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CRR WORKING PAPER SERIES B

Working Paper No. B-6

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by

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July 2005

Keywords and Phrases: Ambiguity, Jump-diffusion information, Knightian uncertainty, Normalization, Stochastic differential utility.

JEL Classification Numbers: C61, D81, D91.

Abstract. This paper presents that robust control-based utility advocated by Anderson, Hansen, and Sargent [1], and Hansen and Sargent [11], admits a normalized representation, where future utility enters the recursion through an aggregator, of stochastic differential utility (Duffie and Epstein [4]) under nonMakovian jump-diffusion information. In virtue of the normalized representation, sufficient conditions are shown for the existence, uniqueness, ambiguity aversion, risk aversion, time consistency, and other properties of the utility.

I would like to thank my adviser Professor Jan Werner for his invaluable advice. This research was supported in part by a grant from the Ministry of Education, Sport, Science and Technology, Japan.

1. Introduction

Against the background of a growing dissatisfaction in Economics toward the expedient assumption that each agent is positive that her or his subjective probability is true, utilities with "Knightian uncertainty" or "ambiguity" have been intensively studied for the past decade[1]. Such utilities with ambiguity are expected to contribute to solving unsolved puzzles in Economics, which includes the Ellsberg paradox (Ellsberg [6]), the equity premium puzzle (Mehra and Prescott [16]), and the home-bias puzzle where investers in many countries invest much less in foreign securities than those implied by traditional economics models. Two promising ones among utilities with ambiguity are the *multi-prior utility* (see Remark 2) developed by Gilboa and Schmeidler [10], Epstein and Wang [8], and Chen and Epstein [3], and the utility introduced by Anderson, Hansen, and Sargent [1], and Hansen and Sargent [11], which a robust control is applied to (see Definition 1). It is shown that these two utilities are closely related to each other (Hansen, Sargent, Turmuhambetva, and Williams [12]). This paper considers the existence, uniqueness, representation, and properties of the robust control-based utility. Anderson, Hansen, and Sargent [1] have shown the existence, uniqueness, and a semigroup representation of the robust control-based utility under Markovian jump-diffusion information, and Skiadas [18] has presented that the existence, uniqueness, and Stochastic Differential Utility (SDU) (Duffie and Epstein [4]) representation of the robust control-based utility under nonMarkovian pure diffusion information. Only few studies have so far been made at properties of the robust control-based utility, in particular, in the case of jump-diffusion information. Market crush is caused by jump information, and the fear of market crush is amplified by ambiguity for jump information. Therefore, it is essential to consider utility with ambiguity under jump-diffusion information to examine the puzzles mentioned above. There are two main results in this paper. One is to show sufficient conditions for the existence, uniqueness and *normalized SDU representation*, where future utility enters the recursion through an aggregator, of the robust control-based utility, and the other is to give sufficient conditions for properties of the utility including ambiguity aversion, risk aversion, and time consistency, both under nonMarkovian jump-diffusion information. Many of the results are derived exploiting the normalized SDU representation.

This paper is summarized as follows. An extended SDU (Duffie and Skiadas [5]) which includes not only usual SDU but also Hindy-Huang-Kreps utility (Hindy, Huang, and Kreps [13]), is first introduced under jump-diffusion information. Then the robust control advocated by Anderson, Hansen, and Sargent [1], and Hansen and Sargent [11], is applied to the extended SDU to introduce "ambiguity aversion" into the utility. This extended SDU which the robust control is applied to, is called the *Robust Control-based SDU* (RCSDU, hereafter) in this paper. First, it is presented that the RCSDU admits an (unnormalized) SDU representation under jump-diffusion information if it exists. While any SDU is normalized under pure diffusion information, this does not apply to the case of jump-diffusion information. Very recently, a sufficient condition for the normalization of SDU has been revealed under jump-diffusion information (Kusuda [14]). It is confirmed that this condition holds for the RCSDU, and therefore the normalized RCSDU is obtained. Then sufficient conditions for the existence and uniqueness of normalized RCSDU are given exploiting the results of Duffie and Epstein [4] for normalized (usual) SDU. Next, attitudes towards ambiguity and risk of an agent with RCSDU are examined

[1] Anderson, Hansen, and Sargent [1], Chen and Epstein [3], Epstein and Wang [8], Epstein and Zhang [9], Hansen and Sargent [11], Hansen, Sargent, Turmuhambetva, and Williams [12], Lazrak and Quenez [15], Skiadas [18], Wang [19], *etc.*

mainly using the notions proposed by Chen and Epstein [3]. Finally, sufficient conditions for the continuity, monotonicity, time consistency, and concavity of the RCSDU are given utilizing the results of Duffie and Epstein [4] for normalized usual SDU.

The remaining of this paper is organized as follows. Section 2 provides a specification of RCSDU under jump-diffusion information. Section 3 presents the existence, uniqueness and normalization of RCSDU. Section 4 shows sufficient conditions for the ambiguity aversion and risk aversion of RCSDU. Section 5 gives sufficient conditions for other properties of RCSDU. Appendix A, B, and C introduces marked point process, Ito's Formula and Girsanov's Theorem, and extensions of Gronwall-Bellman Inequality, respectively. Appendix D shows proofs of lemmas and propositions.

2. Robust Control-based Stochastic Differential Utility under Jump-Diffusion Information

In this section, a specification of RCSDU (Robust Control-based Stochastic Differential Utility) is provided under jump-diffusion information.

A continuous-time economy with time span $\mathbf{T} := [0, T]$ is considered. Agents' common subjective reference probability and information structure is modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbf{T}}$ is the natural filtration generated by a d-dimensional Wiener process W and a jump process called *marked point process* $\nu(dt \times dz)$ on a Lusin space $(\mathbb{Z}, \mathcal{Z})$ (in usual applications, $\mathbb{Z} = \mathbb{R}^{d'}$, or $\mathbb{N}^{d'}$, or a finite set) with the P-intensity kernel $\lambda_t(dz)$ (for marked point process, see Appendix A). The expectation operator under P is denoted by E, and the conditional expectation operator under P given $\mathcal{F}_t$ is denoted by E_t. A space of cumulative consumption process is denoted by $\mathcal{C}$. Let $X : \mathcal{C} \to \mathbf{L} := [\mathbf{L}^p(\Omega, \mathbf{T}), \mathcal{P}, \mu]^{n_0}$ where $1 \le p \le \infty$, the power n_0 denotes a Cartesian product, $\mathcal{P}$ is the predictable σ-algebra, and μ is the product measure of P and the Lebesgue measure on $\mathbf{T}$. Let $\bar{\mathbf{L}}^p$ denote the $\|\cdot\|_{\bar{\mathbf{L}}^p}$-normed space of $\mathcal{P}$-measurable real-valued processes where $\|Y\|_{\bar{\mathbf{L}}^p} := \|\sup_{t\in\mathbf{T}} |Y_t|\,\|_{\mathbf{L}^p}$. Let $n \in \mathbb{N}$. The space of real-valued $\mathcal{P}$-measurable processes Y satisfying the integrability condition $\int_0^T |Y_s|^n\, ds < \infty$ P-a.s., is denoted by $\mathcal{L}^n$. The space of real-valued $\mathcal{P} \otimes \mathcal{Z}$-measurable process H satisfying the integrability condition $\int_0^T\!\int_{\mathbb{Z}} |H_s(z)|^n\, \lambda_s(dz)\, ds < \infty$ P-a.s., is denoted by $\mathcal{L}^n(\lambda_t(dz) \times dt)$.

Let β denote nonnegative constant. We call that a functional $U : \mathcal{C} \to \mathbb{R}$ is an *SDU with its aggregator* (f, X, β) if and only if $U(C) = U_0(C)$ for every $C \in \mathcal{C}$ where $U_t(C)$ is a unique solution in $\bar{\mathbf{L}}^p$ of the recursive equation:

$$U_t(C) = E_t\left[\int_t^T e^{-\beta(s-t)} f(X_s(C), U_s(C))\, ds\right] \qquad \forall t \in \mathbf{T}. \tag{2.1}$$

This extended SDU is first introduced by Duffie and Skiadas [5] as the name of *Dynamic Utility.* Suppose that $C_t = \int_0^t c_s ds$, and that $X_t(C) = c_t$. Then U is the original SDU introduced by Duffien and Epstein [4]. In particular, if $f(c_t, U_t) = u(c_t)$, then U is a standard time additive utility. Suppose that $X_t(C) = \beta_0 e^{-\beta t} + \int_0^t \beta e^{-\beta(t-s)}\, dC_s$ where $\beta_0 > 0$. Then U is a Hindy-Huang-Kreps utility (Hindy, Huang, and Kreps [13]).

Let $\mathbb{P}$ be the set of all probability measures on $(\Omega, \mathcal{F})$ that are *equivalent to* P, *i.e.* they define the same null events as P. It follows from *Girsanov's Theorem* (see

Appendix B.2) that an equivalent measure $P^{v,m}$ is characterized by the Radon-Nikodym derivative $\dfrac{dP^{v,m}}{dP} = \varLambda_T^{v,m}$ defined by

$$\varLambda_T^{v,m} = \exp\left[\int_0^T \left(-\frac{1}{2}\|v_t\|^2 - \int_{\mathbb{Z}} m_t(z)\,\lambda_t(dz)\right) dt + \int_0^T v_t \cdot dW_t + \int_0^T \ln(1+m_t(z))\,\nu(dt\times dz)\right]. \tag{2.2}$$

where $v_t \in \Pi_{i=1}^d \mathcal{L}^n$, $m_t(z) \in \mathcal{L}(\lambda_t(dz) \times dt)$ and $m_t(z) > -1$ P-a.s. for every $(t,z) \in \mathbf{T} \times \mathbf{Z}$. The expectation operator under $P^{v,m}$ is denoted by $E^{v,m}$, and the conditional expectation operator under $P^{v,m}$ given $\mathcal{F}_t$ is denoted by $E_t^{v,m}$.

