SHIGA UNIVERSITY

CRR WORKING PAPER SERIES　B

Working Paper No. B-4

A Jump-Diffusion LIBOR Market Model and General Equilibrium Pricing of Interest Rate Derivatives

Koji Kusuda

May 2005

Center for Risk Research
Faculty of Economics
SHIGA UNIVERSITY

1-1-1 BANBA, HIKONE,
SHIGA 522-8522, JAPAN

CRR WORKING PAPER SERIES B

Working Paper No. B-4

A Jump-Diffusion LIBOR Market Model and General Equilibrium Pricing of Interest Rate Derivatives

Koji Kusuda

May 2005

Center for Risk Research
Faculty of Economics
SHIGA UNIVERSITY

1-1-1 BANBA, HIKONE,
SHIGA 522-8522, JAPAN

A Jump-Diffusion LIBOR Market Model and General Equilibrium Pricing of Interest Rate Derivatives

by

Koji Kusuda

Faculty of Economics
Shiga University
1-1-1 Banba
Hikone, Shiga 522-8522
kusuda@biwako.shiga-u.ac.jp

May 2005

Keywords and Phrases: Approximately complete markets, Equilibrium pricing, Forward martingale measure, Interest rate derivative, Jump-diffusion model, LIBOR, LIBOR market model.

JEL Classification Numbers: C51, D58, E43, G13.

ABSTRACT. The LIBOR market (LM) model (Brace *et al.* [8], Miltersen *et al.* [27], and Jamshidian [16]) is an interest rate version of the Black-Scholes model of stock price. However, a statistical test (Kusuda [22]) rejected the LM model and suggested that a jump process should be introduced into the LM model. This paper presents a jump-diffusion LM model using a general equilibrium security market model (Kusuda [21] [23] [24]) with jump-diffusion information. Approximate general equilibrium pricing formulas for caplet and swaption are derived. Also, a method of specification and estimation of the jump-diffusion LM model is presented.

This paper is a revision and expansion of my paper (Kusuda [19]) and Chapter 5 in my Ph.D. dissertation (Kusuda [22]) at the Department of Economics, University of Minnesota. I would like to thank my adviser Professor Jan Werner for his invaluable encouragement and advice. I am grateful for comments of participants of presentations at Japanese Economic Association of Financial Econometrics and Engineering Summer 2002 Conference, University of Minnesota, Institute for Advanced Studies, Vienna, Hitotsubashi University, and Nagoya City University.

1. Introduction

In international financial markets, most interest rate related contracts refer to LIBOR (London InterBank Offered Rate[1]) rates, forward LIBOR rates, swap rates (a long term version of LIBOR rates), and forward swap rates. The two most frequently traded interest rate derivatives, *i.e.* a *caplet* and a *swaption*, are a European option on a forward LIBOR rate and on a forward swap rate, respectively. It seems reasonable to suppose that an ideal interest rate derivative pricing model should have the following two properties: (1) Arbitrage-free pricing formulas for caplet and swaption are derived in the model. (2) The model is statistically acceptable, *i.e.* the model can capture the dynamics of interest rates in the real markets. (3) The model has a sound theoretical background. The properties (1) and (2) are necessary to price interest rate derivatives speedily and accurately, respectively. The *LIBOR market model*, developed by Brace, Gatarek, and Musiela [8], Miltersen, Sandmann, Sondermann [27], and Jamshidian [16], can be interpreted as an interest rate version of the celebrated Black-Scholes model (Black and Scholes [7]) of stock price. In the Black-Scholes model, the change in stock price is subject to a lognormal distribution under the risk-neutral measure. In the LIBOR market (LM) model, the change in each forward LIBOR rate (resp. forward swap rate) is subject to a lognormal distribution (resp. an approximate lognormal distribution) under the associated equivalent martingale measure. Thus a Black-Scholes-like pricing formula (resp. approximate pricing formula) for each caplet (resp. swaption) is derived. The LM model therefore can be calibrated using the formulas, and other interest rate derivatives can be speedily priced employing the calibrated model. Also, the bond markets are arbitrage-free in the LM model unlike in the *Black model*[2] (Black [6]). It can be safely stated that the LM model has the property (1) and (3), and this has made the LM model currently the most popular interest rate derivative pricing models among both practitioners and researchers. Extended LM models including the *constant elasticity of volatility* (CEV) *model* (Andersen and Andreasen [2]) and the *affine volatility* (AV) *model* (Zühlsdorff [32]) have been proposed in order to account for the observation that the *implied volatilities* of forward LIBOR rates, which are derived by substituting the market quoted prices of caps into the pricing formula, depend on the strike rates. In both CEV and AV models, an arbitrage-free pricing formula (approximate arbitrage-free pricing formula) for each caplet (resp. swaption) is derived.

Now an interesting question is whether the extended LM models have the property (2) or not, *i.e.* the extended LM models are statistically acceptable or not. Unfortunately, a statistical test conducted in Kusuda [25] rejected the extended LM models and showed that the distribution of the estimated discretized Wiener process, which is supposed to be a normal distribution, has much fatter tail than

[1]The LIBOR rate is the interest rate offered by banks on deposits from other banks in Eurocurrency markets and is frequently a reference rate of interest for loans in international financial markets. In the LIBOR market model, the dynamics of forward LIBOR rates are modeled. A representative real example of forward LIBOR rate is a Eurodollar future rate traded on the Chicago Mercantile Exchange. In the case of Eurodollar futures, the underlying instrument of Eurodollar future contracts is the 90-day LIBOR and future rates with 48 different times to maturity, *i.e.*, one month, two month, $\cdots$, one year, one year and three month, one year and six month, $\cdots$, ten years, are traded.

[2]Practitioners had used the Black model in which the change in each forward LIBOR rate and forward swap rate is subject to a lognormal distribution under the associated equivalent martingale measure. However, if the change in a forward LIBOR rate is subject to a log-normal distribution under the associated equivalent martingale measure, then a forward swap rate is not subject to a lognormal distribution under the associated equivalent martingale measure in arbitrage-free markets. Thus, there exists an arbitrage opportunity in the bond markets assumed in the Black model.

the normal distribution. This result suggests that the deterministic volatility in extended LM models with a stochastic one and/or that a jump process should be introduced into the extended LM models. There is a considerable evidence that the dynamics of interest rate processes are better described by pure diffusion processes than jump-diffusion processes (Balduzzi, Elton, and Green [3], Das [10], Johannes [17] *etc.*). The main purpose of this paper is to present a jump-diffusion LM model which is an extension of the LM model and satisfies the properties (1), (2), and (3).

In the jump-diffusion LM model, it is assumed like in most jump-diffusion option pricing models that the jump magnitude of forward LIBOR rate is a continuously distributed random variable at each jump time. Under this assumption, the markets have uncountably infinite number of information sources, and no finite number of securities complete the markets.[3] In incomplete markets, the standard arbitrage-free pricing method cannot be applied. Glasserman and Kou [12] have proposed another jump-diffusion LM model assuming *approximately complete markets* (Björk, Kabanov, and Runggaldier [4]) in which a continuum of bonds are traded in markets, and any contingent claim can be approximately replicated with an arbitrary precision by an admissible portfolio of the bonds. In their model, the market price of risk is rather arbitrarily specified such that arbitrage-free pricing formulas for caplet and swaption can be derived. Here it must be noted that in general equilibrium (GE, hereafter) model, there is a functional relation among the market price of risk and the dynamics of aggregate consumption and commodity price in equilibrium. It is desired to verify that the option pricing model can be embedded in some convincing GE model, or to put it more precisely that the specification of market price of risk can be consistent to the GE functional relation among the market price of risk and the dynamics of aggregate consumption and commodity price in some convincing GE model. In particular, in the case when option prices depend on the market price of risk, it should be verified.

Recently, the author (Kusuda [20]) introduced the notion of *approximate security market equilibrium* in which each agent is allowed to choose a consumption plan that is approximately financed with any prescribed precision by an budgetary admissible portfolio, and presented the existence and uniqueness of approximate security market equilibria in approximately complete markets. This paper presents a jump-diffusion LM model assuming the GE approximately complete market model. Since the nominal bond price processes can be exogenously given in the GE model, they are specified such that the model is an extension of the LM model and that the common jump magnitude of every forward LIBOR rate is analytically tractable. Then the GE dynamics of forward LIBOR rate is derived under the associated equivalent martingale measure called *forward martingale measure* introduced by Jamshidian [15]. The pricing problem of a caplet (resp. swaption) is reduced to the calculation of conditional expectation of the caplet's (resp. swaption's) payoff under the associated forward martingale measure. It is shown that the change in each forward LIBOR rate (resp. forward swap rate) is subject to an approximate Poisson-lognormal distribution under the associated forward martingale measure, and therefore approximate GE pricing formulas for caplet and swaption are derived. The pricing formulas show that the GE prices of caplet and swaption depend on the market price of jump risk while they do not depend on the market price of diffusive risk. Finally, a method of specification and estimation of the jump-diffusion LM

---

[3]Merton [26] assumed that the market price of risk is zero in order to make the number of sources of the market information finite, and to complete the markets. However, an empirical analysis in Pan [29] showed that the market price of risk cannot be regarded as zero.

model is presented. The method is an extension of the method for the extended LM models, which were proposed by the author (Kusuda [25]).

Other jump-diffusion interest rate models (Ahn and Thompson [1], Das and Foresi [11], Heston [13], Naik and Lee [28] *etc.*) have been presented assuming GE incomplete market models. In each of these models, it is assumed that there are homogeneous agents with a common CRRA utility or that there is a representative agent with a CRRA utility. It is needless to say that the assumption of homogeneous agents is restrictive. In order to justify the assumption on the representative agent, it is required to present the existence of security market equilibria and the CRRA utility of the representative agent in some security market equilibrium. However, it is very difficult to do so in incomplete markets.

The remainder of this paper is organized as follows. Section 2 reviews the GE model. Section 3 provides the specification of jump-diffusion LIBOR market model and derives the GE dynamics of forward LIBOR rate. Section 4 and 5 derive approximate GE pricing formulas for caplet and swaption. Section 6 presents the method of specification and estimation.

2. The General Equilibrium Model of Security Markets with Jump-Diffusion Information

In this section, the GE model of security markets with jump-diffusion information is reviewed following Kusuda [21] [23] [24].

2.1. **Security Market Economy with Jump-Diffusion Information.** A continuous-time frictionless security market economy with time span $[0, T^\dagger]$ (abbreviated by $\mathbf{T}$, hereafter) for a fixed horizon time $T^\dagger > 0$ is considered. The agents' common subjective probability and information structure is modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbf{T}}$ is the natural filtration generated by a d-dimensional Wiener process W and a *marked point process* $\nu(dt \times dz)$ (see Appendix A.1) on a Lusin space $(\mathbb{Z}, \mathcal{Z})$ ($\mathbb{Z} = \mathbb{R}^n$ in the jump-diffusion LM model) with the P-intensity kernel $\lambda_t(dz)$. There is a single perishable consumption commodity. The commodity space is a Banach space $\mathbf{L}^\infty = \mathbf{L}^\infty(\Omega \times \mathbf{T}, \mathcal{P}, \mu)$ where $\mathcal{P}$ is the predictable σ-algebra on $\Omega \times \mathbf{T}$, and μ is the product measure of the probability measure P and the Lebesgue measure on $\mathbf{T}$. There are I agents. Each agent $i \in \{1, 2, \cdots, I\}$ (abbreviated by $\mathbf{I}$, hereafter) is represented by $(U^i, \bar{c}^i)$, where U^i is a strictly increasing and continuous utility on the positive cone $\mathbf{L}^\infty_+$ of the consumption process and $\bar{c}^i \in \mathbf{L}^\infty_+$ is an endowment process, which is assumed to be nonzero. The economy mentioned above is described by a collection: $\mathbf{E} = ((\Omega, \mathcal{F}, \mathbb{F}, P), (U^i, \bar{c}^i)_{i\in\mathbf{I}})$. There are markets for the consumption commodity and securities at every date $t \in \mathbf{T}$. The traded securities are nominal-risk-free security (NOT the risk-free security) called the *money market account* and a continuum of zero-coupon bonds whose maturity times are $(0, T^\dagger]$, each of which pays one unit of cash (NOT one unit of the commodity) at its maturity time. Let p, B, and $(B^T)_{T\in(0,T^\dagger]}$ denote the processes of consumption commodity price, nominal money market account price, and nominal bond price, respectively. The collection $(B, (B^T)_{T\in(0,T^\dagger]})$ of security prices is abbreviated by $\mathbf{B}$, and called the *family of bond prices*. Each agent is allowed to hold a portfolio consisting of the money market account and all bonds at one time.

