty}^{\infty} m^{\mathbf{B}}_t(z) m^T_t(z) \phi(z)\, dz$. Then applying Ito's formula to the definition of L^T yields for every $t \in [0, T)$

$$\begin{aligned} dL^T_t = &\frac{1+\delta L^T_{t-}}{\delta} \Bigg[\left\{ r^T_t - r^{T^\delta}_t - v^{T^\delta}_t \cdot (v^T_t - v^{T^\delta}_t) - \lambda_t \int_{-\infty}^{\infty} (m^T_t(z) - m^{T^\delta}_t(z)) \phi(z)\, dz \right\} dt \\ &\qquad + (v^T_t - v^{T^\delta}_t) \cdot dW_t + \int_{-\infty}^{\infty} \frac{m^T_t(z) - m^{T^\delta}_t(z)}{1+m^{T^\delta}_t(z)}\, \nu(dt \times dz) \Bigg] \\ = &\frac{1+\delta L^T_{t-}}{\delta} \Bigg[\Bigg\{ (v^{\mathbf{B}}_t - v^{T^\delta}_t) \cdot (v^T_t - v^{T^\delta}_t) \\ &\qquad - \lambda_t \int_{-\infty}^{\infty} (1 - m^{\mathbf{B}}_t(z))(m^T_t(z) - m^{T^\delta}_t(z)) \phi(z)\, dz \Bigg\} dt \\ &\qquad + (v^T_t - v^{T^\delta}_t) \cdot dW_t + \int_{-\infty}^{\infty} \frac{m^T_t(z) - m^{T^\delta}_t(z)}{1+m^{T^\delta}_t(z)}\, \nu(dt \times dz) \Bigg]. \end{aligned} \tag{D.2}$$

Dividing both sides of (D.2) by L^T_{t-} and substituting (3.8) into the resultant equation yields

$$\frac{dL^T_t}{L^T_{t-}} = \frac{1+\delta L^T_{t-}}{\delta L^T_{t-}}\Bigg[\Bigg\{(\gamma v^{\bar{c}}_t + v^p_t - v^{T^\delta}_t)\cdot(v^T_t - v^{T^\delta}_t) - \lambda_t \int_{-\infty}^{\infty} e^{-(\gamma J_{\bar{c}}(z)+J^p(z))}(m^T_t(z) - m^{T^\delta}_t(z))\phi(z)\,dz\Bigg\}dt + (v^T_t - v^{T^\delta}_t)\cdot dW_t + \int_{-\infty}^{\infty}\frac{m^T_t(z)-m^{T^\delta}_t(z)}{1+m^{T^\delta}_t(z)}\,\nu(dt\times dz)\Bigg]. \quad \text{(D.3)}$$

Substituting (3.2), (3.4), (3.7), and (D.1) into (D.3) give (3.9). Then substituting (3.11) into (3.9) yields (3.10). Moreover, it follows from Ito's formula and Girsanov's Theorem that $W^{T^\delta}_t$ and $\lambda^{T^\delta}_t\,\phi^{T^\delta}_t(z)\,dz)$ are a P^{T^δ}-Wiener process and P^{T^δ}-intensity kernels of marked point processes $\nu(dt\times dz)$, respectively.

D.2. **Proof of Lemma 1.** Since $\psi^{T^\delta}_{t,\tilde{X}^T}(\xi,\tilde{X}_s,s)$ is a P^{T^δ}-martingale, it follows from Ito's formula that $\psi^{T^\delta}_{t,\tilde{X}^T}((\xi_1,0)',\tilde{X}_s,s)$ satisfies