A robust control exploits the discounted relative entropy of P with respect to $P^{v,m}$, which is defined by

$$\mathcal{R}_t^{v,m} = E_t^{v,m}\left[\beta \int_t^T e^{-\beta(s-t)} \ln\left(\frac{\varLambda_s^{v,m}}{\varLambda_t^{v,m}}\right) ds + e^{-\beta(T-t)} \ln\left(\frac{\varLambda_T^{v,m}}{\varLambda_t^{v,m}}\right)\right] \qquad \forall t \in \mathbf{T}. \tag{2.3}$$

The notion of *RCSDU* is defined in the following.

Definition 1. Let ζ denote a positive constant. A functional $\hat{U} : \mathcal{C} \to \mathbb{R}$ is said to be an *RCSDU with its aggregator* (f, X, β, ζ) if and only if $\hat{U}(C) = \hat{U}_0(C)$ for every $C \in \mathcal{C}$ where $\hat{U}_t(C)$ is a unique solution in $\bar{\mathbf{L}}^p$ of the recursive equation:

$$\hat{U}_t(C) = \min_{P^{v,m}\in\mathbb{P}} U_t^{v,m}(C) \qquad \forall t \in \mathbf{T} \tag{2.4}$$

where

$$U_t^{v,m}(C) = E_t^{v,m}\left[\int_t^T e^{-\beta(s-t)} f(X_s(C), U_s^{v,m}(C))\, ds\right] + \frac{1}{\zeta}\mathcal{R}_t^{v,m}. \tag{2.5}$$

Remark 1. It is shown in Section 4 that the parameter ζ represents *ambiguity aversion*. For $\zeta = 0_+$, the minimizer of (2.4) is the reference probability measure P. As ζ increases, the relative entropy of P with respect to the minimizer of (2.4) becomes larger. Refer to Wang [19] for an axiomatic treatment of robust control-based preference orders.

Remark 2. It is suggested by Hansen, Sargent, Turmuhambetva, and Williams [12] that the RCSDU for jump-diffusion information is closely related to a *Recursive Multiple-Priors Utility* (RMPU), also called *Maxmin Expected Utility*, defined by

$$U_t^*(C) = \min_{P^{v,m}\in\mathbb{P}(\eta)} E_t^{v,m}\left[\int_t^T e^{-\beta(s-t)} f(X_s(C), U_s^*(C))\, ds\right] \qquad \forall t \in \mathbf{T} \tag{2.6}$$

where η is a positive constant, and $\mathbb{P}(\eta) = \{P^{v,m} \in \mathbb{P} : \mathcal{R}^{v,m} \le \eta\}$. The notion of MPU was introduced by Gilboa and Schmeidler [10] in atemporal setting, and extended to discrete-time setting (Epstein and Wang [8]), and to continuous-time setting with diffusion information (Chen and Epstein [3]). Recently, Lazrak and Quenez [15] have advocated a *Generalized SDU* (GSDU) under diffusion information, the class of which includes the RMPU and an RCSDU in which the underlying SDU in (2.1) is the standard time additive utility.

3. Existence, Uniqueness, and Normalization of RCSDU

In this section, a sufficient condition is presented for the existence, uniqueness, and normalization of RCSDU.

The following lemma follows from integration by part and Girsanov's Theorem.

Lemma 1. *It follows that for every $P^{v,m} \in \mathbb{P}$ and $C \in \mathcal{C}$,*

$$\mathcal{R}_t^{v,m} = E_t^{v,m}\left[\int_t^T e^{-\beta(s-t)}\left\{\frac{1}{2}\|v_s\|^2 + \int_{\mathbb{Z}}\Big(\frac{m_s(z)}{1+m_s(z)} + \ln(1+m_s(z))\Big)\lambda_s^{v,m}(dz)\right\} ds\right] \tag{3.1}$$

for every $t \in \mathbf{T}$.

Proof. See Appendix D.1. □

Using expression (3.1), equation (2.5) is rewritten to

$$U_t^{v,m}(C) = E_t^{v,m}\left[\int_t^T e^{-\beta(s-t)}\left\{f(X_s(C), U_s^{v,m}(C)) + \frac{1}{\zeta}\left(\frac{1}{2}\|v_s\|^2 + \int_{\mathbb{Z}}\Big(\frac{-m_s(z)}{1+m_s(z)} + \ln(1+m_s(z))\Big)\lambda_s^{v,m}(dz)\right)\right\} ds\right]. \tag{3.2}$$

For every $x \in \mathbb{R}$, let $x^+ := \max\{x, 0\}$ and $x^- := \min\{x, 0\}$. The following condition on f is imposed in order to guarantee the existence of minimizer in utility decision problem (2.4).

Assumption 1. *The aggregator f is $\mathbf{C}^{0,1}$ and satisfies either the following conditions:*

- *f is concave in its second (utility) argument, and such that for every $\mathbb{P}^{v,m} \in \mathbb{P}$ and every $C \in \mathcal{C}$, $E^{v,m}[\int_0^T f_u^-(X_t, U_t)\, dt] < \infty$ and $E^{v,m}[\int_0^T e^{-\beta t} f_u^-(X_t, U_t)$ $(U_t - U_t^{v,m})^+ \, dt] < \infty$.*
- *f is convex in its second (utility) argument, and such that for every $\mathbb{P}^{v,m} \in \mathbb{P}$ and every $C \in \mathcal{C}$, $E^{v,m}[\int_0^T f_u^+(X_t, U_t)\, dt] < \infty$ and $E^{v,m}[\int_0^T e^{-\beta t} f_u^+(X_t, U_t)$ $(U_t - U_t^{v,m})^+ \, dt] < \infty$.*

Here U is a solution in $\bar{\mathbf{L}}^p$ of the backward stochastic differential-difference equation (BSDDE, hereafter) (3.3).

The following proposition extends the result for nonMarkovian pure diffusion information (Skiadas [18]) to nonMarkovian jump-diffusion information, although Skiadas [18] also shows the existence and uniqueness of RCSDU using the result of Schroder and Skiadas [17]. The existence and uniqueness of RCSDU is shown in Theorem 1.

Proposition 1. *Let $C \in \mathcal{C}$. Suppose that for every $P^{v,m} \in \mathbb{P}$, U is a unique solution in $\bar{\mathbf{L}}^p$ of the BSDDE*

$$dU_t = -\left\{f(X_t, U_t) - \beta U_t - \frac{\zeta}{2}\|\sigma_t^U\|^2 - \int_{\mathbb{Z}}\int_{U_{t-}}^{(1+m_t(z))U_{t-}}\Big(1 - e^{-\zeta(u-U_{t-})}\Big)\, du\, \lambda_t(dz)\right\} dt + \sigma_t^U \cdot dW_t + U_{t-}\int_{\mathbb{Z}} m_t^U(z)\{\nu(dt \times dz) - \lambda_t(dz)\, dt\} \qquad \forall t \in \mathbf{T} \tag{3.3}$$

with $U_T = 0$ where $\sigma_t^U \in \Pi_{i=1}^d \mathcal{L}^n$, $m_t^U(z) \in \mathcal{L}(\lambda_t(dz) \times dt)$ and $m_t^U(z) > -1$ P-a.s. for every $(t,z) \in \mathbf{T} \times \mathbf{Z}$. Then $U^{v,m}$ satisfies

$$U_t^{v,m} = U_t + E_t^{v,m}\left[\int_t^T e^{-\beta(s-t)}\{f(X_s, U_s^{v,m}) - f(X_s, U_s) + \mathcal{Q}_s^{v,m}\}\, ds\right] \tag{3.4}$$

where

$$\mathcal{Q}_s^{v,m} = \frac{1}{\zeta}\bigg\{\frac{1}{2}\|v_s - \zeta\sigma_s^U\|^2$$

$$+\int_{\mathbb{Z}}\bigg(\frac{-m_s(z)}{1+m_s(z)} + \ln(1+m_s(z)) + \frac{e^{-\zeta m_s^U(z)U_{s-}} - 1}{1+m_s(z)} + \zeta m_s^U(z)U_{s-}\bigg)\lambda_s^{v,m}(dz)\bigg\}. \tag{3.5}$$

Furthermore, if Assumption 1 is satisfied, then the minimizer of (2.4) is $P^{\hat{v},\hat{m}}$ *where* $(\hat{v}_t, \hat{m}_t(z)) = (-\zeta\sigma_t^U, e^{-\zeta m_t^U(z)U_{t-}} - 1)$*, and* $\hat{U}(C) = U$*.*

Proof. See Appendix D.2. □

Suppose that U is a unique solution in $\bar{\mathbf{L}}^p$ of the BSDDE (3.3). Then a RCSDU process U has the following representation:

$$U_t = E_t\bigg[\int_t^T\bigg\{f^\beta(X_s,U_s) - \frac{\zeta}{2}\|\sigma_s^U\|^2 - \int_{\mathbb{Z}}\int_{U_{s-}}^{(1+m_s(Z))U_{s-}}\Big(1 - e^{-\zeta(u-U_{s-})}\Big)\,du\,\lambda_s(dz)\bigg\}\,ds\bigg] \tag{3.6}$$

for every $t \in \mathbf{T}$ where $f^\beta(x,u) := f(x,u) - \beta u$, or equivalently

$$U_t = E_t\bigg[\int_t^T e^{-\beta(s-t)}\bigg\{f(X_s,U_s) - \frac{\zeta}{2}\|\sigma_s^U\|^2 - \int_{\mathbb{Z}}\int_{U_{s-}}^{(1+m_s(Z))U_{s-}}\Big(1 - e^{-\zeta(u-U_{s-})}\Big)\,du\,\lambda_s(dz)\bigg\}\,ds\bigg] \tag{3.7}$$

for every $t \in \mathbf{T}$. These representations are analytically intractable. In order to obtain an analytically tractable representation, the notion of an *ordinally equivalent utility* (Duffie and Epstein [4]) is exploited, which is defined as follows. A utility $\bar{U} : \mathcal{C} \to \mathbb{R}$ is said to be *ordinally equivalent* to a utility $U : \mathcal{C} \to \mathbb{R}$ if and only if there exists a strictly increasing $\mathbf{C}^2$-function $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi(0) = 0$ such that $\bar{U} = \varphi \circ U$.