Definition 1. A *portfolio* is a stochastic process $\vartheta = (\vartheta^0, \vartheta^1(\cdot))$ that satisfies:

(1) The component ϑ^0 is a real-valued $\mathcal{P}$-measurable process.
(2) The component ϑ^1 is such that:
   (a) For every $(\omega, t) \in \Omega \times \mathbf{T}$, the set function $\vartheta^1_t(\omega, \cdot)$ is a signed finite Borel measure on $[t, T^\dagger]$.

(b) For every Borel set A, the process $\vartheta^1(A)$ is $\mathcal{P}$-measurable.

Then the *value process* $\mathcal{V}_t(\vartheta^i_n)$ *of a portfolio* ϑ^i_n is given by

$$\mathcal{V}_t(\vartheta^i_n) = B_t \vartheta^{i0}_{nt} + \int_t^{T^\dagger} B^T_t \vartheta^{i1}_{nt}(dT) \qquad \forall t \in \mathbf{T}.$$

2.2. **Arbitrage-Free Approximately Complete Markets.** Let $n \in \mathbb{N}$. Let $\mathcal{L}^n$ denote the set of real-valued $\mathcal{P}$-measurable process X satisfying the integrability condition $\int_0^{T^\dagger} |X_s|^n \, ds < \infty$ P-a.s. Also let $\mathcal{L}^n(\lambda_t(dz) \times dt)$ denote the set of real-valued $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$-measurable process H satisfying the integrability condition$\int_0^{T^\dagger} \int_{-\infty}^{\infty} |H_s(z)|^n \, \lambda_s(dz) \, ds < \infty$ P-a.s. The notion of *implementable family of bond prices* is introduced.

Definition 2. A bond price family $\mathbf{B}$ is *implementable* if and only if the following conditions hold:

(1) (a) For every $T \in (0, T^\dagger]$, the dynamics of nominal bond price process B^T satisfies the following stochastic differential-difference equation (SDDE)

$$\frac{dB^T_t}{B^T_{t-}} = r^T_t \, dt + v^T_t \cdot dW_t + \int_{\mathbb{Z}} m^T_t(z) \{ \nu(dt \times dz) - \lambda_t(dz) \, dt \} \qquad \forall t \in [0, T)$$

with $B^T_T = 1$ and $B^T_t = 0$ for every $t \in (T, T^\dagger]$ for some $r^T \in \mathcal{L}^1$, $v^T \in \prod_{j=1}^d \mathcal{L}^2$, and $m^T \in \mathcal{L}^1(\lambda_t(dz) \times dt)$. Moreover, it follows that:

(i) For every $(\omega, t) \in \Omega \times \mathbf{T}$, $r^\cdot_t(\omega), v^\cdot_t(\omega) \in \mathbf{C}^1(\mathbf{T})$, and for every $(\omega, t, z) \in \Omega \times \mathbf{T} \times \mathbb{Z}$, $m^\cdot_t(\omega, z) \in \mathbf{C}^1(\mathbf{T})$.

(ii) For every $T \in (0, T^\dagger]$, B^T is regular enough to allow for the differentiation under the integral sign and the interchange of integration order.

(iii) For every $t \in \mathbf{T}$, bond price curves $B^\cdot_t$ are bounded P-a.e.

(iv) The family of jump magnitude functions $m^\cdot_t(\,\cdot\,)$ is uniformly bounded μ-a.e.

(b) The dynamics of nominal money market account price process B satisfies

$$\frac{dB_t}{B_t} = r^B_t \, dt \qquad \forall t \in [0, T^\dagger)$$

with $B_0 = 1$ where r^B_t is given by $r^B_t = -\left.\frac{\partial \ln B^T_t}{\partial T}\right|_{T=t}$, and $r^B \geq 0$ μ-a.e.

(2) (a) There exists a unique real-valued P-martingale $\Lambda^{\mathbf{B}}$ such that

$$\frac{d\Lambda^{\mathbf{B}}_t}{\Lambda^{\mathbf{B}}_{t-}} = -v^{\mathbf{B}}_t \cdot dW_t - \int_{\mathbb{Z}} m^{\mathbf{B}}_t(z) \{ \nu(dt \times dz) - \lambda_t(dz) \, dt \} \qquad \forall t \in [0, T^\dagger) \tag{2.1}$$

with $\Lambda^{\mathbf{B}}_0 = 1$ for some $v^{\mathbf{B}} \in \prod_{j=1}^{d_1} \mathcal{L}^2$ and $m^{\mathbf{B}} \in \mathcal{L}^1(\lambda_t(dz) \times dt)$.

(b) For every $T \in (0, T^\dagger]$, the following holds:

$$r^T_t = r^B_t + v^{\mathbf{B}}_t \cdot v^T_t + \int_{\mathbb{Z}} m^{\mathbf{B}}_t(z) H^T_t(z) \, \lambda_t(dz) \qquad \forall t \in [0, T^\dagger). \tag{2.2}$$

(c) The process $\frac{\Lambda^{\mathbf{B}}}{B}$ is bounded above and bounded away from zero μ-a.e.

The processes $v^{\mathbf{B}}_t$ and $m^{\mathbf{B}}_t(z)\lambda_t(dz)$ are called *market price of (nominal) diffusive risk* and *market price of (nominal) jump risk*, respectively. It has been shown by Björk, Di Masi, Kabanov, and Runggaldier [5] that risk-neutral measures are unique under the condition 1 in Definition 2 if and only if markets are *approximately* *complete* in the sense that for every *contingent claim* there exists a sequence of *admissible self-financing* portfolios converging to the claim (for these definitions, see Appendices C.1, C.2, and C.3). Let $\bar{\mathcal{B}}$ and $\underline{\Theta}(\tilde{\mathbf{B}})$ denote the set of all implementable bond price families and the space of admissible portfolios, respectively.

2.3. **Approximate Security Market Equilibrium.** The notion of *approximate security market equilibrium* is introduced in which each agent is allowed to choose any consumption plan that is approximately financed with an arbitrary precision by a budgetary admissible portfolio.

Definition 3. A collection $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B}) \in \prod_{i\in\mathbf{I}} \mathbf{L}^\infty_+ \times \mathbf{L}^\infty \times \bar{\mathcal{B}}$ constitutes an *approximate security market equilibrium for* $\mathbf{E}$ if and only if it follows that:

(1) For every $i \in \mathbf{I}$, $\hat{c}^i$ solves the problem

$$\max_{c^i \in \bar{\mathcal{C}}^i(p,\mathbf{B})} U^i(c^i)$$

where

$$\bar{\mathcal{C}}^i(p,\mathbf{B}) = \{c^i \in \mathbf{L}^\infty_+ \mid \exists (\vartheta^i_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \underline{\Theta}(\tilde{\mathbf{B}}) \quad \text{s.t.} \quad \vartheta^i_{n0} = 0 \quad \forall n \in \mathbb{N}$$

$$\mathcal{V}_t(\vartheta^i_n) = \int_0^t \vartheta^{i0}_{ns}\, dB_s + \int_0^t \int_s^{T^\dagger} \vartheta^{i1}_{ns}(dT)\, dB^T_s + \int_0^t p_s(\bar{c}^i_s - c^i_s)\, ds \qquad \forall (n,t) \in \mathbb{N} \times \mathbf{T},$$

$$\lim_{n\to\infty} \mathcal{V}_{T^\dagger}(\vartheta^i_n) = 0\}.$$

(2) The commodity market is cleared as $\sum_{i\in\mathbf{I}} \hat{c}^i = \sum_{i\in\mathbf{I}} \bar{c}^i$.

Hereafter, approximate security market equilibrium is abbreviated by ASM equilibrium. The following assumption is a sufficient condition for the existence of ASM equilibria.

Assumption 1. (1) *Every agent has a common CRRA utility U of the form*

$$U(c) = E\left[\int_0^{T^\dagger} u(t, c_t)\, dt\right]$$

where the von Neumann-Morgenstern utility function u is given by

$$u(t,x) = e^{-\rho t} \frac{\beta}{1-\beta}\left(\left(\frac{x}{\beta}\right)^{1-\beta} - 1\right)$$

for some $\rho > 0$ and $\beta > 0$.

(2) *The aggregate endowment is bounded away from zero μ-a.e.*

3. The Jump-Diffusion LIBOR Market Model

In this section, a specification of jump-diffusion LM model is provided, and the GE dynamics of a forward LIBOR rate under the associated *forward martingale measure* is presented. Here the forward martingale measure is defined in the following.

Definition 4. Let $\mathbf{B} \in \bar{\mathcal{B}}$. For every $T \in (0, T^\dagger]$, a probability measure denoted by P^T on $(\Omega, \mathcal{F})$ is a *T-forward martingale measure at* $\mathbf{B}$ if and only if P^T is equivalent to P, and for every $T' \in (0, T^\dagger]$, $\frac{B^{T'}}{B^T}$ is a local P^T-martingale.

Let the common tenor of forward LIBOR rates be denoted by $\delta \in (0,1]$. For every $T \in (0, T^\dagger - \delta]$, the *$T$-forward LIBOR rate process* L^T is defined by

$$L^T_t = \frac{1}{\delta}\left(\frac{B^T_t}{B^{T+\delta}} - 1\right) \qquad \forall t \in [0,T].$$

3.1. **The Jump-Diffusion LIBOR Market Model.** The integer $\lceil \frac{T-t}{\delta} \rceil - 1$ is denoted by K_t^T, hereafter. The jump-diffusion LM model is specified by the set of Assumption 1 and the following two assumptions.

Assumption 2. (1) *The Lusin space $(\mathbb{Z}, \mathcal{Z})$ is d'-dimensional Euclidean space where $d' \in \mathbb{N}$, i.e. $(\mathbb{Z}, \mathcal{Z}) = (\mathbb{R}^{d'}, \mathcal{B}(\mathbb{R}^{d'}))$.*

(2) *The P-intensity kernel $\lambda_t(dz)$ is given by*

$$\lambda_t(dz) = \lambda_t\, f(z)\, dz \tag{3.1}$$

where λ_t is a $\mathcal{P}$-measurable process and f is given by

$$f(z) = \prod_{i=1}^{d'} \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left[-\frac{1}{2} \sum_{i=1}^{d'} \left(\frac{z_i - \mu_i}{\sigma_i} \right)^2 \right]. \tag{3.2}$$

(3) *The dynamics of the aggregate endowment process follows the SDDE*

$$\frac{d\bar{c}_t}{\bar{c}_{t-}} = r_t^{\bar{c}}\, dt + v_t^{\bar{c}} \cdot dW_t + \int_{\mathbb{R}} (e^{m_{\bar{c}} \cdot z} - 1) \{ \nu(dt \times dz) - \lambda_t(dz)\, dt \} \qquad \forall t \in [0, T^\dagger)$$

for some $r^{\bar{c}} \in \mathcal{L}^1$, $v^{\bar{c}} \in \prod_{j=1}^{d_1} \mathcal{L}^2$, and $m_{\bar{c}} \in \mathbb{R}^{d'}$.

Assumption 3. *The family $\mathbf{B}$ of nominal bond price processes satisfies the following conditions:*

(1) *$\mathbf{B} \in \bar{\mathcal{B}}$, and the jump magnitude of the process $\Lambda^{\mathbf{B}}$ satisfies*

$$m_t^{\mathbf{B}}(z) = 1 - e^{m_{\mathbf{B}} \cdot z} \qquad \forall (t, z) \in \mathbf{T} \times \mathbb{R}^{d'}$$

for some constant vector $m_{\mathbf{B}} \in \mathbb{R}^{d'}$.