$$\frac{\partial\psi^{T^\delta}_{t,\tilde{X}^T}}{\partial s} + \frac{\partial\psi^{T^\delta}_{t,\tilde{X}^T}}{\partial x_1}\Big\{-\tilde{\iota}^{T^\delta}_s\lambda_0 - \big(\tilde{\iota}^{T^\delta}_s\lambda_1 + \frac{1}{2}\|\tilde{\varsigma}(T-s)\|^2\big)\tilde{V}_s\Big\} + \frac{\partial\psi^{T^\delta}_{t,\tilde{X}^T}}{\partial x_2}\Big[\kappa\bar{V} - \big\{\kappa + \varsigma_V\cdot\big(\gamma\varsigma_{\bar{c}} + \varsigma_p - \tilde{\varsigma}^{T^\delta}_s\big)\big\}\tilde{V}_s\Big] + \frac{1}{2}\frac{\partial^2\psi^{T^\delta}_{t,\tilde{X}^T}}{\partial x_1^2}\|\tilde{\varsigma}(T-s)\|^2\tilde{V}_s + \frac{\partial^2\psi^{T^\delta}_{t,\tilde{X}^T}}{\partial x_1\partial x_2}\tilde{\varsigma}(T-s)\cdot\varsigma_V\tilde{V}_s + \frac{1}{2}\frac{\partial^2\psi^{T^\delta}_{t,\tilde{X}^T}}{\partial x_2^2}\|\varsigma_V\|^2\tilde{V}_s + E^{T^\delta}_s\big[\Delta\psi^{T^\delta}_{t,\tilde{X}^T}\big] = 0. \quad \text{(D.4)}$$

Thus, we have

$$\psi^{T^\delta}_{t,\tilde{X}^T}((\xi_1,0)',(x_1,x_2)',s) = e^{\alpha(\tau;\xi_1)+\xi_1 x_1+\beta(\tau;\xi_1)x_2} \quad \text{(D.5)}$$

where $\tau = T-s$ and $(\alpha,\beta)'$ satisfies the following system of ODEs

$$\begin{aligned}\alpha'(\tau) &= -\tilde{\iota}^{T^\delta}_s\lambda_0\xi_1 + \kappa\bar{V}\beta(\tau) + \tilde{\iota}^{T^\delta}_s\lambda_0\left(\int_{-\infty}^{\infty} e^{\xi_1(\tilde{\sigma}(\tau)z+\tilde{\mu}(\tau))}\tilde{\phi}^{T^\delta}_s(z)\,dz - 1\right),\\ \beta'(\tau) &= -\left(\tilde{\iota}^{T^\delta}_s\lambda_1 + \frac{1}{2}\|\tilde{\varsigma}(\tau)\|^2\right)\xi_1 - \big\{\kappa + \varsigma_V\cdot\big(\gamma\varsigma_{\bar{c}} + \varsigma_p - \tilde{\varsigma}^{T^\delta}_s\big)\big\}\beta(\tau)\\ &\quad + \frac{1}{2}\|\tilde{\varsigma}(\tau)\|^2\xi_1^2 + \tilde{\varsigma}(\tau)\cdot\varsigma^V\xi_1\beta(\tau) + \frac{1}{2}\|\varsigma^V\|^2\beta^2(\tau)\\ &\quad + \tilde{\iota}^{T^\delta}_s\lambda_1\left(\int_{-\infty}^{\infty} e^{\xi_1(\tilde{\sigma}(\tau)z+\tilde{\mu}(\tau))}\tilde{\phi}^{T^\delta}_s(z)\,dz - 1\right).\end{aligned} \quad \text{(D.6)}$$

with the initial condition $(\alpha(0;\xi_1),\beta(0;\xi_1))' = (0,0)'$. This system of ODE is rewritten as

$$\begin{aligned}\alpha'(\tau) &= \kappa\bar{V}\beta(\tau) + \tilde{\iota}^{T^\delta}_s\lambda_0\left(\exp\left[\frac{1}{2}\tilde{\sigma}(\tau)\xi_1^2 + (\tilde{\mu}(\tau) + \bar{z}^{T^\delta}_s\tilde{\sigma}(\tau))\xi_1\right] - \xi_1 - 1\right)\\ \beta'(\tau) &= \frac{1}{2}\Big(a\beta^2(\tau) + 2b_n\beta(\tau) + c_n\Big),\end{aligned} \quad \text{(D.7)}$$

where

$$a = \|\varsigma_V\|^2, \qquad b_n = \varsigma(\tau_n) \cdot \varsigma_V \xi_1 - \left\{\kappa + \varsigma_V \cdot \left(\gamma \varsigma_{\bar{c}} + \varsigma_p - \tilde{\varsigma}_s^{T^\delta}\right)\right\},$$

$$c_n = \|\varsigma(\tau_n)\|^2 \xi_1^2 - \left(2\tilde{\iota}_s^{T^\delta} \lambda_1 + \|\varsigma(\tau_n)\|^2\right) \xi_1 + 2\tilde{\iota}_s^{T^\delta} \lambda_1 \left(\exp\left[\frac{1}{2}\tilde{\sigma}(\tau)\xi_1^2 + \left(\tilde{\mu}(\tau) + \bar{z}_s^{T^\delta}\tilde{\sigma}(\tau)\right)\xi_1\right] - 1\right).$$

Thus, the solution for this system of ODEs is given by (4.11).