Definition 2. An RCSDU $U : \mathcal{C} \to \mathbb{R}$ is *normalizable* if and only if there exists an ordinally equivalent utility $\bar{U}$ such that for every $C \in \mathcal{C}$, $\bar{U}(C) = \bar{U}_0(C)$ where $\bar{U}(C)$ is a unique solution in $\bar{\mathbf{L}}^p$ of the equation:

$$\bar{U}_t(C) = E_t\left[\int_t^T \bar{f}(X_s(C), \bar{U}_s(C))\,ds\right] \qquad \forall t \in \mathbf{T}. \tag{3.8}$$

The RCSDU $\bar{U}$ in (2) is called the *normalized RCSDU*. It is shown by Duffie and Epstein [4] that any SDU is normalized under pure diffusion information. However, this does not apply to the case of jump-diffusion information. Recently, Kusuda [14] has presented a necessary and sufficient condition for an SDU to be normalized under jump-diffusion information. The following lemma immediately follows from the result.

Lemma 2. *Suppose that for every* $C \in \mathcal{C}$*, an SDU process* $U_t := U_t(C)$ *satisfies the BSDDE*

$$dU_t = -\mu_t^U\,dt + \sigma_t^U \cdot dW_t + U_{t-}\int_{\mathbb{Z}} m_t^U(z)\,\{\,\nu(dt\times dz) - \lambda_t(dz)\,dt\,\} \qquad \forall t \in \mathbf{T} \tag{3.9}$$

with $U_T = 0$ where $\mu^U \in \mathcal{L}^1$, $\sigma^U \in \prod_{j=1}^d \mathcal{L}^2$, $m^U \in \mathcal{L}^1(\lambda_t(dz)\times dt)$ and $m_t^U(z) > -1$ P-a.s. for every $(t,z) \in \mathbf{T}\times\mathbb{Z}$, and satisfy

$$\mu_t^U = f(X_{t-}, U_{t-}) - \frac{1}{2}\,\psi(U_t)\,\|\sigma_t^U\|^2 \\ - \int_{\mathbb{Z}}\int_{U_{t-}}^{(1+m_t^U(z))U_{t-}}\Big(1-\exp\Big(-\int_{U_{t-}}^{u}\psi(u')\,du'\Big)\Big)du\,\lambda_t(dz) \quad (3.10)$$

where $\psi : \mathbb{R}\to\mathbb{R}$ is a continuous function such that $\psi \geq 0$. Define a transform $\varphi : \mathbb{R}\to\mathbb{R}$ by

$$\varphi(u) = \int_0^u \exp\Big(-\int_{u_0}^{u''}\psi(u')\,du'\Big)\,du'' \qquad \forall u\in\mathbb{R} \quad (3.11)$$

where $u_0 \in \mathbb{R}$. Suppose that the function $\bar{f} : \mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ defined by

$$\bar{f}(x,\bar{u}) = \varphi'(\varphi^{-1}(\bar{u}))f(x,\varphi^{-1}(\bar{u})) \qquad \forall(x,\bar{u})\in\mathbb{R}_+\times\mathbb{R}. \quad (3.12)$$

satisfies the following conditions:

- *$\bar{f}$ is uniformly Lipschitz in its utility argument: There exists a constant k such that for every $x\in\mathbb{R}_+$ and every $(\bar{u}_1,\bar{u}_2)\in\mathbb{R}^2$, it follows that $|\bar{f}(x,\bar{u}_1)-\bar{f}(x,\bar{u}_2)| \leq k\,|\bar{u}_1-\bar{u}_2|$.*
- *$\bar{f}$ satisfies a growth condition in its second argument: There exist constants k_0 and k_1 such that for every $x\in\mathbb{R}_+$, it follows that $|\bar{f}(x,0)| \leq k_0 + k_1\|x\|$.*

Then $\bar{U} = \varphi(U)$ is a normalized RCSDU with its aggregator $(\bar{f}, X)$, i.e. $\bar{U}$ is a unique solution in $\bar{\mathbf{L}}^p$ of the recursive equation

$$\bar{U}_t = E_t\left[\int_t^T \bar{f}(X_s,\bar{U}_s)\,ds\right] \qquad \forall t\in\mathbf{T}. \quad (3.13)$$

Proof. See Appendix D.3. □

Specifying the function ψ in the BSDDE (3.10) by $\psi(x) = \zeta$ leads to the BSDDE (3.3). Thus, the RCSDU U is normalized by the transform φ defined by

$$\varphi(u) = \zeta^{-1}(1-e^{-\zeta u}) \qquad \forall u\in\mathbb{R}. \quad (3.14)$$

Then it follows from Lemma 2 that the functional form of normalized aggregator $\bar{f}$ and a sufficient condition for the existence, uniqueness and normalization of BSDDE (3.3) are given as follows.

Assumption 2. *The function $\bar{f}$ defined by*

$$\bar{f}(x,\bar{u}) = -\zeta\bar{u}\,f^\beta(x, -\zeta^{-1}\ln(1-\zeta\bar{u})) \qquad \forall(x,\bar{u})\in\mathbb{R}^2, \quad (3.15)$$

is uniformly Lipschitz in its first (utility) argument , and satisfies the growth condition in its first argument.

Theorem 1. *Under Assumption 1 and 2, $\hat{U}$ is normalized by the ordinally equivalent trasnsform φ defined by (3.14), and the normalized RCSDU $\bar{U}$ is characterized by its aggregator $(\bar{f}, X)$, i.e. $\bar{U}$ is a unique solution in $\bar{\mathbf{L}}^p$ of the recursive equation*

$$\bar{U}_t(C) = E_t\Big[\int_t^T \bar{f}(X_s(C),\bar{U}_s(C))\,ds\Big] \qquad \forall t\in\mathbf{T}. \quad (3.16)$$

4. Aversions for Ambiguity and Risk

In this section, attitudes towards ambiguity and risk of an agent with RCSDU are examined mainly using the notions advocated by Epstein and Wang [8], Epstein [7], and Chen and Epstein [3].

An aggregator (f, X) is said to be *regular* if and only if the following conditions hold:

- f is continuous and satisfies Assumption 2, *i.e.* f is uniformly Lipschitz in its second (utility) argument and satisfies the growth condition in its first argument.
- X is continuous and satisfies the growth condition, and $X(\bar{C})$ is a deterministic process for every deterministic consumption process $\bar{C} \in \mathcal{C}$.

The following lemma is useful to explore comparative aversions for ambiguity and risk.

Lemma 3. *Let $C \in \mathcal{C}$. Let ψ be given in Lemma 2. Suppose that $U^* := U^*(C)$ is a unique solution in $\bar{\mathbf{L}}^p$ of the equation*

$$U^*_t = E_t\bigg[\int_t^T\bigg\{f(X_s, U^*_s) - \frac{1}{2}\psi^*(U^*_s)\,\|\sigma^{U^*}_s\|^2$$

$$-\int_{\mathbb{Z}}\int_{U^*_{t-}}^{(1+m^{U^*}_t(z))U^*_{t-}}\Big(1-\exp\Big(-\int_{U^*_{t-}}^{u}\psi^*(u')\,du'\Big)\Big)du\,\lambda_s(dz)\bigg\}\,ds\bigg] \quad \forall t \in \mathbf{T} \quad (4.1)$$

where (f, X) is regular, and $\psi^* : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\psi^* \geq \psi$. Let φ be defined by (3.11) and $\bar{U}^* := \varphi(U^*)$. Then $\bar{U}^*$ satisfies

$$\bar{U}^*_t = E_t\bigg[\int_t^T\Big\{\bar{f}(X_s, \bar{U}^*_s) - Z_s(U^*, \psi^*, \psi)\Big\}\,ds\bigg]. \quad (4.2)$$

where

$$Z_s(U^*, \psi^*, \psi) = \frac{1}{2}\varphi'(U^*_s)\Big(\psi^*(U^*_s) - \psi(U^*_s)\Big)\,\|\sigma^{\bar{U}^*}_s\|^2$$

$$+\varphi'(U^*_{s-})\int_{\mathbb{Z}}\int_{U^*_{s-}}^{(1+m^{U^*}_s(z))U^*_{s-}} e^{-\int_{U^*_{t-}}^{u}\psi(u')\,du'}\Big(1-e^{-\int_{U^*_{t-}}^{u}(\psi^*(u')-\psi(u'))\,du'}\Big)du\,\lambda_s(dz). \quad (4.3)$$

Furthermore, $Z(U^*, \psi^*, \psi) \geq 0$.

Proof. Equations (4.2) and (4.3) immediately follow from applying *Ito's Formula* (see Appendix B.1) to $\bar{U}^* := \varphi(U^*)$. It is easy to see that $Z(U^*, \psi^*, \psi) \geq 0$. □

4.1. Comparative Ambiguity Aversion.

Chen and Epstein [3] define the notion of *comparative ambiguity aversion* in the following way (for formal arguments, see Epstein [7] and Epstein and Zhang [9]). An event $A \in \mathcal{F}_T$ is said to be *unambiguous* if and only if $\tilde{P}(A) = P(A)$ for every $\tilde{P} \in \mathbb{P}$. Let $\bar{\mathcal{A}}$ denote the class of unambiguous events. Let $\bar{\mathcal{A}}_t := \bar{\mathcal{A}} \cap \mathcal{F}_t$ for every $t \in \mathbf{T}$. A consumption process C is said to be *unambiguous* if and only if C_t is $\bar{\mathcal{A}}_t$-measurable for every $t \in \mathbf{T}$. Let $\mathcal{C}_{\bar{\mathcal{A}}}$ denote the set of all unambiguous consumption processes. The notion of *comparative ambiguity aversion* is defined in the following.

Definition 3. Let $\hat{U}$ and $\hat{U}^*$ be RCSDUs with corresponding classes $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}^*$ of unambiguous events. It is said that $\hat{U}^*$ is more ambiguity averse than $\hat{U}$ if and only if $\bar{\mathcal{A}} \supset \bar{\mathcal{A}}^*$ and for every consumption process $C \in \mathcal{C}$ and every $\bar{\mathcal{A}}^*$-unambiguous consumption process $C^* \in \mathcal{C}_{\bar{\mathcal{A}}^*}$, the following holds:

$$\hat{U}(C) \leq \hat{U}(C^{\bar{\mathcal{A}}^*}) \quad \Longrightarrow \quad \hat{U}^*(C) \leq \hat{U}^*(C^{\bar{\mathcal{A}}^*}).$$

The condition $\bar{\mathcal{A}} \supset \bar{\mathcal{A}}^*$ in the definition of comparative ambiguity aversion, means that the more ambiguous averse agent views more events as ambiguous.