(2) *There exists a function γ on $\mathbf{T}^2$ such that for every $T \in (0, T^\dagger - \delta]$ and $t \in [0, T)$,*

$$v_t^T \begin{cases} = v_t^{T - K_t^T \delta} - \sum_{k=1}^{K_t^T} \frac{\delta L_{t-}^{T-k\delta}}{1 + \delta L_{t-}^{T-k\delta}} \gamma(t, T - k\delta) & \forall t \in [0, T - \delta) \\ \approx 0 & \forall t \in [T - \delta, T). \end{cases} \tag{3.3}$$

(3) *There exists a constant vector $\eta \in \mathbb{R}^{d'}$ such that $\|\eta\| = 1$, and for every $T \in (0, T^\dagger - \delta]$ and $t \in [0, T)$,*

$$m_t^T(z) \begin{cases} = \frac{1 + m^{T - K_t^T \delta}(z)}{\prod_{k=1}^{K_t^T} \left(1 + \frac{\delta L_{t-}^{T-k\delta}}{1 + \delta L_{t-}^{T-k\delta}} (e^{\eta \cdot z} - 1) \right)} - 1 & \forall (t, z) \in [0, T - \delta) \times \mathbb{R}^{d'} \\ \approx 0 & \forall (t, z) \in [T - \delta, T) \times \mathbb{R}^{d'}. \end{cases} \tag{3.4}$$

Remark 1. As shown later, the GE dynamics of LIBOR rates includes information of the dynamics of the aggregate consumption and the commodity price, and therefore the jump-diffusion LM model can be efficiently and accurately estimated using the data of the aggregate consumption and the commodity price as well as the data of future LIBOR rates. This is a main reason why the Lusin space is assumed to be d'-dimensional Euclidean space. When the jump-diffusion LM model is estimated using only the data of future LIBOR rates, d' should be set one.

Remark 2. Let the function $\gamma(t, T)$ be denoted by γ_t^T, hereafter. Assumption 3.2 is the same as in the LM model, and γ_t^T is the volatility of L_t^T. Also, it is shown in Proposition 1 that $(e^{\eta \cdot z} - 1)$ in (3.4) is the common jump magnitude of every forward LIBOR rate.

3.2. **The GE Dynamics of Forward LIBOR Rates.** As shown in Section 4, the pricing problem of a caplet on L^T is reduced to the calculation of expectation of the caplet's payoff under the $(T+\delta)$-forward martingale measure. Let $T^\delta = T+\delta$, hereafter. The following proposition presents that the bond price family $\mathbf{B}$ is supported as an ASM equilibrium, and the GE dynamics of T-forward LIBOR rate process under P^{T^δ} as well as under P.

Proposition 1. *Under Assumptions 1-3, the following holds:*

(1) *The collection* $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ *is an ASM equilibrium for* $\mathbf{E}$ *where* $p_t = \frac{\Lambda_t^{\mathbf{B}}}{B_t} u_c(t, \bar{c}_t)$ *In addition, if* $\beta \le 1$*, then the ASM equilibrium is unique.*

(2) *The market prices of nominal diffusive risk and nominal jump risk satisfy for every* $t \in [0, T^\dagger)$,

$$v_t^{\mathbf{B}} = \beta\, v_t^{\bar{c}} + v_t^p, \qquad m_t^{\mathbf{B}}(z)\,\lambda_t(dz) = \lambda(1 - e^{-(\beta m_{\bar{c}} + m_p)\cdot z})\, dz, \tag{3.5}$$

in the equilibrium where v^p *and* $(e^{m_p\cdot z} - 1)$ *are the volatility and the jump magnitude of commodity price process, respectively.*

(3) *Let* $T \in (0, T^\dagger - \delta]$. *The dynamics of* T*-forward LIBOR rate process satisfies for every* $t \in [0, T)$,

$$\frac{dL_t^T}{L_{t-}^T} = \Big\{ \gamma_t^T \cdot (\beta\, v_t^{\bar{c}} + v_t^p - v_t^{T^\delta}) - \lambda \int_{\mathbb{R}^{d'}} (e^{\eta\cdot z} - 1) \times (1 + m_t^{T^\delta}(z)) e^{-(\beta m_{\bar{c}} + m_p)\cdot z} f(z)\, dz \Big\}\, dt + \gamma_t^T \cdot dW_t + \int_{\mathbb{R}^{d'}} (e^{\eta\cdot z} - 1)\, \nu(dt\times dz), \tag{3.6}$$

or equivalently

$$\frac{dL_t^T}{L_{t-}^T} = \gamma_t^T \cdot dW_t^{T^\delta} + \int_{\mathbb{R}^{d'}} (e^{\eta\cdot z} - 1)\, \{\, \nu(dt\times dz) - \lambda_t^{T^\delta} f_t^{T^\delta}(z)\, dz\, dt \,\}, \tag{3.7}$$

in the equilibrium where

$$\begin{aligned} W_t^{T^\delta} &= W_t^T + \int_0^t (\beta\, v_s^{\bar{c}} + v_s^p - v_s^{T^\delta})\, ds, \\ \lambda_t^{T^\delta} &= \iota_t^{T^\delta} \lambda, \qquad f_t^{T^\delta}(z) = \frac{1}{\iota_t^{T^\delta}} (1 + m_t^{T^\delta}(z)) e^{-(\beta m_{\bar{c}} + m_p)\cdot z} f(z), \end{aligned} \tag{3.8}$$

where

$$\iota_t^{T^\delta} = \int_{\mathbb{R}^{d'}} (1 + m_t^{T^\delta}(z)) e^{-(\beta m_{\bar{c}} + m_p)\cdot z} f(z)\, dz,$$

and $W_t^{T^\delta}$ *and* $\lambda_t^{T^\delta} f_t^{T^\delta}(z)\, dz$ *are a* P^{T^δ}*-Wiener process and the* P^{T^δ}*-intensity kernel of* $\nu(dt\times dz)$*, respectively.*

Remark 3. As shown in (3.5), in equilibrium, the market price of diffusive risk is a function of the volatilities of the aggregate consumption and the commodity price, and the market price of jump risk is a function of the jump magnitudes of the aggregate consumption and the commodity price. In most conventional option pricing models, the market price of risk is rather arbitrarily specified. It is desired to verify that the option pricing model can be embedded in some convincing GE model, or to put it more precisely that the specification of market price of risk can be consistent to the GE functional relation among the market price of risk and the dynamics of the aggregate consumption and the commodity price in some convincing GE model. In particular, in the case when option prices depend on the market price of risk, it should be verified. As shown in Section 4 and 5, the GE prices of caplet and swaption depend on the market price of jump risk in the jump-diffusion LM model.

Remark 4. As shown in (3.6), the GE dynamics of LIBOR rates includes information of the dynamics of the aggregate consumption and the commodity price. Thus the jump-diffusion LM model can be efficiently and accurately estimated using the data of the aggregate consumption and the commodity price as well as the data of future LIBOR rates.

Proof. For proofs of 1 and 2, see Kusuda [23] [24] and Kusuda [21], respectively. Let $T \in (0, T^\dagger - \delta]$. It follows from (3.3) and (3.4) in Assumption 3 that

$$\gamma_t^T = \frac{1+\delta L_{t-}^T}{\delta L_{t-}^T}(v_t^T - v_t^{T^\delta}), \qquad e^{\eta \cdot z} - 1 = \frac{1+\delta L_{t-}^T}{\delta L_{t-}^T}\left(\frac{1+m_t^T(z)}{1+m_t^{T^\delta}(z)} - 1\right). \tag{3.9}$$

Applying Ito's formula to the definition of L^T yields for every $t \in [0, T)$,

$$\begin{aligned} dL_t^T &= \frac{1+\delta L_{t-}^T}{\delta}\Bigg[\left\{r_t^T - r_t^{T^\delta} - v_t^{T^\delta}\cdot(v_t^T - v_t^{T^\delta}) - \lambda\int_{\mathbb{R}^{d'}}(m_t^T(z) - m_t^{T^\delta}(z))f(z)\,dz\right\}dt \\ &\qquad + (v_t^T - v_t^{T^\delta})\cdot dW_t + \int_{\mathbb{R}^{d'}}\frac{m_t^T(z) - m_t^{T^\delta}(z)}{1+m_t^{T^\delta}(z)}\,\nu(dt\times dz)\Bigg] \\ &= \frac{1+\delta L_{t-}^T}{\delta}\Bigg[\left\{(v_t^{\mathbf{B}} - v_t^{T^\delta})\cdot(v_t^T - v_t^{T^\delta}) - \lambda\int_{\mathbb{R}^{d'}}(1-m_t^{\mathbf{B}}(z))(m_t^T(z) - m_t^{T^\delta}(z))f(z)\,dz\right\}dt \\ &\qquad + (v_t^T - v_t^{T^\delta})\cdot dW_t + \int_{\mathbb{R}^{d'}}\frac{m_t^T(z) - m_t^{T^\delta}(z)}{1+m_t^{T^\delta}(z)}\,\nu(dt\times dz)\Bigg]. \end{aligned} \tag{3.10}$$

Substituting (3.5) and (3.9) into (3.10) yields (3.6). Next, it follows from Ito's formula and Girsanov's Theorem that $W_t^{T^\delta}$ and $\lambda_t^{T^\delta} f_t^{T^\delta}(z)\,dz$ are a P^{T^δ}-Wiener process and the P^{T^δ}-intensity kernel of the marked point process $\nu(dt \times dz)$, respectively. Finally, (3.7) is obtained substituting (3.8) into (3.6).

□

4. Approximate GE Pricing Formula for Caplet

In this section, an approximate GE pricing formula for a caplet in the jump-diffusion LM model is derived exploiting the forward martingale measure approach developed by Jamshidian [15]. Here a caplet is a European call option on a forward LIBOR rate, and is defined in the following.

Let $T \in (0, T^\dagger - \delta]$ and $K > 0$. A *caplet on T-forward LIBOR rate L^T with strike rate K* is a contingent claim with payoff $\delta\,|L_T^T - K|$ at time T^δ.

Let $\mathrm{Cpl}_t(L^T, K)$ denote the GE price of the caplet on T-forward LIBOR rate L^T with strike rate K at time t in an ASM equilibrium $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ for $\mathbf{E}$ in the jump-diffusion LM model. Since the security markets are approximately complete in the jump-diffusion LM model, there may not exist a replicable claim for the T^δ-contingent claim $\delta\,|L_T^T - K|$, but there exists a sequence of replicable claims converging to the T^δ-contingent claim. Let $(\vartheta_n)_{n\in\mathbb{N}}$ denote the corresponding sequence of replicating portfolios. Since the value process of every replicating portfolio discounted by B^{T^δ} is a P^{T^δ}-martingale, the following holds:

$$\frac{\mathcal{V}_t(\vartheta_n)}{B_t^{T^\delta}} = E_t^{T^\delta}\left[\frac{\mathcal{V}_{T^\delta}(\vartheta_n)}{B_{T^\delta}^{T^\delta}}\right] = E_t^{T^\delta}[\mathcal{V}_{T^\delta}(\vartheta_n)] \tag{4.1}$$

where $E_t^{T^\delta}[\,\cdot\,] = E^{T^\delta}[\,\cdot\,|\,\mathcal{F}_t\,]$ and $E^{T^\delta}[\,\cdot\,]$ is the expectation operator under P^{T^δ}. Taking the limit of the both sides of (4.1) yields

$$\mathrm{Cpl}_t(L^T, K) = \delta B_t^{T^\delta} E_t^{T^\delta}[\,|L_T^T - K|\,]. \tag{4.2}$$

4.1. **Approximation of Conditional Distribution of Forward LIBOR Rate.** In order too calculate $E_t^{T^\delta}[\,|L_T^T - K|\,]$, it is desired that the conditional distribution of L_T^T, given $\mathcal{F}_t$, under P^{T^δ}, is derived in analytic form. It follows from (3.7) that $L_T^T|\mathcal{F}_t$ is solved in the form

$$L_T^T = L_t^T \exp\Bigg[-\int_t^T \left\{ \frac{1}{2}\|\gamma_s^T\|^2 + \lambda_s^{T^\delta} \int_{\mathbb{R}^{d'}} (e^{\eta\cdot z} - 1)\, f_s^{T^\delta}(z)\, dz \right\} ds + \int_t^T \gamma_s^T \cdot dW_s^{T^\delta} + \int_t^T \int_{\mathbb{R}^{d'}} \eta\cdot z\, \nu(ds\times dz) \Bigg]. \quad (4.3)$$