References

[1] Ahn, C.-M., and H. E. Thompson (1988): “Jump-Diffusion Processes and the Term-Structure of Interest Rates,” *Journal of Finance*, 43, 155-174.

[2] Andersen, L., and J. Andreasen (2000): “Volatility Skews and Extensions of the LIBOR Market Model,” *Applied Mathematical Finance*, 7, 1-32.

[3] Bates, D. S. (1991): “The Crash of 87: Was it expected? The Evidence from Options Markets,” *Journal of Finance*, 46, 1009-1044.

[4] Bates, D. S. (1996): “Jump and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options,” *Review of Financial Studies*, 9, 69-107.

[5] Björk, T., Y. Kabanov, and W. Runggaldier (1997): “Bond Market Structure in the Presence of Marked Point Processes,” *Mathematical Finance*, 7, 211-239.

[6] Björk, T., G. D. Masi, and Y. Kabanov, W. Runggaldier (1997): “Towards a General Theory of Bond Markets,” *Finance and Stochastics*, 1, 141-174.

[7] Black, F., and M. Scholes (1973): “The Pricing of Options and Corporate Liabilities,” *Journal of Political Economy*, 81, 637-654.

[8] Brace, A., D. Gatarek, and M. Musiela (1997): “The Market Model of Interest Rate Dynamics,” *Mathematical Finance*, 7, 127-155.

[9] Brémaud, T. (1981): *Point Processes and Queues: Martingale Dynamics*. New York: Springer-Verlag.

[10] Carrasco, M., and J.-P. Florens (2000): “Generalization of GMM to a Continuum of Moment Conditions,” *Economic Theory*, 16, 797-834.

[11] Carrasco, M., M. Chernov, J.-P. Florens, and E. Ghysels (2003): “Efficient Estimation of Jump-Diffusions and General Dynamic Models with a Continuum of Moment Conditions,” Scientific Series 2003s-02, CIRANO.

[12] Das, S. R., and S. Foresi (1996): “Exact solutions for bond and option prices with systematic jump risk,” *Review of Derivative Studies*, 1, 7-24.

[13] Duffie, D., J. Pan, and K. Singleton (2000): “Transform Analysis and Asset Pricing for Affine Jump-Diffusions,” *Econometrica*, 68, 1343-1376.

[14] Glasserman, P., and S. G. Kou (2003): “The Term Structure of Simple Forward Rates with Jump Risk,” *Mathematical Finance*, 13, 383-410.

[15] Heath, D., R. Jarrow, and A. Morton (1992): “Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claim Valuation,” *Econometrica*, 60, 77-106.

[16] Heston, S. L. (1993): “A Closed-Form Solution of Options with Stochastic Volatility with Applications to Bond and Currency Options,” *Review of Financial Studies*, 6, 327-343.

[17] Jamshidian, F. (1989): “An Exact Bond Option Formula,” *Journal of Finance*, 44, 205-209.

[18] Jamshidian, F. (1997): “LIBOR and Swap Market Models and Measures,” *Finance and Stochastics*, 293-330.

[19] Kusuda, K. (2003): *Term Structure Models of Interest Rates with Jump-Diffusion Information: Equilibrium, CAPM, and Derivative Asset Pricing*, Ph.D. Dissertation, Department of Economics, University of Minnesota.

[20] Merton, V. (1976): “Option Pricing when Underlying Stock Returns are Discontinuous,” *Journal of Financial Economics*, 3, 125-144.

[21] Miltersen, K. R., K. Sandmann, and D. Sondermann (1997): “Closed Form Solutions for Term Structure Derivatives with Log-Normal Interest Rates,” *Journal of Finance*, 52, 409-430.

[22] Naik, V., and M. Lee (1990): “General Equilibrium Pricing of Options on the marMet Portfolio with Discontinuous Returns,” *Review of Financial Studies*, 3, 493-521.

[23] Pan, J. (2002): “The Jump-Risk Premia Implicit in Options: Evidence from an Integrated Time-Series Study,” *Journals of Financial Economics*, 63, 3-50.