Under Assupmption 1, for every unambiguous consumption process, an RCSDU is reduced to the corresponding SDU.

Lemma 4. *Let $\hat{U}$ be an RCSDU with its aggregator (f, X, β, ζ). If f satisfies Assumption 1, then for every $C^{\bar{\mathcal{A}}} \in \mathcal{C}_{\bar{\mathcal{A}}}$, $\hat{U}(C^{\bar{\mathcal{A}}})$ satisfies*

$$\hat{U}_t(C^{\bar{\mathcal{A}}}) = E_t\left[\int_t^T e^{-\beta(s-t)} f(X_s(C^{\bar{\mathcal{A}}}), \hat{U}_s(C^{\bar{\mathcal{A}}}))\, ds\right] \qquad \forall t \in \mathbf{T}, \tag{4.4}$$

or equivalently

$$\hat{U}_t(C^{\bar{\mathcal{A}}}) = E_t\left[\int_t^T f^\beta(X_s(C^{\bar{\mathcal{A}}}), \hat{U}_s(C^{\bar{\mathcal{A}}}))\, ds\right] \qquad \forall t \in \mathbf{T} \tag{4.5}$$

where $f^\beta(x, u) = f(x, u) - \beta u$.

Proof. See Appendix D.4. □

Suppose that an agent has an RCSDU with its aggregator (f, X, β, ζ). The following proposition shows that the larger the parameter ζ is, the more the agent is ambiguous averse.

Proposition 2. *Let $\hat{U}$ and $\hat{U}^*$ be RCSDUs with corresponding aggregators (f, X, β, ζ) and (f, X, β, ζ^*) in which f satisfies Assumption 1. Suppose that the normalized aggregator $\bar{f}$ is regular. If $\zeta^* \geq \zeta$, then $\hat{U}^*$ is more ambiguity averse than $\hat{U}$.*

Proof. It is obvious that $\bar{\mathcal{A}} = \bar{\mathcal{A}}^*$. Since the result is clear in the case $\zeta^* = \zeta$, assume that $\zeta^* > \zeta$. It follows by Lemma 4 that $\hat{U}_t(C^{\bar{\mathcal{A}}^*}) = \hat{U}^*_t(C^{\bar{\mathcal{A}}^*})$ for every $C^{\bar{\mathcal{A}}} \in \mathcal{C}_{\bar{\mathcal{A}}}$. Let $C \in \mathcal{C}$. Let φ be defined by (3.14), and let $\bar{U} = \varphi(\hat{U})$ and $\bar{U}^* = \varphi(\hat{U}^*)$. It is sufficient to show that $\bar{U}(C) \geq \bar{U}^*(C)$. The transform φ normalizes $\hat{U}$ to $\bar{U}$, and obtains the normalized representation (3.16). Since $\zeta^* > \zeta$, it follows from Lemma 3 that $\bar{U}^* := \bar{U}^*(C)$ satisfies

$$\bar{U}^*_t = E_t\left[\int_t^T \left\{\bar{f}(X_s, \bar{U}^*_s) - Z_s(\hat{U}^*, \zeta^*, \zeta)\right\} ds\right] \qquad t \in \mathbf{T} \tag{4.6}$$

where $Z(\hat{U}^*, \zeta^*, \zeta)$ is given in (4.3). Subtracting (3.16) from (4.6) yields

$$\bar{U}_t - \bar{U}^*_t = E_t\left[\int_t^T \left\{\bar{f}(X_s, \bar{U}_s) - \bar{f}(X_s, \bar{U}^*_s) - Z_s(\hat{U}^*, \zeta^*, \zeta)\right\} ds\right] \quad \forall t \in \mathbf{T}. \tag{4.7}$$

By the Lipschitz condition in utility, the integrand dominates $-k\,|\bar{U}_s - \bar{U}^*_s| + Z_s(\hat{U}^*, \zeta^*, \zeta)$, while $Z_s(\hat{U}^*, \zeta^*, \zeta) \geq 0$. Then it follows from Lemma 12 in Appendix C that $\bar{U}(C) \geq \bar{U}^*(C)$. □

4.2. **Comparative Risk Aversion.** Let $\bar{C}_t := E[C_t]$ for every $t \in \mathbf{T}$. The following notion of *comparative risk aversion* is advocated by Chen and Epstein [3].

Definition 4. Let $\hat{U}$ and $\hat{U}^*$ be RCSDUs with corresponding classes $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}^*$ of unambiguous events. It is said that U^* is more risk averse than U if and only if $\bar{\mathcal{A}} \subset \bar{\mathcal{A}}^*$ and for every unambiguous comsumption process $C^{\bar{\mathcal{A}}} \in \mathcal{C}_{\bar{\mathcal{A}}}$ and every deterministic process $\bar{C}$, the following holds:

$$\hat{U}(C^{\bar{\mathcal{A}}}) \leq \hat{U}(\bar{C}) \quad \Longrightarrow \quad \hat{U}^*(C^{\bar{\mathcal{A}}}) \leq \hat{U}^*(\bar{C}).$$

Proposition 3. *Let $\hat{U}$ and $\hat{U}^*$ be RCSDUs with corresponding aggregators (f, X, β, ζ) and (f^*, X, β, ζ^*) such that (f, X) is regular and satisfy Assumption 1. Let $\check{U}$ and*

$\check{U}^*$ be corresponding ordinally equivalent utilities such that for every $C^{\bar{\mathcal{A}}} \in \mathcal{C}_{\bar{\mathcal{A}}}$ and every $t \in \mathbf{T}$, $\check{U} := \check{U}(C^{\bar{\mathcal{A}}})$ and $\check{U}^* := \check{U}^*(C^{\bar{\mathcal{A}}})$ satisfy

$$\check{U}_t = E_t\bigg[\int_t^T\bigg\{\check{f}(X_s, \check{U}_s) - \frac{1}{2}\psi(\check{U}_s)\,\|\sigma_s^{\check{U}}\|^2 \\ - \int_{\mathbb{Z}}\int_{\check{U}_{t-}}^{(1+m_t^{\check{U}}(z))\check{U}_{t-}}\bigg(1 - \exp\Big(-\int_{\check{U}_{t-}}^{u}\psi(u')\,du'\Big)\bigg)du\,\lambda_s(dz)\bigg\}\,ds\bigg], \quad (4.8)$$

$$\check{U}_t^* = E_t\bigg[\int_t^T\bigg\{\check{f}(X_s, \check{U}_s^*) - \frac{1}{2}\psi^*(\check{U}_s^*)\,\|\sigma_s^{\check{U}^*}\|^2 \\ - \int_{\mathbb{Z}}\int_{\check{U}_{t-}^*}^{(1+m_t^{\check{U}^*}(z))\check{U}_{t-}^*}\bigg(1 - \exp\Big(-\int_{\check{U}_{t-}^*}^{u}\psi^*(u')\,du'\Big)\bigg)du\,\lambda_s(dz)\bigg\}\,ds\bigg], \quad (4.9)$$

respectively. If $\psi^*(x) \geq \psi(x)$ for every $x \in \mathbb{R}$, then $\hat{U}^*$ is more risk averse than $\hat{U}$.

Proof. See Appendix D.5. □

4.3. **Absolute Ambiguity Aversion.** In order to define the absolute ambiguity aversion of a utility, Chen and Epstein [3] introduces the notion of *probabilistically sophisticated utility for timeless prospects* (see Chen and Epstein [3]) as the "ambiguity neutral version" of the ambiguity averse utility. In this paper, the following notion is introduced.

Definition 5. The *ambiguity neutral version of an RCSDU $\hat{U}$ with its aggregator* (f, X, β, ζ) is the SDU whose aggregator is (f, X, β).

It is conjectured that an RCSDU $\hat{U}$ with its aggregator (f, X, β, ζ) is probabilistically sophisticated utility for timeless prospects if and only if $\hat{U}$ is the SDU with (f, X, β), but it is beyond the scope of this paper to explore the conjecture. Let $\tilde{U}$ denote the ambiguity neutral version of RCSDU $\hat{U}$. Suppose that $\hat{U}$ is an RCSDU with its aggregator (f, X, β, ζ), and that the normalized aggregator $(\bar{f}, X)$ is regular. Since $\tilde{U}$ can be interpreted as the RCSDU with its aggregator $(f, X, \beta, 0_+)$, it follows from the proof of Proposition 2 that

$$\hat{U}(C) \leq \tilde{U}(C) \qquad \forall C \in \mathcal{C}. \quad (4.10)$$

The meaning of the following definition is intuitively clear, which states that whenever the ambiguity neutral version of an ambiguous averse RCSDU rejects an ambiguous consumption plan against an unambiguous consumption plan, the RCSDU rejects the ambiguous consumption plan against the unambiguous consumption plan.

Definition 6. An RCSDU $\hat{U}$ is *ambiguity averse* if and only if for every unambiguos consumption process $C^{\bar{\mathcal{A}}} \in \mathcal{C}_{\bar{\mathcal{A}}}$ and every consumption process $C \in \mathcal{C}$, the following holds:

$$\tilde{U}(C) \leq \tilde{U}(C^{\bar{\mathcal{A}}}) \quad \Longrightarrow \quad \hat{U}(C) \leq \hat{U}(C^{\bar{\mathcal{A}}}). \quad (4.11)$$

Proposition 4. *Let $\hat{U}$ be an RCSDU with its aggregator (f, X, β, ζ) such that f satisfy Assumption 1. Suppose that the normalized aggregator $(\bar{f}, X)$ is regular. Then $\hat{U}$ is ambiguity averse.*

Proof. Suppose that $C^{\bar{\mathcal{A}}} \in \mathcal{C}_{\bar{\mathcal{A}}}$ and $C \in \mathcal{C}$ are such that $\tilde{U}(C) \leq \tilde{U}(C^{\bar{\mathcal{A}}})$. Then it follows from (4.10) and $\tilde{U}(C^{\bar{\mathcal{A}}}) = \hat{U}(C^{\bar{\mathcal{A}}})$ that $\hat{U}(C) - \hat{U}(C^{\bar{\mathcal{A}}}) = \hat{U}(C) - \tilde{U}(C) + \tilde{U}(C) - \tilde{U}(C^{\bar{\mathcal{A}}}) + \tilde{U}(C^{\bar{\mathcal{A}}}) - \hat{U}(C^{\bar{\mathcal{A}}}) \leq 0$.