As shown in (4.3), the conditional distribution of $L_T^T|\mathcal{F}_t$ under P^{T^δ} cannot be derived in analytic form. Hence, the conditional distribution is approximated in order that the approximate conditional distribution is derived in analytic form. Let $\tilde{K}_t^T = \lceil \frac{K_t^T}{2} \rceil$ and $t \le s \le T$. First, $m_s^{T^\delta}(z)$ is approximated as follows:

$$m_s^{T^\delta}(z) = \frac{1 + m_s^{T^\delta - K_s^{T^\delta}\delta}(z)}{\prod_{k=1}^{K_s^{T^\delta}} \left(1 + \frac{\delta L_{s-}^{T^\delta - k\delta}}{1+\delta L_{s-}^{T^\delta - k\delta}} (e^{\eta\cdot z} - 1) \right)} - 1$$

$$\approx \exp\left[-\sum_{k=1}^{\tilde{K}_t^{T^\delta}} \ln\left(1 + \frac{\delta L_{t-}^{T^\delta - k\delta}}{1 + \delta L_{t-}^{T^\delta - k\delta}} (e^{\eta\cdot z} - 1) \right) \right] - 1 \approx e^{-\delta \sum_{k=1}^{\tilde{K}_t^{T^\delta}} L_{t-}^{T^\delta - k\delta} \eta\cdot z} - 1.$$

Let $\tilde{m}_t^{T^\delta}(z) = e^{-\delta \sum_{k=1}^{\tilde{K}_t^{T^\delta}} L_{t-}^{T^\delta - k\delta} \eta\cdot z} - 1$. Then $\lambda_s^{T^\delta}$ and $f_s^{T^\delta}(z)$ are approximated to

$$\tilde{\lambda}_t^{T^\delta} = \tilde{\iota}_t^{T^\delta} \lambda, \qquad \tilde{f}_t^{T^\delta}(z) = \frac{1}{\tilde{\iota}_t^{T^\delta}} (1 + \tilde{m}_t^{T^\delta}(z)) e^{-(\beta m_{\bar{c}} + m_p)\cdot z} f(z), \quad (4.4)$$

respectively, where

$$\tilde{\iota}_t^{T^\delta} = \int_{\mathbb{R}^{d'}} (1 + \tilde{m}_t^{T^\delta}(z)) e^{-(\beta m_{\bar{c}} + m_p)\cdot z} f(z)\, dz.$$

Substituting $\tilde{m}_t^{T^\delta}(z) = e^{-\delta \sum_{k=1}^{\tilde{K}_t^{T^\delta}} L_{t-}^{T^\delta - k\delta} \eta\cdot z} - 1$ into (4.4) yields

$$\tilde{\lambda}_t^{T^\delta} = e^{\frac{1}{2}\sum_{j=1}^{d'} \tilde{m}_{tj}^{T^\delta} \left(2\mu_j + \tilde{m}_{tj}^{T^\delta}\sigma_j^2\right)} \lambda, \qquad \tilde{f}_t^{T^\delta}(z) = \prod_{i=1}^{d'} \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left[-\frac{1}{2}\sum_{i=1}^{d'} \left(\frac{z_i - \tilde{\mu}_{ti}^{T^\delta}}{\sigma_i} \right)^2 \right], \quad (4.5)$$

where

$$\tilde{m}_{tj}^{T^\delta} = \delta \sum_{k=1}^{\tilde{K}_t^{T^\delta}} L_{t-}^{T^\delta - k\delta} \eta_j + \beta m_{\bar{c}j} + m_{pj}, \qquad \tilde{\mu}_{ti}^{T^\delta} = \mu_i - \tilde{m}_{ti}^{T^\delta} \sigma_i^2. \quad (4.6)$$

Using the approximation (4.5), define an approximate GE caplet price $\widetilde{\mathrm{Cpl}}_t(L^T, K)$ by

$$\widetilde{\mathrm{Cpl}}_t(L^T, K) = \delta B_t^{T^\delta} E_t^{T^\delta}[\,|\tilde{L}_T^T - K|\,] \quad (4.7)$$

where $\tilde{L}_T^T$ is an approximate T-forward LIBOR rate at time T given by

$$\tilde{L}_T^T = L_t^T \exp\Bigg[-\int_t^T \left\{ \frac{1}{2}\|\gamma_s^T\|^2 + \tilde{\lambda}_t^{T^\delta} \int_{\mathbb{R}^{d'}} (e^{\eta\cdot z} - 1)\, \tilde{f}_t^{T^\delta}(z)\, dz \right\} ds + \int_t^T \gamma_s^T \cdot dW_s^{T^\delta} + \int_t^T \int_{\mathbb{R}^{d'}} \eta\cdot z\, \tilde{\nu}_t^{T^\delta}(ds\times dz) \Bigg] \quad (4.8)$$

where $\tilde{\nu}_t^{T^\delta}(ds\times dz)$ is the marked point process with P^{T^δ}-intensity kernel $\tilde{\lambda}_t^{T^\delta}\, \tilde{f}_t^{T^\delta}(z)\,dz$.

Remark 5. The approximation $\tilde{m}_t^{T^\delta}$ for $m_s^{T^\delta}$ looks quite rough at a glance. However, it seems reasonable to suppose that $m_s^{T^\delta}$ is fairly close to zero under forecast values of parameters. Therefore, $\tilde{m}_t^{T^\delta}$ can be regarded as a fairly good approximation for $m_s^{T^\delta}$ from the viewpoint of deriving an approximate price of $\mathrm{Cpl}_t(L^T, K)$.

4.2. **The Approximate GE Pricing Formula for Caplet.** The following proposition is obtained.

Proposition 2. *Let $T \in (0, T^\dagger - \delta]$. Under Assumptions 1-3, the approximate GE caplet price $\widetilde{\mathrm{Cpl}}_t(L^T, K)$ is*

$$\widetilde{\mathrm{Cpl}}_t(L^T, K) = \delta B_t^{T^\delta} \sum_{n=0}^{\infty} e^{-\tilde{\lambda}_t^{T^\delta}(T-t)} \frac{\{\tilde{\lambda}_t^{T^\delta}(T-t)\}^n}{n!} \times \left\{ L_t^T e^{\zeta_n} \Phi\left(\frac{\ln \frac{L_t^T}{K} + \zeta_n + \frac{1}{2}\tilde{\sigma}_n^2}{\tilde{\sigma}_n} \right) - K\,\Phi\left(\frac{\ln \frac{L_t^T}{K} + \zeta_n - \frac{1}{2}\tilde{\sigma}_n^2}{\tilde{\sigma}_n} \right) \right\} \tag{4.9}$$

where

$$\begin{aligned} \zeta_n &= \left(1 - e^{-\frac{1}{2}\sum_{i=1}^{d'} \eta_i (2\tilde{\mu}_{ti}^{T^\delta} - \eta_i \sigma_i^2)}\right) \tilde{\lambda}_t^{T^\delta}(T-t) + \sum_{i=1}^{d'} \eta_i (\tilde{\mu}_{ti}^{T^\delta} + \eta_i \sigma_i^2)\, n, \\ \tilde{\sigma}_n &= \sqrt{\int_t^T \|\gamma_s^T\|^2\,ds + \sum_{i=1}^{d'} \eta_i^2 \sigma_i^2\, n}. \end{aligned} \tag{4.10}$$

In particular, if $d' = 1$, then (4.10) is rewritten as

$$\begin{aligned} \zeta_n &= \left(1 - e^{-\frac{1}{2}(2\tilde{\mu}_{t1}^{T^\delta} - \sigma_1^2)}\right) \tilde{\lambda}_t^{T^\delta}(T-t) + (\tilde{\mu}_{t1}^{T^\delta} + \sigma_1^2)\, n, \\ \tilde{\sigma}_n &= \sqrt{\int_t^T \|\gamma_s^T\|^2\,ds + \sigma_1^2\, n}. \end{aligned} \tag{4.11}$$

Remark 6. As shown in (4.6), the term $\tilde{\mu}_{ti}^{T^\delta}$ depends on the market price of jump risk. Therefore, the GE caplet price depends on the market price of jump risk while it does not depend on the market price of diffusive risk.

Proof. Let $Y = \ln \frac{\tilde{L}_T^T}{L_t^T}$. It follows from (4.5) and (4.8) that

$$\begin{aligned} Y = -\frac{1}{2}\int_t^T \|\gamma_s^T\|^2\,ds + \left(1 - e^{-\frac{1}{2}\sum_{i=1}^{d'} \eta_i (2\tilde{\mu}_{ti}^{T^\delta} - \eta_i \sigma_i^2)}\right) \tilde{\lambda}_t^{T^\delta}(T-t) \\ + \int_t^T \gamma_s^T \cdot dW_s^{T^\delta} + \int_t^T \int_{\mathbb{R}^{d'}} \eta \cdot z\, \tilde{\nu}_t^{T^\delta}(ds \times dz). \end{aligned}$$

Let $n \in \{0, 1, \cdots\}$ and write $E_{t,n}^{T^\delta}[\,\cdot\,] = E^{T^\delta}[\,\cdot\,|\,\mathcal{F}_t, N_T = N_t + n\,]$. Let $\tilde{\mu}_n = E_{t,n}^{T^\delta}[Y]$ and $\tilde{\sigma}_n = \sqrt{E_{t,n}^{T^\delta}[\,(Y - \tilde{\mu}_n)^2]}$. It is straightforward to see that

$$\tilde{\mu}_n = -\frac{1}{2}\int_t^T \|\gamma_s^T\|^2\,ds + \left(1 - e^{-\frac{1}{2}\sum_{i=1}^{d'} \eta_i (2\tilde{\mu}_{ti}^{T^\delta} - \eta_i \sigma_i^2)}\right) \tilde{\lambda}_t^{T^\delta}(T-t) + \sum_{i=1}^{d'} \eta_i \tilde{\mu}_{ti}^{T^\delta}\, n,$$

and $\tilde{\sigma}_n$ satisfies (4.10). Note that $\zeta_n = \tilde{\mu}_n + \frac{1}{2}\tilde{\sigma}_n$. Let $Z = \frac{Y - \tilde{\mu}_n}{\tilde{\sigma}_n}$ and $z_0 = \frac{\ln \frac{K}{L_t^T} - \tilde{\mu}_n}{\tilde{\sigma}_n}$. Then one obtains

$$\begin{aligned} E_{t,n}^{T^\delta}[\,|\tilde{L}_T^T - K|\,] &= L_t^T E_{t,n}^{T^\delta}[\,e^Y 1_{\{Y \geq \ln \frac{K}{L_t^T}\}}\,] - K\, E_{t,n}^{T^\delta}[\,1_{\{Y \geq \ln \frac{K}{L_t^T}\}}\,] \\ &= L_t^T E_{t,n}^{T^\delta}[\,e^{\tilde{\sigma}_n Z + \tilde{\mu}_n} 1_{\{Z \geq z_0\}}\,] - K\, E_{t,n}^{T^\delta}[\,1_{\{Z \geq z_0\}}\,] \\ &= L_t^T \int_{z_0}^{\infty} e^{\tilde{\sigma}_n z + \tilde{\mu}_n} \phi(z)\, dz - K\, P_{t,n}^{T^\delta}(Z \geq z_0) \\ &= L_t^T e^{\tilde{\mu}_n + \frac{1}{2}\tilde{\sigma}_n^2} \Phi(\tilde{\sigma}_n - z_0) - K\, \Phi(-z_0) \\ &= L_t^T e^{\zeta_n} \Phi\left(\frac{\ln \frac{L_t^T}{K} + \zeta_n + \frac{1}{2}\tilde{\sigma}_n^2}{\tilde{\sigma}_n}\right) - K\, \Phi\left(\frac{\ln \frac{L_t^T}{K} + \zeta_n - \frac{1}{2}\tilde{\sigma}_n^2}{\tilde{\sigma}_n}\right). \end{aligned}$$

□

5. Approximate GE Pricing Formula for Swaption

In this section, an approximate GE pricing formula for a swaption in the jump-diffusion LM model is derived. Here a swaption is a European option on a forward swap rate. Thus an approximate GE pricing formula for a swaption is derived in the similar way as done in Sections 3 and 4. First, the GE dynamics of forward swap rate process under the associated forward martingale measure is derived. Next, the conditional distribution of forward swap rate under the associated forward martingale measure is approximated. Finally, an approximate GE pricing formula for swaption is derived based on the approximate conditional distribution.