□

4.4. **Absolute Risk Aversion.** A utility $\hat{U}$ is *risk averse* if and only if for every $C^{\bar{\mathcal{A}}} \in \mathcal{C}_{\bar{\mathcal{A}}}$, the following holds.

$$\hat{U}(C^{\bar{\mathcal{A}}}) \leq \hat{U}(\bar{C}^{\bar{\mathcal{A}}}). \tag{4.12}$$

Lemma 5. *Let $\hat{U}$ be an RCSDU with its aggregator (f, X, β, ζ) such that (f, X) is regular and satisfies Assumption 1. Suppose that X is concave and that $f(\,\cdot\,, u)$ is concave for every $u \in \mathbb{R}$, then $\hat{U}$ is risk averse.*

Proof. See Appendix D.6. □

5. Other Properties

In this section, sufficient conditions for the continuity, monotonicity, time consistency, and concavity of RCSDU are presented. Sufficient conditions for these properties of normalized SDU are shown by Duffie and Epstein [4]. In virtue of the normalized representation of RCSDU, the results of Duffie and Epstein [4] are easily extended to the RCSDU.

5.1. **Continutiy and Monotonicity.**

Lemma 6. *Let $\bar{U}$ be a normalized RCSDU with its regular aggregator $(\bar{f}, X)$. Then $\bar{U}$ is continuous.*

Proof. See Appendix D.7. □

Lemma 7. *Let $\bar{U}$ be a normalized RCSDU with its regular aggregator $(\bar{f}, X)$. If X is increasing in consumption and f is increasing in its first argument, then $\bar{U}$ is increasing. If X is strictly increasing in consumption and $\bar{f}$ is strictly increasing in its first argument, then $\bar{U}$ is strictly increasing.*

Proof. Let $C, C' \in \mathcal{C}$ such that $C \geq C'$. For every $t \in \mathbf{T}$, $\bar{U}^C := \bar{U}(C)$ and $\bar{U}^{C'} := \bar{U}(C')$ satisfy

$$\bar{U}_t^C - \bar{U}_t^{C'} = E_t\left[\int_t^T \left\{\bar{f}(X_s^C, \bar{U}_s^C) - \bar{f}(X_s^{C'}, \bar{U}_s^{C'})\right\} ds\right],$$

and since $\bar{f}$ is Lipschitz in utility,

$$\begin{aligned}\bar{f}(X_s^C, \bar{U}_s^C) - \bar{f}(X_s^{C'}, \bar{U}_s^{C'}) &= \bar{f}(X_s^C, \bar{U}_s^C) - \bar{f}(X_s^{C'}, \bar{U}_s^C) + \bar{f}(X_s^{C'}, \bar{U}_s^C) - \bar{f}(X_s^{C'}, \bar{U}_s^{C'} \\ &\geq \bar{f}(X_s^C, \bar{U}_s^C) - \bar{f}(X_s^{C'}, \bar{U}_s^C) + k\,|\bar{U}_s^C - \bar{U}_s^{C'}|.\end{aligned}$$

The result follows by Lemma 12 in Appendix C. □

5.2. **Time Consistency.** Consider a family $\succcurlyeq := \{\succcurlyeq_{\omega,t} \colon (\omega, t) \in \Omega \times \mathbf{T}\}$ of binary orders on $\mathcal{C}$. A binary order $\succcurlyeq$ is said to be *adapted* if and only if $C \succcurlyeq_t C' \in \mathcal{F}_t$ for every (C, C') in $\mathcal{C}^2$ and every $t \in \mathbf{T}$.

Definition 7. An adapted binary family $\succcurlyeq := \{\succcurlyeq_{\omega,t}\}$ of binary orders on $\mathcal{C}$ is said to be *time consistent* if and only if the following conditions hold. For every stopping time τ and every pair C and C' of consumption processes in $\mathcal{C}$ such that the restrictions of C and C' to $[0, \tau]$ conincide, then

$$P(C \succcurlyeq_\tau C') = 1 \quad \Longrightarrow \quad C \succcurlyeq_0 C', \tag{5.1}$$

and

$$P(C \succcurlyeq_\tau C') = 1 \quad \text{and} \quad P(C \succ_\tau C') > 0 \quad \Longrightarrow \quad C \succ_0 C'. \tag{5.2}$$

The definition of RCSDU is momentarily extended in order that there is a terminal reward at a $\mathbf{T}$-valued stopping time τ. The terminal reward is defined by a $\mathcal{F}_\tau$-measurable $Y \in \mathbf{L}^1(P)$. Proprosition A1 in Duffie and Epstein [4] implies that there exists a unique solution for the recursive equation

$$U_t^{C,Y} = E_t\left[\int_t^\tau \bar{f}(X_s^C, U_s^{C,Y})\,ds + Y\right] \qquad \forall t \in \mathbf{T}. \tag{5.3}$$

Lemma 8. *Let τ be a $\mathbf{T}$-valued stopping time τ. Suppose that $Y \geq Y'$ for $\mathcal{F}_\tau$-measurable random variable $Y, Y' \in \mathbf{L}^1(P)$. For every $C \in \mathcal{C}$, let $U^{C,Y}$ and $U^{C,Y'}$ be defined by (5.3). Then $U^{C,Y} > U^{C,Y'}$.*

Proof. See Appendix D.8.

□

It is an immediate consequence of monotonicity for terminal value that the normalized RCSDU is time consistent.

Lemma 9. *Let $\bar{U}$ be a normalized RCSDU with its regular aggregator $(\bar{f}, X)$. Then $\bar{U}$ is time consistent.*

Proof. Let τ, C, and C' as in the definition of consistency. Let U^{C,U_τ^C} and $U^{C,U_\tau^{C'}}$ be defined by (5.3). Then the result follows from Lemma 8. □

5.3. **Concavity.** The concavity is important partly because it is one of sufficient conditions for the existence of general equilibria.

Lemma 10. *Let $\bar{U}$ be an RCSDU with its regular aggregator $(\bar{f}, X)$. If $\bar{f}$ and X are concave, then $\bar{U}$ is concave.*

Proof. See Appendix D.9. □

Appendix A. Marked Point Process

A.1. **Definitions.** A double sequence $(s_n, Z_n)_{n\in\mathbb{N}}$ is considered, where s_n is the occurrence time of an nth jump and Z_n is a random variable taking its values on a measurable space $(\mathbb{Z}, \mathcal{Z})$ at time s_n. Define a random counting measure $\nu(dt \times dz)$ by

$$\nu([0,t] \times A) = \sum_{n\in\mathbb{N}} 1_{\{s_n \leq t,\, Z_n \in A\}} \qquad \forall (t, A) \in \mathbf{T} \times \mathcal{Z}.$$

This counting measure $\nu(dt \times dz)$ is called the $\mathbb{Z}$-*marked point process*.

Let λ be such that

(1) For every $(\omega, t) \in \Omega \times \mathbf{T}$, the set function $\lambda_t(\omega, \cdot)$ is a finite Borel measure on $\mathbb{Z}$.
(2) For every $A \in \mathcal{Z}$, the process $\lambda(A)$ is $\mathcal{P}$-measurable and satisfies $\lambda(A) \in \mathcal{L}^1$.

The marked point process $\nu(dt \times dz)$ is said to have the P-*intensity kernel* $\lambda_t(dz)$ if and only if the equation $E\left[\int_0^T X_s\,\nu(ds \times A)\right] = E\left[\int_0^T X_s \lambda_s(A)\,ds\right]$ holds for every $A \in \mathcal{Z}$ for any nonnegative $\mathcal{P}$-measurable process X, then it is said that the marked point process $\nu(dt \times dz)$ has the P-*intensity kernel* $\lambda_t(dz)$.

A.2. **Integration Theorem.** Let $\nu(dt \times dz)$ be a $\mathbb{Z}$-marked point process with the P-intensity kernel $\lambda_t(dz)$. Let m be a $\mathcal{P} \otimes \mathcal{Z}$-measurable function. It follows that:

(1) If the integrability condition $E\left[\int_0^T\int_{\mathbb{Z}} |m_s(z)|\lambda_s(z)\,ds\right] < \infty$ holds, then the process $\int_0^t \int_{\mathbb{Z}} m_s(z)\{\,\nu(ds \times dz) - \lambda_s(dz)\,ds\,\}$ is a P-martingale.
(2) If $m \in \mathcal{L}^1(\lambda_t(dz) \times dt)$, then the process $\int_0^t \int_{\mathbb{Z}} m_s(z)\{\,\nu(ds \times dz) - \lambda_s(dz)\,ds\,\}$ is a local P-martingale.

Proof. See p. 235 in Brémaud [2]. □

Appendix B. Ito's Formula and Girsanov's Theorem

B.1. **Ito's Formula.** Let $X=(X^1,\cdots,X^d)'$ be a d-dimensional semimartingale, and g be a real-valued $\mathbf{C}^2$ function on $\mathbb{R}^{d+1}$. Then $g(X)$ is a semimartingale of the form

$$g(t,X_t)=g(0,X_0)\,+\int_0^t\frac{\partial}{\partial s}g(s,X_s)\,ds$$

$$\sum_{i=1}^d\int_0^t\frac{\partial}{\partial x_i}g(s-,X_{s-})\,dX_s^i\,+\,\frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d\int_0^t\frac{\partial^2}{\partial x_i\partial x_j}g(s-,X_{s-})\,d\langle X^{ic},X^{jc}\rangle$$

$$+\sum_{0\le s\le t}\left\{g(s-,X_s)-g(s-,X_{s-})-\sum_{i=1}^d\frac{\partial}{\partial x_i}g(s-,X_{s-})\,\Delta X_s^i\right\}$$

where X^{ic} is the continuous part of X^{ic} and $\langle X^{ic},X^{jc}\rangle$ is the quadratic covariation of X^{ic} and X^{jc}.

B.2. **Girsanov's Theorem.**

(1) Let $v\in\prod_{j=1}^d\mathcal{L}^2$ and $m\in\mathcal{L}^1(\lambda_t(dz)\times dt)$ such that $m_t(z)>-1$ P-a.s. for every $(t,z)\in\mathbf{T}\times\mathbb{Z}$. Define a process $\mathit{\Lambda}$ by

$$\frac{d\mathit{\Lambda}_t}{\mathit{\Lambda}_{t-}}=v_t\cdot dW_t+\int_{\mathbb{Z}}m_t(z)\,\{\,\nu(dt\times dz)-\lambda_t(dz)\,dt\}\qquad\forall t\in\mathbf{T}$$

with $\mathit{\Lambda}_0=1$, and suppose $E\,[\mathit{\Lambda}_T]=1$. Then there exists a probability measure $\tilde{P}$ on $(\Omega,\mathcal{F},\mathbb{F})$ given by the Radon-Nikodym derivative

$$d\tilde{P}=\mathit{\Lambda}_T\,dP$$

such that:

(a) The measure $\tilde{P}$ is equivalent to P.

(b) The process given by $\tilde{W}_t=W_t-\int_0^t v_s\,ds$ for every $t\in\mathbf{T}$, is a $\tilde{P}$-Wiener process.

(c) The marked point process $\nu(dt\times dz)$ has the $\tilde{P}$-intensity kernel such that $\tilde{\lambda}_t(dz)=(1+m_t(z))\lambda_t(dz)$ for every $(t,z)\in\mathbf{T}\times\mathbb{Z}$.

(2) Every probability measure equivalent to P has the structure above.

Appendix C. Extensions of Gronwall-Bellman Inequality

Let $(\Omega,\mathcal{F},\mathbb{F},P)$ be a filtered probability whose filtration $\mathbb{F}:=(\mathcal{F}_t)_{t\in\mathbf{T}}$ satisfies the usual conditions. The following lemma is called the *Stochastic Gronwall-Bellman Inequality.*

Lemma 11. *Suppose Y and X are integrable optional processes, and k is a constant. Suppose the map defined by $s\mapsto E_t[Y_s]$ is continuous P-a.s. for every $t\in\mathbf{T}$. If $Y_t\le E_t[\int_t^T(X_s+kY_s)\,ds]+Y_T$ for every $t\in\mathbf{T}$, then*

$$Y_t\le e^{k(T-t)}E_t[Y_T]+E_t[\int_t^T e^{k(s-t)}X_s\,ds]\qquad P\text{-a.s.}\qquad t\in\mathbf{T}.$$

Alternatively, if $Y_t\ge E_t[\int_t^T(X_s+kY_s)\,ds]+Y_T$ for every , then

$$Y_t\ge e^{k(T-t)}E_t[Y_T]+E_t[\int_t^T e^{k(s-t)}X_s\,ds]\qquad P\text{-a.s.}\qquad t\in\mathbf{T}.$$

Proof. See Duffie and Epstein [4]. $\square$

The following lemma was shown by Skiadas in Duffie and Epstein [4].

Lemma 12. *Suppose Y and G are integrable optional processes, and k is a constant. Suppose the map defined by $s \mapsto Y_s$ is right continuous, and the map defined by $s \mapsto E_t[Y_s]$ is continuous P-a.s. for every $t \in \mathbf{T}$. If $Y_T \geq 0$ P-a.s., and for every $t \in \mathbf{T}$, $Y_t = E_t[\int_t^T G_s\, ds + Y_T]$ P-a.s., and $G_t \geq -k|Y_t|$ P-a.s., then $Y_t \geq 0$ P-a.s. for every $t \in \mathbf{T}$.*

Proof. See Duffie and Epstein [4]. □

For every $x \in \mathbb{R}$, let $x^+ := \max\{x, 0\}$. The following lemma was shown by Schroeder and Skiadas [17].

Lemma 13. *Suppose Y and Z are optional processes satisfying $E[\int_0^T Z_t^+\, dt] < \infty$ and $E[\int_0^T Z_t^+ Y_t^+\, dt] < \infty$. Suppose Z is right continuous. If $Y_T \leq 0$ P-a.s., and for every $t \in \mathbf{T}$,*

$$Y_t \leq E\left[\int_t^T Z_s Y_s\, ds + Y_T\right] \qquad P\text{-a.s.},$$

then $Y_t \leq 0$ P-a.s. for every $t \in \mathbf{T}$.

Proof. See the proof for Lemma C3 in Schroeder and Skiadas [17]. □

Appendix D. Proofs

D.1. **Proof of Lemma 1.** Applying integration by parts to (2.3) yields

$$\mathcal{R}_t^{v,m} = E_t^{v,m}\left[\int_t^T e^{-\beta(s-t)} d\ln\frac{\Lambda_s^{v,m}}{\Lambda_t^{v,m}} ds\right] \qquad \forall t \in \mathbf{T}. \tag{D.1}$$

It follows from (2.2) that

$$\begin{aligned} \ln\frac{\Lambda_s^{n,m}}{\Lambda_t^{v,m}} &= \int_t^s \left\{-\frac{1}{2}\|v_{s'}\|^2 - \int_{\mathbb{Z}} m_{s'}(z)\,\lambda_{s'}(dz)\right\} ds' \\ &\quad + \int_t^s v_{s'}\cdot dW_{s'} + \int_t^s \ln(1+m_{s'}(z))\,\nu(ds'\times dz) \qquad \forall s \in [t,T] \end{aligned} \tag{D.2}$$

for every $t \in \mathbf{T}$. It follows from Girsanov's Theorem that a $P^{v,m}$-Wiener process $W^{v,m}$ and the $P^{v,m}$-intensity kernel $\lambda^{v,m}$ of ν satisfies

$$dW_t = dW_t^{v,m} + v_t\, dt \qquad \forall t \in \mathbf{T}, \tag{D.3a}$$

$$\lambda_t(dz) = \frac{\lambda_t^{v,m}(dz)}{1+m_t(z)} \qquad \forall (t,z) \in \mathbf{T}\times\mathbb{Z}, \tag{D.3b}$$

respectively. Substituting (D.3a) and (D.3b) into (D.2) yields

$$\begin{aligned} \ln\frac{\Lambda_s^{v,m}}{\Lambda_t^{v,m}} &= \int_t^s \left\{\frac{1}{2}\|v_{s'}\|^2 + \int_{\mathbb{Z}}\Big(\frac{-m_{s'}(z)}{1+m_{s'}(z)} + \ln(1+m_{s'}(z))\Big)\,\lambda_{s'}^{v,m}(dz)\right\} ds \\ &\quad + \int_t^s v_{s'}\cdot dW_{s'}^{v,m} + \int_t^s\int_{\mathbb{Z}} \ln(1+m_{s'}(z))\{\nu(ds'\times dz) - \lambda_{s'}^{v;m}(dz)\,ds'\}. \end{aligned} \tag{D.4}$$

for every $s \in [t,T]$ and every $t \in \mathbf{T}$. Substituting (D.2) into (D.1) gives (3.1).

D.2. **Proof of Proposition 1.** For every process Y, let $\tilde{Y}$ denote its discounted process, *i.e.* $\tilde{Y}_t := e^{-\beta t} Y_t$ for every $t \in \mathbf{T}$. Substituting (D.3a) and (D.3b) into (3.3) gives the BSDDE

$$\begin{aligned} d\tilde{U}_t = -e^{-\beta t}\Bigg\{ & f(X_t, U_t) - \Big(\frac{\zeta}{2}\|\sigma_t^U\|^2 + \sigma_t^U \cdot v_t\Big) \\ & + \int_{\mathbb{Z}} \Big(\frac{1}{\zeta}\,\frac{\big(1 - e^{-\zeta m_t^U(z) U_{t-}}\big)}{1 + m_t(z)} - m_t^U(z) U_{t-}\Big)\lambda_t^{v,m}(dz)\Bigg\}\, dt \\ & + \tilde{\sigma}_t^U \cdot dW_t^{v,m} + \tilde{U}_{t-}\int_{\mathbb{Z}} m_t^U(z)\,\{\nu(dt \times dz) - \lambda_t^{v,m}(dz)\, dt\} \qquad \forall t \in \mathbf{T} \end{aligned} \tag{D.5}$$

with $\tilde{U}_T = 0$. It follows from (3.2) and Martingale Representation Theorem that there exists a $(\sigma^{v,m}, m_t^{v,m}(z))$ such that $\tilde{U}^{v,m}$ is subject to the BSDDE

$$\begin{aligned} d\tilde{U}_t^{v,m} &= -e^{-\beta t}\left\{ f(X_t, U_t^{v,m}) + \frac{1}{\zeta}\Big(\frac{1}{2}\|v_t\|^2 + \int_{\mathbb{Z}}\Big(\frac{-m_t(z)}{1 + m_t(z)} + \ln(1 + m_t(z))\Big)\lambda_t^{v,m}(dz)\Big)\right\} dt \\ & + \tilde{\sigma}_t^{v,m} \cdot dW_t^{v,m} + \tilde{U}_{t-}^{v,m}\int_{\mathbb{Z}} m_t^{v,m}(z)\,\{\nu(dt \times dz) - \lambda_t^{v,m}(dz)\, dt\} \qquad \forall t \in \mathbf{T} \end{aligned} \tag{D.6}$$

with $\tilde{U}_T^{v,m} = 0$. Combining (D.5) with (D.6) yields

$$\begin{aligned} d(\tilde{U}_t - \tilde{U}_t^{v,m}) = e^{-\beta t}\Bigg[& f(X_t, U_t^{v,m}) - f(X_t, U_t) + \frac{1}{\zeta}\Bigg\{\frac{1}{2}\|v_t + \zeta\sigma_t^U\|^2 \\ + \int_{\mathbb{Z}} \Big(\frac{-m_t(z)}{1 + m_t(z)} & + \ln(1 + m_t(z)) + \frac{e^{-\zeta m_t^U(z) U_{t-}} - 1}{1 + m_t(z)} + \zeta m_t^U(z) U_{t-}\Big)\lambda_t^{v,m}(dz)\Bigg\}\Bigg] dt + dM_t^{v,m} \end{aligned} \tag{D.7}$$

for every $t \in \mathbf{T}$ where

$$M_t^{v,m} = (\tilde{\sigma}_t^U - \tilde{\sigma}_t^{v,m}) \cdot dW_t^{v,m} + \int_{\mathbb{Z}} \big(\tilde{U}_{t-} m_t^U(z) - \tilde{U}_{t-}^{v,m} m_t^{v,m}(z)\big)\,\{\nu(dt \times dz) - \lambda_t^{v,m}(dz)\, dt\}.$$