Let $N \in \mathbb{N}$ and $T \in (0, T^\dagger - N\delta]$. An N-period *T-forward swap rate process* $L^{T,N}$ is defined by

$$L_t^{T,N} = \frac{1}{\delta}\left(\frac{B_t^{T,N}}{B_t^{T^\delta,N}} - 1\right) \qquad \forall t \in [0, T^\delta) \tag{5.1}$$

where $B_t^{T,N} = \sum_{j=1}^N B_t^{T+(j-1)\delta}$. The N-period T-forward swap rate process is called (T,N)-forward swap rate, hereafter. A *payer swaption on (T,N)-forward swap rate $L^{T,N}$ with strike rate $K > 0$* is a contingent claim with fixed payoffs $\delta|L_T^{T,N} - K|$ at time $T+\delta, T+2\delta, ..., T+N\delta$.

In order to derive the GE price of the swaption on (T,N)-forward swap rate with strike rate K, a forward martingale measure called *(T^δ,N)-forward martingale measure* is exploited.

Definition 5. Let $\mathbf{B} \in \bar{\mathcal{B}}$. For every $N \in \mathbb{N}$ and $T \in (0, T^\dagger - N\delta]$, a probability measure denoted by $P^{T,N}$ on $(\Omega, \mathcal{F})$ is a *(T,N)-forward martingale measure at* $\mathbf{B}$ if and only if $P^{T,N}$ is equivalent to P, and $\frac{\mathbf{B}}{B^{T,N}}$ is a local $P^{T,N}$-martingale.

Let $\mathrm{PS}_t(L^{T,N}, K)$ denote the GE price of the (T,N)-swaption with strike rate K at time t in an ASM equilibrium $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ for $\mathbf{E}$ in the jump-diffusion LM Model. The GE price $\mathrm{PS}_t(L^{T,N}, K)$ satisfies

$$\mathrm{PS}_t(L^{T,N}, K) = \delta B_t^{T^\delta,N} E_t^{T^\delta,N}[\,|L_T^{T,N} - K|\,] \tag{5.2}$$

where $E^{T^\delta,N}$ is the expectation operator under the (T,N)-forward martingale measure $P^{T^\delta,N}$.

5.1. **The GE Dynamics of Forward Swap Rate.** First, it is easy to show that the dynamics of $B^{T,N}$ satisfies the following SDDE:

$$\frac{dB_t^{T,N}}{B_{t-}^{T,N}} = r^{T,N}\,dt + v_t^{T,N}\cdot dW_t + \int_{\mathbb{R}^{d'}} m_t^{T,N}(z)\,\{\nu(dt\times dz) - \lambda\, f(z)\,dz\,dt\,\} \quad (5.3)$$

where

$$r^{T,N} = r_t^B + v_t^{\mathbf{B}}\cdot v_t^{T,N} + \lambda\int_{\mathbb{R}^{d'}} m_t^{\mathbf{B}}(z)m_t^{T,N}(z)f(z)\,dz,$$

$$v_t^{T,N} = \sum_{j=1}^N \frac{B_t^{T+(j-1)\delta}}{B_t^{T,N}} v_t^{T+(j-1)\delta}, \qquad m_t^{T,N}(z) = \sum_{j=1}^N \frac{B_t^{T+(j-1)\delta}}{B_t^{T,N}} m_t^{T+(j-1)\delta}(z). \quad (5.4)$$

Then the following proposition is obtained in the same way as done in Proposition 1.

Proposition 3. *Let $N\in\mathbb{N}$ and $T\in(0,T^\dagger-(N+1)\delta]$. Under Assumptions 1-3, the GE dynamics of (T,N)-forward swap rate process satisfies for every $t\in[0,T)$,*

$$\frac{dL_t^{T,N}}{L_{t-}^{T,N}} = \gamma_t^{T,N}\cdot dW_t^{T^\delta,N} + \int_{\mathbb{R}^{d'}} \eta^{T,N}(z)\,\{\nu(dt\times dz) - \lambda_t^{T^\delta,N}\, f^{T^\delta,N}(z)\,dz\,dt\,\}. \quad (5.5)$$

where

$$\gamma_t^{T,N} = \frac{1+\delta L_{t-}^{T,N}}{\delta L_{t-}^{T,N}}(v_t^{T,N} - v_t^{T^\delta,N}), \qquad \eta_t^{T,N}(z) = \frac{1+\delta L_{t-}^{T,N}}{\delta L_{t-}^{T,N}}\left(\frac{1+m_t^{T,N}(z)}{1+m_t^{T^\delta,N}(z)} - 1\right),$$

$$W_t^{T^\delta,N} = W_t^T + \int_0^t (\beta\, v_s^{\bar{c}} + v_s^p - v_s^{T^\delta,N})\,ds,$$

$$\lambda_t^{T^\delta,N} = \iota_t^{T^\delta,N}\lambda, \qquad f_t^{T^\delta,N}(z) = \frac{1}{\iota_t^{T^\delta,N}}(1+m_t^{T^\delta,N}(z))e^{-(\beta m_{\bar{c}}+m_p)\cdot z}f(z)$$

where

$$\iota_t^{T^\delta,N} = \int_{\mathbb{R}^{d'}} (1+m_t^{T^\delta,N}(z))e^{-(\beta m_{\bar{c}}+m_p)\cdot z}f(z)\,dz, \quad (5.6)$$

and $W_t^{T^\delta}$ and $\lambda_t^{T^\delta}\, f_t^{T^\delta}(z)\,dz$ are a $P^{T^\delta,N}$-Wiener process and the $P^{T^\delta,N}$-intensity kernel of $\nu(dt\times dz)$, respectively.

5.2. **Approximation of Conditional Distribution of Forward Swap Rate.** It follows from (5.5) that $L_T^{T,N}$ is solved in the form

$$L_T^{T,N} = L_t^{T,N}\exp\Bigg[-\int_t^T\left\{\frac{1}{2}\|\gamma_s^{T,N}\|^2 + \lambda_s^{T^\delta,N}\int_{\mathbb{R}^{d'}}\eta_s^{T,N}(z)f_s^{T^\delta,N}(z)\,dz\right\}ds + \int_t^T \gamma_s^{T,N}\cdot dW_s^{T^\delta,N} + \int_t^T\int_{\mathbb{R}^{d'}} \ln\eta_s^{T,N}(z)\,\nu(ds\times dz)\Bigg].$$

As conducted in the previous section, the conditional distribution of $L_T^{T,N}|\mathcal{F}_t$ under $P^{T^\delta,N}$ is approximated in order that the approximate conditional distribution is derived in analytic form. Let $t\le s\le T$. First, $m_s^{T^\delta,N}(z)$ is approximated as follows:

$$m_s^{T^\delta,N}(z) = \sum_{j=1}^N \frac{B_s^{T^\delta+(j-1)\delta}}{B_s^{T^\delta,N}} m_s^{T^\delta+(j-1)\delta}(z) \approx \sum_{j=1}^N \frac{B_s^{T^\delta+(j-1)\delta}}{B_s^{T^\delta,N}} \tilde{m}_t^{T^\delta}(z) = \tilde{m}_t^{T^\delta}(z).$$

Thus $\lambda_s^{T^\delta,N}$ and $f_s^{T^\delta,N}$ are approximated to $\tilde{\lambda}_t^{T^\delta}$ and $\tilde{f}_t^{T^\delta}$, respectively. Here $\tilde{\lambda}_t^{T^\delta}$ and $\tilde{f}_t^{T^\delta}$ are given in (4.5). Next, $\gamma_s^{T,N}$ is approximated in the following:

$$
\begin{aligned}
\gamma_s^{T,N} &= \frac{1+\delta L_{s-}^{T,N}}{\delta L_{s-}^{T,N}} \sum_{j=1}^{N} \left(\frac{B_s^{T+(j-1)\delta}}{B_s^{T,N}} v_s^{T+(j-1)\delta} - \frac{B_s^{T^\delta+(j-1)\delta}}{B_s^{T^\delta,N}} v_s^{T+j\delta} \right) \\
&\approx \frac{1+\delta L_{s-}^{T,N}}{\delta L_{s-}^{T,N}} \sum_{j=1}^{N} \frac{B_s^{T+(j-1)\delta}}{B_s^{T,N}} (v_s^{T+(j-1)\delta} - v_s^{T+j\delta}) \\
&= \frac{1+\delta L_{s-}^{T,N}}{\delta L_{s-}^{T,N}} \sum_{j=1}^{N} \frac{B_s^{T+(j-1)\delta}}{B_s^{T,N}} \frac{\delta L_{s-}^{T+(j-1)\delta}}{1+\delta L_{s-}^{T+(j-1)\delta}} \gamma_s^{T+(j-1)\delta} \\
&\approx \frac{1+\delta L_{s-}^{T,N}}{\delta L_{s-}^{T,N}} \sum_{j=1}^{N} \frac{B_s^{T+(j-1)\delta}}{B_s^{T,N}} \frac{\delta L_{s-}^{T,N}}{1+\delta L_{s-}^{T,N}} \gamma_s^{T+(j-1)\delta} \approx \sum_{j=1}^{N} \frac{B_t^{T+(j-1)\delta}}{B_t^{T,N}} \gamma_s^{T+(j-1)\delta}.
\end{aligned}
\tag{5.7}
$$

Here the following approximations were used.

$$
\frac{B_s^{T+(j-1)\delta}}{B_s^{T,N}} \approx \frac{B_s^{T^\delta+(j-1)\delta}}{B_s^{T^\delta,N}}, \qquad \frac{\delta L_{s-}^{T+(j-1)\delta}}{1+\delta L_{s-}^{T+(j-1)\delta}} \approx \frac{\delta L_{s-}^{T,N}}{1+\delta L_{s-}^{T,N}}, \qquad \frac{B_s^{T+(j-1)\delta}}{B_s^{T,N}} \approx \frac{B_t^{T+(j-1)\delta}}{B_t^{T,N}}.
$$

In the similar way, $\eta_s^{T,N}$ is approximated as follows:

$$
\begin{aligned}
\eta_s^{T,N}(z) &= \frac{1+\delta L_{s-}^{T,N}}{\delta L_{s-}^{T,N}} \frac{m_s^{T,N}(z) - m_s^{T^\delta,N}(z)}{1+m_s^{T^\delta,N}(z)} \\
&\approx \frac{1+\delta L_{s-}^{T,N}}{\delta L_{s-}^{T,N}} \frac{1}{1+m_s^{T^\delta,N}(z)} \sum_{j=1}^{N} \frac{B_s^{T^\delta+(j-1)\delta}}{B_s^{T^\delta,N}} (m_s^{T+(j-1)\delta}(z) - m_s^{T+j\delta}(z)) \\
&= \frac{1+\delta L_{s-}^{T,N}}{\delta L_{s-}^{T,N}} \frac{1}{1+m_s^{T^\delta,N}(z)} \sum_{j=1}^{N} \frac{B_s^{T^\delta+(j-1)\delta}}{B_s^{T^\delta,N}} \frac{\delta L_{s-}^{T+(j-1)\delta}}{1+\delta L_{s-}^{T+(j-1)\delta}} (1+m_s^{T+j\delta}(z))\,(e^{\eta\cdot z}-1) \\
&\approx \frac{1+\delta L_{s-}^{T,N}}{\delta L_{s-}^{T,N}} \frac{1}{1+m_s^{T^\delta,N}(z)} \sum_{j=1}^{N} \frac{B_s^{T^\delta+(j-1)\delta}}{B_s^{T^\delta,N}} \frac{\delta L_{s-}^{T,N}}{1+\delta L_{s-}^{T,N}} (1+m_s^{T^\delta+(j-1)\delta}(z))\,(e^{\eta\cdot z}-1) \\
&= \frac{1}{1+m_s^{T^\delta,N}(z)} \sum_{j=1}^{N} \frac{B_s^{T^\delta+(j-1)\delta}}{B_s^{T^\delta,N}} (1+m_s^{T^\delta+(j-1)\delta}(z))\,(e^{\eta\cdot z}-1) = (e^{\eta\cdot z}-1).
\end{aligned}
\tag{5.8}
$$

Let $\bar{\gamma}_s^{T,N} = \sum_{j=1}^{N} \frac{B_t^{T+(j-1)\delta}}{B_t^{T,N}} \gamma_s^{T+(j-1)\delta}$.