Equation (3.4) follows from (D.7). It is easy to see that $\mathcal{Q}^{v,m}$ in (3.4) is nonnegative, and attains zero if and only if $(v_t, m_t(z)) = (\hat{v}_t, \hat{m}_t(z)) := (-\zeta\sigma_t^U, e^{-\zeta m_t^U(z) U_{t-}} - 1)$. It is obvious that $U^{\hat{v},\hat{m}} = U$. Finally, it is claimed that $U_t^{v,m} \geq U_t$ $P^{v,m}$-a.s. for every $t \in \mathbf{T}$. If f is concave in its second argument, then the gradient inequality and the fact that $\mathcal{Q}^{v,m}$ is nonnegative, gives

$$U_t^{v,m} - U_t \geq E_t^{v,m}\left[\int_t^T e^{-\beta(s-t)}\big(-f_u(X_s, U_s)\big)(U_s^{v,m} - U_s)\, ds\right]. \tag{D.8}$$

Similarly, if f is convex in its second argument, then the gradient inequality and the fact that $\mathcal{Q}^{v,m}$ is nonnegative, yields

$$U_t^{v,m} - U_t \geq E_t^{v,m}\left[\int_t^T e^{-\beta(s-t)} f_u(X_s, U_s)(U_s^{v,m} - U_s)\, ds\right]. \tag{D.9}$$

In either case, Lemma 13 in Appendix C implies that $U_t^{v,m} \geq U_t$ $P^{v,m}$-a.s. for every $t \in \mathbf{T}$.

D.3. **Proof of Lemma 2.** Applying Ito's Formula to $\bar{U}_t = \varphi(U_t)$ yields

$$\begin{aligned} d\bar{U}_t &= -\mu_t^{\bar{U}}\, dt + \varphi'(U_{t-})\,\sigma_t^{\bar{U}} \cdot dW_t \\ &+ \int_{\mathbb{Z}} \big\{\varphi((1 + m_{t-}^U(z)) U_{t-}) - \varphi(U_{t-})\big\}\,\{\nu(dt \times dz) - \lambda_t(dz)\, dt\} \qquad \forall t \in \mathbf{T} \end{aligned} \tag{D.10}$$

with $\bar{U}_T = \varphi(U_T) = 0$ where

$$\mu_t^{\bar{U}} = \varphi'(U_{t-})\Big\{\mu_{t-}^U + U_{t-}\int_{\mathbb{Z}} m_t^U(z)\,\lambda_t(dz)\Big\} - \frac{1}{2}\,\varphi''(U_{t-})\,\|\sigma_t^U\|^2 - \int_{\mathbb{Z}}\big\{\varphi((1+m_t^U(z))U_{t-}) - \varphi(U_{t-})\big\}\lambda_t(dz). \quad \text{(D.11)}$$

Substituting (3.10) and (3.12), *i.e.* $\varphi'(U_{t-})f(X_{t-},U_{t-}) = \bar{f}(X_{t-},\bar{U}_{t-})$, into (D.11) gives

$$\begin{aligned}
\mu_t^{\bar{U}} =& \varphi'(U_{t-})f(X_{t-},U_{t-}) - \frac{1}{2}\,\varphi'(U_{t-})\,\psi(U_{t-})\,\|\sigma_t^U\|^2 \\
&-\varphi'(U_{t-})\int_{\mathbb{Z}}\int_{U_{t-}}^{(1+m_t^U(z))U_{t-}}\Big(1-\exp\Big(-\int_{U_{t-}}^{u}\psi(u')\,du'\Big)\Big)du\,\lambda_t(dz) \\
&-\frac{1}{2}\,\varphi''(U_{t-})\,\|\sigma_t^U\|^2 - \int_{\mathbb{Z}}\Big\{\varphi((1+m_t^U(z))U_{t-}) - \varphi(U_{t-}) - m_t^U(z)U_{t-}\varphi'(U_t)\Big\}\,\lambda_t(dz) \\
&= f(X_{t-},\bar{U}_{t-}) - \frac{1}{2}\,\{\varphi'(U_{t-})\,\psi(U_{t-}) + \varphi''(U_{t-})\}\,\|\sigma_t^U\|^2 \\
&+\int_{\mathbb{Z}}\Big[\varphi'(U_{t-})\Big\{\int_{U_{t-}}^{(1+m_t^U(z))U_{t-}}\exp\Big(-\int_{U_{t-}}^{u}\psi(u')\,du'\Big)\,du\Big\} - \Big\{\varphi((1+m_t^U(z))U_{t-}) - \varphi(U_{t-})\Big\}\Big]\,\lambda_t(dz) \\
=& f(X_{t-},\bar{U}_{t-}).
\end{aligned} \quad \text{(D.12)}$$

Substituting (D.12) into (D.10) and integrating from t to T yield

$$\bar{U}_t = \int_t^T f(X_s,\bar{U}_s)\,ds - \int_t^T \varphi'(U_s)\,\sigma_s^{\bar{U}}\cdot dW_s - \int_t^T\int_{\mathbb{Z}}\{\varphi((1+m_s^U(z))U_{s-}) - \varphi(U_{s-})\}\,\{\,\nu(ds\times dz) - \lambda_s(dz)\,ds\,\}. \quad \text{(D.13)}$$

The normalized representation (3.16) follows from (D.13).

D.4. **Proof of Lemma 4.** For every process Y, let $\tilde{Y}$ denote its discounted process, *i.e.* $\tilde{Y}_t := e^{-\beta t}Y_t$ for every $t \in \mathbf{T}$. It follows from (4.4) and Martingale Representation Theorem that there exists a $(\sigma', m_t'(z))$ such that $\tilde{U} := \tilde{U}(C^{\bar{\mathcal{A}}})$ satisfies the BSDDE

$$d\tilde{U}_t = -e^{-\beta t}f(X_t,U_t)\,dt + \tilde{\sigma}_t'^U\cdot dW_t + \tilde{U}_{t-}\int_{\mathbb{Z}} m_t'(z)\,\{\,\nu(dt\times dz) - \lambda_t(dz)\,dt\}\quad \forall t\in\mathbf{T} \quad \text{(D.14)}$$

with $\tilde{U}_T = 0$. It follows from expression (3.2) and definition of $C^{\bar{\mathcal{A}}} \in \mathcal{C}_{\bar{\mathcal{A}}}$ that $U^{v,m} := U^{v,m}(C^{\bar{\mathcal{A}}})$ satisfies

$$U_t^{v,m} = E_t\left[\int_t^T e^{-\beta(s-t)}\Big\{f(X_s,U_s^{v,m}) + \frac{1}{\zeta}\left(\frac{1}{2}\|v_s\|^2 + \int_{\mathbb{Z}}\Big(\frac{-m_s(z)}{1+m_s(z)} + \ln(1+m_s(z))\Big)\lambda_s^{v,m}(dz)\right)\Big\}\,ds\right] \quad \text{(D.15)}$$

for every $t \in \mathbf{T}$. It follows from (D.15) and Martingale Representation Theorem that that $U_t^{v,m}$ satisfies the BSDDE

$$d\tilde{U}_t^{v,m} = -e^{-\beta t}\left\{f(X_t,U_t^{v,m}) + \frac{1}{\zeta}\Big(\frac{1}{2}\|v_t\|^2 + \int_{\mathbb{Z}}\Big(\frac{-m_t(z)}{1+m_t(z)} + \ln(1+m_t(z))\Big)\lambda_t(dz)\Big)\right\}dt + \tilde{\sigma}_t\cdot dW_t + \tilde{U}_{t-}^{v,m}\int_{\mathbb{Z}} m_t^{v,m}(z)\,\{\,\nu(dt\times dz) - \lambda_t(dz)\,dt\}\quad \forall t\in\mathbf{T} \quad \text{(D.16)}$$

with $U_T^{v,m} = 0$. Combining (D.14) with (D.16) gives

$$d(\tilde{U}_t - \tilde{U}_t^{v,m}) = e^{-\beta t}\Bigg[f(X_t, U_t^{v,m}) - f(X_t, U_t) + \frac{1}{\zeta}\bigg\{\frac{1}{2}\|v_t\|^2 + \int_{\mathbb{Z}}\bigg(\frac{-m_t(z)}{1+m_t(z)} + \ln(1+m_t(z))\bigg)\lambda_t^{v,m}(dz)\bigg\}\Bigg]dt + dM_t'^{v,m} \quad \text{(D.17)}$$

where

$$M_t'^{v,m} = (\tilde{\sigma}_t'^U - \tilde{\sigma}_t^{v,m})\cdot dW_t^{v,m} + \int_{\mathbb{Z}}\big(\tilde{U}_{t-}m_t'^U(z) - \tilde{U}_{t-}^{v,m}m_t^{v,m}(z)\big)\,\{\nu(dt\times dz) - \lambda_t^{v,m}(dz)\,dt\}.$$

It follows from Assumption 1 that the result is derived in the similar way as in the proof of Proposition 1.