Using the approximations (5.7) and (5.8), define an approximate GE swaption price $\widetilde{\mathrm{PS}}_t(L^{T,N},K)$ by

$$
\widetilde{\mathrm{PS}}_t(L^{T,N},K) = \delta B_t^{T^\delta,N} E_t^{T^\delta,N}[\,|\tilde{L}_T^{T,N} - K|\,] \tag{5.9}
$$

where $\tilde{L}_T^{T,N}$ is an approximate (T,N)-forward swap rate at time T given by

$$
\begin{aligned}
\tilde{L}_T^{T,N} = L_s^{T,N} \exp\Bigg[-\int_t^T \left\{ \frac{1}{2}\|\bar{\gamma}_s^{T,N}\|^2 + \tilde{\lambda}_t^{T^\delta} \int_{\mathbb{R}^{d'}} (e^{\eta\cdot z}-1) \tilde{f}_t^{T^\delta}(z)\,dW \right\} ds \\
+ \int_t^T \gamma_s^{T,N} \cdot dW_s^{T^\delta} + \int_t^T \int_{\mathbb{R}^{d'}} \eta\cdot z\, \tilde{\mu}_t^{T^\delta,N}(ds\times dz) \Bigg]
\end{aligned}
\tag{5.10}
$$

where $\tilde{\mu}_t^{T^\delta,N}$ is the Poisson process with the $P^{T^\delta,N}$-intensity kernel $\tilde{\lambda}_t^{T^\delta}\, \tilde{f}_t^{T^\delta}(z)\,dz$.

5.3. **The Approximate GE Pricing Formula for Swaption.** Finally, the following proposition is obtained in the same way as done in Proposition 2.

Proposition 4. *Let $N \in \mathbb{N}$ and $T \in (0, T^\dagger - (N+1)\delta]$. Under Assumptions 1-3, the GE approximate swaption price* $\widetilde{\mathrm{PS}}_t(L^{T,N}, K)$ *is*

$$\widetilde{\mathrm{PS}}_t(L^{T,N}, K) = \delta B_t^{T^\delta, N} \sum_{n=0}^{\infty} e^{-\tilde{\lambda}_t^{T^\delta}(T-t)} \frac{\{\tilde{\lambda}_t^{T^\delta}(T-t)\}^n}{n!}$$
$$\times \left\{ L_t^{T,N} e^{\zeta_n} \Phi\left(\frac{\ln \frac{L_t^{T,N}}{K} + \zeta_n + \frac{1}{2}\bar{\sigma}_n^2}{\bar{\sigma}_n} \right) - K\,\Phi\left(\frac{\ln \frac{L_t^{T,N}}{K} + \zeta_n - \frac{1}{2}\bar{\sigma}_n^2}{\bar{\sigma}_n} \right) \right\}. \quad (5.11)$$

where ζ_n and $\bar{\sigma}_n$ are given by (4.10) and by

$$\bar{\sigma}_n = \sqrt{\int_t^T \|\bar{\gamma}_s^{T,N}\|^2\, ds + \sum_{i=1}^{d'} \eta_i^2 \sigma_i^2\, n}, \quad (5.12)$$

respectively. In particular, if $d' = 1$, then ζ_n and $\bar{\sigma}_n$ are given by (4.11) and by

$$\bar{\sigma}_n = \sqrt{\int_t^T \|\bar{\gamma}_s^{T,N}\|^2\, ds + \sigma_1^2\, n}, \quad (5.13)$$

respectively.

Remark 7. It should be noted that as well as the GE caplet price, the GE swaption price depends on the market price of jump risk while it does not depend on the market price of diffusive risk.

6. Method of Specification and Estimation

In this section, a method of specification and estimation for the jump-diffusion LM model is presented. A method proposed by Kusuda [25] for the extended LM models is extended to the jump-diffusion LM model.

6.1. **Approximation of Conditional Likelihood of Forward LIBOR Rates.** Let $N^* \in \mathbb{N}$ and $\Delta = \frac{\delta}{N^*}$. It follows from the GE dynamics (3.6) of T-forward LIBOR rate that

$$\ln L_{t+\Delta}^T = \ln L_t^T + \int_t^{t+\Delta} \Big\{ \gamma_s^T \cdot (\beta v_s^{\bar{c}} + v_s^p - v_s^{T^\delta}) - \frac{1}{2}\|\gamma_s^T\|^2 -$$
$$\lambda_s^{T^\delta} \int_{\mathbb{R}^{d'}} (e^{\eta \cdot z} - 1)\, f_s^{T^\delta}(z)\, dz \Big\}\, ds + \int_t^{t+\Delta} \gamma_s^T \cdot dW_s$$
$$+ \int_t^{t+\Delta} \int_{\mathbb{R}^{d'}} \eta \cdot z\, \nu(ds \times dz). \quad (6.1)$$

It is easy to see from (6.1) that the conditional likelihood of forward LIBOR rates cannot be derived in analytic form. Therefore, the system of SDDEs is discretized and approximated in order that the approximate conditional likelihood is derived in analytic form following Kusuda [25]. First, the following approximations are conducted.

$$v_s^T \approx \check{v}_s^T, \qquad \lambda_s^{T^\delta} \approx \check{\lambda}_s^{T^\delta}, \qquad f_s^{T^\delta}(\,\cdot\,) \approx \check{f}_s^{T^\delta}(\,\cdot\,),$$

where

$$\begin{aligned}
\check{v}_s^T &= \begin{cases} -\sum_{k=1}^{K_s^T} \frac{\delta L_{s-}^{T-k\delta}}{1+\delta L_{s-}^{T-k\delta}} \gamma_s^{T-k\delta} & \forall s \in [0, T-\delta) \\ 0 & \forall s \in [T-\delta, T), \end{cases} \\
\check{\lambda}_s^{T^\delta} &= e^{\frac{1}{2} \sum_{j=1}^{d'} \check{m}_{sj}^{T^\delta} \left(2\mu_j + \check{m}_{sj}^{T^\delta} \sigma_j^2\right)} \lambda, \qquad (6.2) \\
\check{f}_s^{T^\delta}(z) &= \prod_{i=1}^{d'} \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left[-\frac{1}{2} \sum_{i=1}^{d'} \left(\frac{z_i - \check{\mu}_{si}^{T^\delta}}{\sigma_i} \right)^2 \right],
\end{aligned}$$

where

$$\check{m}_{sj}^{T^\delta} = \delta \sum_{k=1}^{K_s^{T^\delta}} L_{s-}^{T^\delta - k\delta} \eta_j + \beta m_{\bar{c}j} + m_{pj}, \qquad \check{\mu}_{si}^{T^\delta} = \mu_i - \check{m}_{si}^{T^\delta} \sigma_i^2. \qquad (6.3)$$

Then the following approximation is obtained for $\ln L_{t+\Delta}^T$.

$$\ln L_{t+\Delta}^T \approx \ln L_t^T + \int_t^{t+\Delta} \Big\{ \gamma_s^T \cdot (\beta v_s^{\bar{c}} + v_s^p - \check{v}_s^{T^\delta}) - \frac{1}{2} \|\gamma_s^T\|^2 - (e^{-\frac{1}{2} \sum_{i=1}^{d'} \eta_i (2\check{\mu}_{ti}^{T^\delta} - \eta_i \sigma_i^2)} - 1)\, \check{\lambda}_s^{T^\delta} \Big\} ds + \int_t^{t+\Delta} \gamma_s^T \cdot dW_s + \int_t^{t+\Delta} \int_{\mathbb{R}^{d'}} \eta \cdot z\, \nu(ds \times dz). \qquad (6.4)$$

Here suppose that there exists $q^* \in \mathbb{N}$ such that $(\lambda\Delta)^{q^*+1} \approx 0$. Then the Euler-Maruyama discretization of the SDDE (6.4) is

$$\ln L_{t+\Delta}^T \approx \ln L_t^T + \Big\{ \gamma_t^T \cdot (\beta v_t^{\bar{c}} + v_t^p - \check{v}_t^{T^\delta}) - \frac{1}{2} \|\gamma_t^T\|^2 - (e^{-\frac{1}{2} \sum_{i=1}^{d'} \eta_i (2\check{\mu}_{ti}^{T^\delta} - \eta_i \sigma_i^2)} - 1)\, \check{\lambda}_t^{T^\delta} \Big\} \Delta + \gamma_t^T \cdot (W_{t+\Delta} - W_t) + 1_{\{\check{\mathbf{q}}_{t+\Delta} \neq 0\}} \sum_{q=1}^{\max\{1, \check{\mathbf{q}}_{t+\Delta}\}} \eta \cdot \check{\mathbf{z}}_{t+\Delta, q} \qquad (6.5)$$

where $\check{\mathbf{q}}_t$ is an identically and independently distributed with

$$\check{\mathbf{q}}_t = \begin{cases} 0 & \text{w.p.} \quad 1 - \frac{\lambda\Delta}{1-\lambda\Delta} \\ q & \text{w.p.} \quad (\lambda\Delta)^q \qquad \forall q \in \{1, 2, \cdots, q^* - 1\} \\ q^* & \text{w.p.} \quad \frac{(\lambda\Delta)^{q^*}}{1-\lambda\Delta}, \end{cases} \qquad (6.6)$$

$\check{\mathbf{z}}_{tq} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \Sigma)$ where

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \sigma_{d'} \end{pmatrix}.$$

Suppose that the estimation period is $[T_0, T_{I^\dagger M}]$ where $T_0 = 0$ and $I^\dagger, M \in \mathbb{N}$ and that the estimation period $[T_0, T_{I^\dagger M}]$ is divided into the following $I^\dagger$ subperiods $[T_0, T_M), [T_M, T_{2M}), \cdots, [T_{(I^\dagger - 1)M}, T_{I^\dagger M})$. The subperiod $[T_{(i-1)M}, T_{iM})$ is called the *i-th estimation subperiod* for every $i \in \{1, 2, \cdots, I^\dagger\}$, and the period $[T_{m-1}, T_m)$ is called the *i-th unit period*. Note that every estimation subperiod consists of M unit periods.

Suppose that during any unit period $[T_{m-1}, T_m)$, K future LIBOR rates with maturity dates $T_m, T_{m+1}, \cdots, T_{m+K}$ are traded. Let $t_n = T_0 + n\Delta$, and

$$\mathbf{q}_n = \check{\mathbf{q}}_{t_n}, \qquad \mathbf{z}_{nq} = \check{\mathbf{z}}_{t_n q}.$$

It is assumed that the volatility of T_m-forward LIBOR rates is a parametrized function of the time to maturity during each estimation subperiod, *i.e.*,

$$\gamma_{t_n}^{T_m} = \sum_{i=1}^{I^\dagger} 1_{\{t_n \in [T_{(i-1)M}, T_{iM})\}} \tilde{\gamma}_{i,n}^m \tag{6.7}$$

where $\tilde{\gamma}_{i,n}^m$ is a function of the time $(T_m - t_n)$ to maturity for every $i \in \{1, 2, \cdots, I^\dagger\}$. It is also assumed that the volatilities $v^{\bar{c}}$ and v^p are constant during each estimation subperiod, *i.e.*,

$$v_{t_n}^{\bar{c}} = \sum_{i=1}^{I^\dagger} 1_{\{t_n \in [T_{(i-1)M}, T_{iM})\}} v_{\bar{c}i}, \qquad v_{t_n}^{p} = \sum_{i=1}^{I^\dagger} 1_{\{t_n \in [T_{(i-1)M}, T_{iM})\}} v_{pi}, \tag{6.8}$$

where $v_{\bar{c}i}, v_{pi} \in \mathbb{R}^d$ for every $i \in \{1, 2, \cdots, I^\dagger\}$. Under the above assumption, the model can be estimated subperiod by subperiod. Let $i \in \{1, 2, \cdots, I^\dagger\}$ and consider the estimation for certain i-th estimation subperiod $(T_{(i-1)M}, T_{iM}]$. For convenience, the suffix i is omitted, hereafter.

It is computationally infeasible and unnecessary to compute the likelihood on all the K traded future rates. Therefore, the subset $\mathbf{K}'$ of $K'(< K)$ future rates are selected among the K traded future rates for computing the likelihood in each unit period. Let the index set of future rates be denoted by $\mathbf{K}' = \{k_1, k_2, \cdots, k_{K'}\}$ where $k_l \in \{1, 2, \cdots, K\}$ for every $l \in \{1, 2, \cdots, K'\}$ and $k_1 < k_2 < \cdots < k_{K'}$. Here note that the term $\check{v}_t^{T^\delta}$ in the right-hand side of (6.5) includes all the future rates with maturity dates between t and T^δ. Thus some suitable interpolation is conducted for the future rates that do not belong to $\mathbf{K}'$. Let $\tilde{L}_{t_n-}^{T_m-k\delta}$ be such that $\tilde{L}_{t_n-}^{T_m-k\delta}$ is some interpolation if the future rate does not belong to $\mathbf{K}'$, and $\tilde{L}_{t_n-}^{T_m-k\delta} = L_{t_n-}^{T_m-k\delta}$ otherwise. Let

$$\begin{aligned}
\tilde{v}_n^m &= \begin{cases} -\sum_{k=1}^{K_{t_n}^{T_m}} \frac{\delta \tilde{L}_{t_n-}^{T_m-k\delta}}{1+\delta L_{t_n-}^{T_m-k\delta}} \gamma_s^{T_m-k\delta} & \forall t_n \in [0, T_m - \delta) \\ 0 & \forall t_n \in [T_m - \delta, T_m), \end{cases} \\
\tilde{\lambda}_n^{m+1} &= e^{\frac{1}{2}\sum_{j=1}^{d'} \tilde{m}_{nj}^{m+1}\left(2\mu_j + \tilde{m}_{nj}^{m+1}\sigma_j^2\right)} \lambda, \\
\tilde{\mu}_{ni}^{m+1} &= \mu_i - (\delta \sum_{k=1}^{K_{t_n}^{T_{m+1}}} \tilde{L}_{s-}^{T_{m+1}-k\delta} \eta_i + \beta m_{\bar{c}i} + m_{pi})\sigma_i^2,
\end{aligned} \tag{6.9}$$

where

$$\tilde{m}_{nj}^{m+1} = \delta \sum_{k=1}^{K_{t_n}^{T_{m+1}}} \tilde{L}_{t_n-}^{T_{m+1}-k\delta} \eta_j + \beta m_{\bar{c}j} + m_{pj}. \tag{6.10}$$

Finally, the number d of common diffusion factors is set. If one sets $d = K'$ then the number of parameters of the d-dimensional parametrized volatility function becomes too many to estimate. However, if one sets $d < K'$ then the likelihood on the set $\mathbf{K}'$ of future rates is not defined since the variance-covariance matrix becomes singular in this case. Therefore, $K''(< K')$ future rates are selected among the set $\mathbf{K}'$ of future rates, and error terms are introduced into the discretized equations (6.5) for the set $\mathbf{K}''$ of future rates. Then the number of common diffusion factors is set such that $d = \#\mathbf{K}' - \#\mathbf{K}''$. Let $\mathbf{K}'' = \{k_{l_1}, k_{l_2}, \cdots, k_{l_{K''}}\}$ where $k_{l_m} \in \{k_1, k_2, \cdots, k'_K\}$ for $m \in \{1, 2, \cdots, K''\}$ and $k_{l_1} < k_{l_2} < \cdots < k_{l_{K''}}$.

Let m_n denote $t_n \in [T_{m_n}, T_{m_n+1})$, and $\mathbf{y}_n^m$ denote $\ln L_{t_n}^{T_m} - \ln L_{t_{n-1}}^{T^m}$. Now the approximate conditional likelihood is computed based on the following system of equations.

$$
\begin{aligned}
\mathbf{y}_n^{m_n+k_1} &= \tilde{\mu}_{n-1}^{m_n+k_1}\Delta + \sqrt{\Delta}\tilde{\gamma}_{n-1}^{m_n+k_1}\cdot \mathbf{w}_n + 1_{\{\mathbf{q}_n\neq 0\}}\sum_{q=1}^{\max\{1,\mathbf{q}_n\}}\eta\cdot \mathbf{z}_{nq} + \epsilon_{1n}\\
\mathbf{y}_n^{m_n+k_2} &= \tilde{\mu}_{n-1}^{m_n+k_2}\Delta + \sqrt{\Delta}\tilde{\gamma}_{n-1}^{m_n+k_2}\cdot \mathbf{w}_n + 1_{\{\mathbf{q}_n\neq 0\}}\sum_{q=1}^{\max\{1,\mathbf{q}_n\}}\eta\cdot \mathbf{z}_{nq} + \epsilon_{2n}\\
&\cdots\cdots\\
\mathbf{y}_n^{m_n+k_{K'}} &= \tilde{\mu}_{n-1}^{m_n+k_{K'}}\Delta + \sqrt{\Delta}\tilde{\gamma}_{n-1}^{m_n+k_{K'}}\cdot \mathbf{w}_n + 1_{\{\mathbf{q}_n\neq 0\}}\sum_{q=1}^{\max\{1,\mathbf{q}_n\}}\eta\cdot \mathbf{z}_{nq} + \epsilon_{K'n}
\end{aligned}
\tag{6.11}
$$

where $\mathbf{w}_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0_d, \mathrm{I}_d)$,

$$
\begin{aligned}
\tilde{\mu}_n^{m_n+k} &= \tilde{\gamma}_n^{m_n+k}\cdot(\beta v_{\bar{c}} + v_p - \tilde{v}_n^{m_n+1+k})\\
&\quad - \frac{1}{2}\|\tilde{\gamma}_n^{m_n+k}\|^2 - \tilde{\lambda}_n^{m_n+1+k}(e^{-\frac{1}{2}\sum_{i=1}^{d'}\eta_i(2\tilde{\mu}_{ni}^{m_n+1+k}-\eta_i\sigma_i^2)} - 1),
\end{aligned}
\tag{6.12}
$$

$$
\epsilon_{kn}\begin{cases}\overset{\text{i.i.d.}}{\sim} N(0,\psi_{m_n}) & \forall k\in\mathbf{K}''\\ = 0 & \forall k\notin\mathbf{K}'',\end{cases}
\tag{6.13}
$$

and ϵ_{kn} and $\epsilon_{k'n}$ are also independent for every $k, k' \in \mathbf{K}''$.

Remark 8. The likelihood for the approximate model (6.11) is unbounded as indicated by Honoré [14]. In estimating the jump-diffusion LM model, a modified ML method proposed in Honoré [14] can be exploited.

6.2. **Specification of the Approximate Jump-Diffusion LM Model.** In the approximate jump-diffusion LM Model, the functional form of volatility function $\tilde{\gamma}_{i,n}^m$, the sets $\mathbf{K}'$ and $\mathbf{K}''$ of future rates are unspecified. It should be noted that the terms $\tilde{\mu}$ and $\tilde{\gamma}$ in (6.11) can be regarded as constants during every estimation subperiod. Thus if the data affected by jumps can be eliminated then the approximate jump-diffusion LM model (6.11) can be regarded as a factor analysis model. This suggests that methods of factor analysis can be exploited to specify the approximate jump-diffusion LM model. The specification method based on factor analysis is summarized as follows:

Step 1: Conduct the factor analysis of the set $\mathbf{K}$ of future rates using the ML method.

Step 2: If there are outliers in the common factors' estimates then eliminate the corresponding data assuming that they were caused by jumps and repeat Step 1, otherwise proceed to Step 3.

Step 3: Decide the number K' of common factors based on a certain information criterion, and select the set $\mathbf{K}'$ using certain variables selection method such as a method of Tanaka and Kodake [30] or of Yanai [31].

Step 4: Conduct the factor analysis of the set $\mathbf{K}'$ of future rates using the ML method.

Step 5: If there are outliers then eliminate the corresponding data assuming that they were caused by jumps and repeat Step 5, otherwise proceed to Step 7.

Step 6: Decide the number $d = K' - K''$ of common factors based on certain information criterion, and select $\mathbf{K}''$ referring to the communality estimates (for details, see Kusuda [25]).

Step 7: Specify the functional form of volatility function $\tilde{\gamma}^m_{i,n}$ based on the factor loading matrix estimates.

Appendix A. Marked Point Process and Integration Theorem

A.1. **Marked Point Process.** A double sequence $(s_n, Z_n)_{n\in\mathbb{N}}$ is considered where s_n is the occurrence time of nth jump and Z_n is a random variable taking its values on a measurable space $(\mathbb{Z}, \mathcal{Z})$ at time s_n. Define a random counting measure $\nu(dt \times dz)$ by

$$\nu([0,t] \times A) = \sum_{n\in\mathbb{N}} 1_{\{s_n \le t,\, Z_n \in A\}} \qquad \forall (t,A) \in \mathbf{T} \times \mathcal{Z}.$$

This counting measure $\nu(dt \times dz)$ is called the $\mathbb{Z}$-*marked point process*.

Let λ be such that

1. For every $(\omega, t) \in \Omega \times (0, T^\dagger]$, the set function $\lambda_t(\omega, \cdot)$ is a finite Borel measure on $\mathbb{Z}$.
2. For every $A \in \mathcal{Z}$, the process $\lambda(A)$ is $\mathcal{P}$-measurable and satisfies $\lambda(A) \in \mathcal{L}^1$.

If the equation

$$E\left[\int_0^{T^\dagger} Y_s\, \nu(ds \times A)\right] = E\left[\int_0^{T^\dagger} Y_s \lambda_s(A)\, ds\right] \qquad \forall A \in \mathcal{Z}$$

holds for any nonnegative $\mathcal{P}$-measurable process Y, then it is said that the marked point process $\nu(dt \times dz)$ has the P-*intensity kernel* $\lambda_t(dz)$.

A.2. **Integration Theorem.** Let $\nu(dt \times dz)$ be a $\mathbb{Z}$-marked point process with the P-intensity kernel $\lambda_t(dz)$. Let H be a $\mathcal{P} \otimes \mathcal{Z}$-measurable process. It follows that:

1. If we have
$$E\left[\int_0^{T^\dagger}\int_{\mathbb{Z}} |H_s(z)|\lambda_s(z)\, ds\right] < \infty,$$
then the process $\int_0^t \int_{\mathbb{Z}} H_s(z)\{\, \nu(ds \times dz) - \lambda_s(dz)\, ds\,\}$ is a P-martingale.
2. If $H \in \mathcal{L}(\lambda_t(dz))$, then the process $\int_0^t \int_{\mathbb{Z}} H_s(z)\{\, \nu(ds \times dz) - \lambda_s(dz)\, ds\,\}$ is a local P-martingale.

Proof. See p.235 in Brémaud [9]. □

Appendix B. Ito's Formula and Girsanov's Theorem

B.1. **Ito's Formula.** Let $X = (X^1, ..., X^n)'$ be a n-dimensional semimartingales, and g be a real-valued $\mathbf{C}^2$-function on $\mathbb{R}^n$. Then $g(X)$ is a semimartingale of the form

$$g(X_t) = g(X_0) + \sum_{i=1}^n \int_0^t \frac{\partial}{\partial x_i} g(X_{s-})\, dX^i_s + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} g(X_{s-})\, d\langle X^{ic}, X^{jc}\rangle$$
$$+ \sum_{0\le s\le t}\left\{ g(X_s) - g(X_{s-}) + \sum_{i=1}^n \frac{\partial}{\partial x_i} g(X_{s-})\, \Delta X^i_s \right\}$$

where X^{ic} is the continuous part of X^{ic} and $\langle X^{ic}, X^{jc}\rangle$ is the quadratic covariation of X^{ic} and X^{jc}.