D.5. **Proof of Proposition 3.** It is obvious that $\bar{\mathcal{A}}^* = \bar{\mathcal{A}}$. Since the result is clear in the case $\psi^* = \psi$, assume that $\psi^* > \psi$, *i.e.* $\psi^*(x) \geq \psi(x)$ for every $x \in \mathbf{R}$ and that $\psi^*(x) > \psi(x)$ for some $x \in \mathbf{R}$. It is clear that $\check{U}(\bar{C}) = \check{U}^*(\bar{C})$ for every deterministic process $\bar{C}$. Let $C^{\bar{\mathcal{A}}} \in \mathcal{C}_{\bar{\mathcal{A}}}$. It is sufficient to show that $\check{U}(C^{\bar{\mathcal{A}}}) \geq \check{U}^*(C^{\bar{\mathcal{A}}})$. Let φ be defined by (3.11). Applying φ to $\check{U}$ yields $\hat{U}$, and $\hat{U} := \hat{U}(C^{\bar{\mathcal{A}}})$ satisfies

$$\hat{U}_t = E_t\left[\int_t^T f^\beta(X_s, \hat{U}_s)\,ds\right] \qquad \forall t \in \mathbf{T} \quad \text{(D.18)}$$

where $f^\beta(x,u) = f(x,u) - \beta u$. Since $\psi^*(x) \geq \psi(x)$ for every $x \in \mathbf{R}$ and that $\psi^*(x) > \psi(x)$ for some $x \in \mathbf{R}$. It follows from Lemma 3 that $\dot{U}^* := \varphi(U^*)(C^{\bar{\mathcal{A}}})$ satisfies

$$\dot{U}_t^* = E_t\bigg[\int_t^T \Big\{f^\beta(X_s, \dot{U}_s^*) - Z_s(\check{U}^*, \psi^*, \psi)\Big\}\,ds\bigg] \quad \text{(D.19)}$$

for every $t \in \mathbf{T}$. Subtracting (D.18) from (D.19) yields

$$\hat{U}_t - \dot{U}_t^* = E_t\bigg[\int_t^T \Big\{f^\beta(X_s, \hat{U}_s) - f^\beta(X_s, \dot{U}_s^*) + Z_s(\check{U}^*, \psi^*, \psi)\Big\}\,ds\bigg] \quad \text{(D.20)}$$

for every $t \in \mathbf{T}$. By the Lipschitz condition in utility, the integrand in (D.20) dominates $-k\,|\hat{U}_s - \dot{U}_s^*| + Z_s(\check{U}, \psi^*, \psi)$, while $Z_s(\check{U}^*, \psi^*, \psi) \geq 0$. Then it follows from Lemma 12 in Appendix C that $\hat{U}(C^{\bar{\mathcal{A}}}) \geq \dot{U}(C^{\bar{\mathcal{A}}})$, and therefore $\check{U}(C^{\bar{\mathcal{A}}}) \geq \check{U}^*(C^{\bar{\mathcal{A}}})$.

D.6. **Proof of Lemma 5.** Let $\hat{U}$ be an RCSDU with its aggregator (f, X, β, ζ). Suppose (f, X) is regular, X is concave, and $f(\,\cdot\,, u)$ is concave for every $u \in \mathbb{R}$. Then (f^β, X) is also regular, and $f^\beta(\,\cdot\,, u)$ is concave for every $u \in \mathbb{R}$. Let $C^{\bar{\mathcal{A}}} \in \mathcal{C}_{\bar{\mathcal{A}}}$. Then it follows from Lemma 4 that

$$\hat{U}_t(\bar{C}^{\bar{\mathcal{A}}}) - \hat{U}_t(C^{\bar{\mathcal{A}}}) = E_t\bigg[\int_t^T \Big\{f^\beta(X_s(\bar{C}^{\bar{\mathcal{A}}}), \hat{U}_s(\bar{C}^{\bar{\mathcal{A}}})) - f^\beta(X_s(C^{\bar{\mathcal{A}}}), \hat{U}_s(C^{\bar{\mathcal{A}}}))\Big\}\,ds\bigg]. \quad \text{(D.21)}$$

It follows from the assumption that $X(\bar{C})$ is a deterministic process for every deterministic consumption process $\bar{C} \in \mathcal{C}$, and Fubini's Theorem for conditional expectations to (D.21) that

$$\hat{U}_t(\bar{C}^{\bar{\mathcal{A}}}) - \hat{U}_t(C^{\bar{\mathcal{A}}}) = E_t\bigg[\int_t^T \Big\{E_t\Big[f^\beta(X_s(\bar{C}^{\bar{\mathcal{A}}}), \hat{U}_s(\bar{C}^{\bar{\mathcal{A}}})) - f^\beta(X_s(C^{\bar{\mathcal{A}}}), \hat{U}_s(\bar{C}^{\bar{\mathcal{A}}}))\Big] + f^\beta(X_s(C^{\bar{\mathcal{A}}}), \hat{U}_s(\bar{C}^{\bar{\mathcal{A}}})) - f^\beta(X_s(C^{\bar{\mathcal{A}}}), \hat{U}_s(C^{\bar{\mathcal{A}}}))\Big\}\,ds\bigg].$$

By the Lipschitz condition in utility, the intgrand dominates

$$E_t\Big[f^\beta(X_s(\bar{C}^{\bar{\mathcal{A}}}), \hat{U}_s(\bar{C}^{\bar{\mathcal{A}}})) - f^\beta(X_s(C^{\bar{\mathcal{A}}}), \hat{U}_s(\bar{C}^{\bar{\mathcal{A}}}))\Big] - k\,|\hat{U}(\bar{C}^{\bar{\mathcal{A}}}) - \hat{U}(C^{\bar{\mathcal{A}}})|$$

while it follows from Jensen's Inequality for conditional expectations and concavities of X and f in its first argument that

$$E_t\Big[f^\beta(X_s(\bar{C}^{\bar{\mathcal{A}}}),\bar{U}_s(\bar{C}^{\bar{\mathcal{A}}})) - f^\beta(X_s(C^{\bar{\mathcal{A}}}),\bar{U}_s(\bar{C}^{\bar{\mathcal{A}}}))\Big] \geq 0.$$

Therefore, it follows from Lemma 12 in Appendix C that $\hat{U}(\bar{C}^{\bar{\mathcal{A}}}) \geq \hat{U}(C^{\bar{\mathcal{A}}})$.

D.7. **Proof of Lemma 6.** Let $(C_n)_{n\in\mathbb{N}}$ be a convergence sequence in $\mathcal{C}$ to C. For every $n \in \mathbb{N}$ and $t \in \mathbf{T}$, $\bar{U}^{C_n} := \bar{U}(C_n)$ and $\bar{U}^C := \bar{U}(C)$ satisfy

$$\begin{aligned}
\bar{U}_t^{C_n} - \bar{U}_t^C &\leq E_t\left[\int_t^T |\bar{f}(X_s^{C_n},\bar{U}_s^{C_n}) - \bar{f}(X_s^C,\bar{U}_s^C)|\, ds\right]. \\
&\leq E_t\left[\int_t^T \Big(|\bar{f}(X_s^{C_n},\bar{U}_s^{C_n}) - \bar{f}(X_s^C,\bar{U}_s^{C_n})| + |\bar{f}(X_s^C,\bar{U}_s^{C_n}) - \bar{f}(X_s^C,\bar{U}_s^C)|\Big)\, ds\right]. \\
&\leq E_t\left[\int_t^T \Big(|\bar{f}(X_s^{C_n},\bar{U}_s^{C_n}) - \bar{f}(X_s^C,\bar{U}_s^{C_n})| + k\,|\bar{U}_s^{C_n} - \bar{U}_s^C|\Big)\, ds\right].
\end{aligned}$$

By Lemma 11 in Appendix C, the following holds:

$$|\bar{U}^{C_n} - \bar{U}^C| \leq E_t\left[\int_0^T e^{kt}|\bar{f}(X_s^{C_n},\bar{U}_s^{C_n}) - \bar{f}(X_s^C,\bar{U}_s^{C_n})|\, ds\right].$$

Since C_n converges to C, X and $\bar{f}$ are continuous and satisfy the growth condition, Dominated Convergence Theorem and Hölder's Inequality imply that the integral on the right-hand side converges to zero. Therefore, $\bar{U}$ is continuous.

D.8. **Proof of Lemma 8.** First, suppose $\tau = T$. It follows that

$$\bar{U}_t^{C,Y} - \bar{U}_t^{C,Y'} = E_t\left[\int_t^T \Big\{\bar{f}(X_s^C,\bar{U}_s^{C,Y}) - \bar{f}(X_s^C,\bar{U}_s^{C,Y'})\Big\}\, ds + Y - Y'\right],$$

and

$$\bar{f}(X_s^C,\bar{U}_s^{C,Y}) - \bar{f}(X_s^C,\bar{U}_s^{C,Y'}) \geq -k\,|\bar{U}_s^{C,Y} - \bar{U}_s^{C,Y'}|.$$

The result follows by Lemma 12 in Appendix C. For general τ, $\bar{f}(X_s^C,\bar{U}_s^{C,Y})$ can be replaced with $1_{\{s\leq\tau\}}\bar{f}(X_s^C,\bar{U}_s^{C,Y})$ throughout the above, and the same answer is obtained.

D.9. **Proof of Lemma 10.** Let $C^0, C^1 \in \mathcal{C}$ and for every $\alpha \in [0,1]$, let $C^\alpha := \alpha C^1 + (1-\alpha)C^0$, $\bar{U}^\alpha := \bar{U}(C^\alpha)$, and $X^\alpha = X(C^\alpha)$. Let $t \in \mathbf{T}$ and $Y_t := \bar{U}_t^\alpha - \{\alpha\bar{U}^1 + (1-\alpha)\bar{U}^0\}$. Then

$$\begin{aligned}
Y_t &= E_t\left[\int_t^T \Big\{\bar{f}(X_s^\alpha,\bar{U}_s^\alpha) - \alpha\bar{f}(X_s^1,\bar{U}_s^1) - (1-\alpha)\bar{f}(X_s^0,\bar{U}_s^0)\Big\}\, ds\right] \\
&= E_t\left[\int_t^T \Big\{\bar{f}(X_s^\alpha,\bar{U}_s^\alpha) - \bar{f}(X_s^\alpha,\alpha\bar{U}_s^1 + (1-\alpha)\bar{U}_s^0) + Z_s\Big\}\, ds\right]
\end{aligned}$$

where

$$Z_s = \bar{f}(X_s^\alpha,\alpha\bar{U}_s^1 + (1-\alpha)\bar{U}_s^0) - \alpha\bar{U}^1 - (1-\alpha)\bar{U}^0.$$

Then the integrand in the last expression dominates $Z_s - k|Y_s|$, and by concavities of f and X, $Z_s \geq 0$. The result follows by Lemma 12 in Appendix C.

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