B.2. **Girsanov's Theorem.**

(1) Let $v \in \prod_{j=1}^{d} \mathcal{L}^2$ and $m \in \mathcal{L}^1(\lambda_t(dz) \times dt)$. Define a real-valued process Λ by

$$\frac{d\Lambda_t}{\Lambda_{t-}} = -v_t \cdot dW_t - \int_{\mathbb{Z}} m_t(z) \{ \nu(dt \times dz) - \lambda_t(dz)\, dt\} \qquad \forall t \in [0, T^\dagger)$$

with $\Lambda_0 = 1$. If $E[\Lambda_{T^\dagger}] = 1$, then there exists a probability measure $\tilde{P}$ on $(\Omega, \mathcal{F}, \mathbb{F})$ given by the Radon-Nikodym derivative

$$d\tilde{P} = \Lambda_{T^\dagger}\, dP$$

such that:

(a) The measure $\tilde{P}$ is equivalent to P.

(b) The process given by

$$\tilde{W} = W_t + \int_0^t v_s\, ds \qquad \forall t \in \mathbf{T}$$

is a $\tilde{P}$-Wiener process.

(c) The marked point process $\nu(dt \times dz)$ has the $\tilde{P}$-intensity kernel $\tilde{\lambda}_t(dz)$ such that

$$\tilde{\lambda}_t(dz) = (1 - m_t(z))\lambda_t(dz) \qquad \forall (t, z) \in \mathbf{T} \times \mathbb{Z}. \tag{B.1}$$

(2) Every probability measure equivalent to P has the structure above.

Appendix C. Definitions in Approximately Complete Bond markets

C.1. **Feasible, Self-Financing, and Admissible Portfolios.** Let X denote a real-valued $\mathcal{P}$-measurable process. The *discounted process of* X is defined by $\tilde{X}^{\mathbf{B}} = \frac{X}{B}$. Let $\tilde{\mathbf{B}}$ denote the discounted bond price family $(1, (\tilde{B}^{T\mathbf{B}})_{T \in \mathbf{T}})$. Notions of *feasible*, *self-financing*, and *admissible portfolios* in approximately complete bond markets are defined as follows.

Definition 6. Let $\mathbf{B} \in \mathcal{B}$.

(1) A portfolio ϑ is a *feasible portfolio at* $\mathbf{B}$ if and only if it follows that:

$$\int_t^{T^\dagger} |B_t^T|\, |\vartheta_t^1(dT)| < \infty \qquad P\text{-a.s.} \qquad \forall t \in \mathbf{T},$$

$$B_t r_t^B \vartheta_t^0, \int_t^{T^\dagger} |B_t^T r_t^T|\, |\vartheta_t^1(dT)| \in \mathcal{L}^1, \qquad \int_t^{T^\dagger} |B_t^T v_t^T| \vartheta_t^1(dT)| \in \mathcal{L}^2,$$

$$\int_t^{T^\dagger} |B_t^T m_t^T(z)|\, |\vartheta_t^1(dT)| \in \mathcal{L}^1(\lambda_t(dz) \times dt).$$

Let $\Theta(\mathbf{B})$ denote the space of feasible portfolios at $\mathbf{B}$.

(2) A feasible portfolio $\vartheta \in \Theta(\mathbf{B})$ at $\mathbf{B}$ is a *self-financing portfolio at* $\mathbf{B}$ if and only if its value process satisfies

$$\mathcal{V}_t(\vartheta) = \mathcal{V}_0(\vartheta) + \int_0^t \vartheta_s^0\, dB_s + \int_0^t \int_s^{T^\dagger} \vartheta_s^1(dT)\, dB_s^T \qquad \forall t \in \mathbf{T}.$$

(3) A feasible portfolio $\vartheta \in \Theta(\mathbf{B})$ at $\mathbf{B}$ is an *admissible portfolio at* $\mathbf{B}$ if and only if $\tilde{\mathcal{V}}^{\mathbf{B}}(\vartheta) \overset{\text{def}}{=} \frac{\mathcal{V}(\vartheta)}{B}$ is bounded below P-a.s.

C.2. **Arbitrage-Free Markets and Spot Martingale Measures.** Definitions of *arbitrage portfolio*, *arbitrage-free*, and *spot martingale measure* are given in the following.

Definition 7. Let $\mathbf{B} \in \mathcal{B}$.

(1) A self-financing portfolio $\vartheta \in \Theta(\mathbf{B})$ at $\mathbf{B}$ is an *arbitrage portfolio at* $\mathbf{B}$ if and only if there exist $0 \leq t < T \leq T^\dagger$ such that $\vartheta_s = 0$ for every $s \in [0, t)$ and either of the following:
   (a) $\mathcal{V}_t(\vartheta) \leq 0$ P-a.s., and $\mathcal{V}_T(\vartheta) > 0$, *i.e.* $\mathcal{V}_T(\vartheta) \geq 0$ P-a.s. and $P(\{\mathcal{V}_T(\vartheta) > 0\}) > 0$.
   (b) $\mathcal{V}_t(\vartheta) < 0$, and $\mathcal{V}_T(\vartheta) \geq 0$ P-a.s.

(2) Markets are *arbitrage-free at* $\mathbf{B}$ if and only if there exists no arbitrage portfolio in the space of admissible portfolios.

(3) A probability measure $\tilde{P}^{\mathbf{B}}$ on $(\Omega, \mathcal{F})$ is a *spot martingale measure at* $\mathbf{B}$ if and only if $\tilde{P}^{\mathbf{B}}$ is equivalent to P, and $\tilde{\mathbf{B}}$ is a local $\tilde{P}^{\mathbf{B}}$-martingale.

C.3. **Approximately Complete Markets.** Definitions of *contingent claim*, *replicable claim*, and *approximately complete* are given as follows.

Definition 8. Let $\mathbf{B} \in \mathcal{B}$.

(1) For every $T \in (0, T^\dagger]$, a *contingent T-claim at* $\mathbf{B}$ is a $\mathcal{F}_T$-measurable random variable X_T such that $\frac{X_T}{B_T} \in \mathbf{L}^\infty_+(\Omega, \mathcal{F}_T)$ where $\mathbf{L}^\infty(\Omega, \mathcal{F}_T)$ is the space of almost surely bounded $\mathcal{F}_T$-measurable random variables.

(2) A contingent T-claim X_T is *replicable at* $\mathbf{B}$ if and only if there exists an admissible self-financing portfolio $\vartheta \in \underline{\Theta}(\tilde{\mathbf{B}})$ such that its value process satisfies $V_T(\vartheta) = X_T$.

(3) Markets are *approximately complete at* $\mathbf{B}$ if and only if for any $T \in (0, T^\dagger]$ and any T-contingent claim X_T there exists a sequence of replicable claims $(X_{Tn})_{n\in\mathbb{N}}$ converging to X_T in $\mathbf{L}^2(\Omega, \mathcal{F}_T, \tilde{P}^{\mathbf{B}})$.

References

[1] Ahn, C.-M., and H. E. Thompson (1988): "Jump-Diffusion Processes and the Term-Structure of Interest Rates," *Journal of Finance*, 43, 155-174.

[2] Andersen, L., and J. Andreasen (2000): "Volatility Skews and Extensions of the LIBOR Market Model," *Applied Mathematical Finance*, 7, 1-32.

[3] Balduzzi, P., E. J. Elton, and T. C. Green (2001): "Economic News and Bod Prices: Evidence from the U.S. Treasury Market," *Journal of Financial and Quantitative Analysis*, 32, 523-543.

[4] Björk, T., Y. Kabanov, and W. Runggaldier (1997): "Bond Market Structure in the Presence of Marked Point Processes," *Mathematical Finance*, 7, 211-239.

[5] Björk, T., G. Di Masi, Y. Kabanov, and W. Runggaldier (1997): "Towards a General Theory of Bond Markets," *Finance and Stochastics*, 1, 141-174.

[6] Black, F. (1976): "The Pricing of Commodity Contracts," *Journal of Financial Economics*, 3, 167-179.

[7] Black, F., and M. Scholes (1973): "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637-654.

[8] Brace, A., D. Gatarek, and M. Musiela (1997): "The Market Model of Interest Rate Dynamics," *Mathematical Finance*, 7, 127-155.

[9] Brémaud, T. (1981): *Point Processes and Queues: Martingale Dynamics*. New York: Springer-Verlag.

[10] Das, S. R. (2002): "The Surprise Element: Jumps in Interest Rates," *Journal of Econometrics*, 106, 27-65.

[11] Das, S. R., and S. Foresi (1996): "Exact Solutions for Bond and Option Prices with Systematic Jump Risk," *Review of Derivative Studies*, 1, 7-24.

[12] Glasserman, P., and S. G. Kou (2003): "The Term Structure of Simple Forward Rates with Jump Risk," *Mathematical Finance*, 13, 383-410.

[13] Heston, S. L. (1995): "A Model of Discontinuous Interest Rate Behavior, Yield Curves and Volatility," Working Paper, Washington University.

[14] Honoré, P. (1998): "Pitfalls in Estimating Jump-Diffusion Models," Working Paper, Department of Finance, Aarhus School of Business.

[15] Jamshidian, F. (1989): "An Exact Bond Option Formula," *Journal of Finance*, 44, 205-209.

[16] Jamshidian, F. (1997): "LIBOR and Swap Market Models and Measures," *Finance and Stochastics*, 1, 293-330.

[17] Johannes, M. (2004): "The Statistical and Economic Role of Jumps in Continuous-Time Interest Rate Models," *Journal of Finance*, 59, 227-260.

[18] Jorion, P. (1989): "On Jump Processes in the Foreign Exchange and Stock Markets," *Review of Financial Studies*, 1, 427-445.

[19] Kusuda, K. (2002a): "Jump-Diffusion LIBOR rate model and Equilibrium Pricing of Interest rate Derivatives," Proceedings of Japanese Association of Financial Econometric and Engineering Summer 2002 Conference, 340-361.

[20] Kusuda, K. (2002b): "Existence, Uniqueness, and Determinacy of Equilibria in Complete Security Markets with Infinite Dimensional Martingale Generator," Discussion Paper 316, Center for Economic Research, Department of Economics, University of Minnesota.

[21] Kusuda, K. (2003a): "Consumption-Based CAPM and Option Pricing under Jump-Diffusion Uncertainty," Discussion Paper 317, Center for Economic Research, Department of Economics, University of Minnesota.

[22] Kusuda, K. (2003b): *Term Structure Models of Interest Rates with Jump-Diffusion Information: Equilibrium, CAPM, and Derivative Asset Pricing*, Ph.D. Dissertation, Department of Economics, University of Minnesota.

[23] Kusuda, K. (2004a): "Implementing Arrow-Debreu Equilibria in Security Markets with Infinite Dimensional Martingale Generator," Working Paper B-1, Center for Risk Research, Faculty of Economics, Shiga University.

[24] Kusuda, K. (2004b): "General Equilibrium Analysis in Security Markets with Infinite Dimensional Martingale Generator," Working Paper B-2, Center for Risk Research, Faculty of Economics, Shiga University.

[25] Kusuda, K. (2004c): "Specification and Test of Extended LIBOR Market Models," Working Paper B-3, Center for Risk Research, Faculty of Economics, Shiga University.

[26] Merton, V. (1976): "Option Pricing when Underlying Stock Returns are Discontinuous," *Journal of Financial Economics*, 3, 125-144.

[27] Miltersen, K. R., K. Sandmann, and D. Sondermann (1997): "Closed Form Solutions for Term Structure Derivatives with Log-Normal Interest Rates," *Journal of Finance*, 52, 409-430.

[28] Naik, V., and M. Lee (1995): "The Yield Curve and Bond Option Prices with Discrete Shifts in Economic Regimes," Working Paper, University of British Columbia.

[29] Pan, J. (2002): "The Jump-Risk Premia Implicit in Options: Evidence from an Integrated Time-Series Study," *Journals of Financial Economics*, 63, 3-50.

[30] Tanaka, Y., and K. Kodake (1981): "A Method of Variable Selection in Factor Analysis and its Numerical Investigation," *Behaviormetrika*, 10, 49-61.

[31] Yanai, H. (1980): "A Proposition of Generalized Method for Forward Selection of Variables," *Behaviormetrika*, 7, 95-107.

[32] Zühlsdorff, C. (2002): "Extended LIBOR Market Models with Affine and Quadratic Volatility," Discussion Paper 6/2002, Bonn Graduate School of Business, University of Bonn.