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Specification and Test of Extended LIBOR Market Models

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Specification and Test of Extended LIBOR Market Models

by

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Abstract. LIBOR market (LM) model is an interest rate version of the Black-Scholes model of stock price. Extended LM models including constant elasticity of volatility models and affine volatility models have been proposed to explain the observation that implied volatilities of forward LIBOR rates depend on caps' rates in the LM model. This paper proposes methods of specifying the dimensionality of Wiener process and the functional form on forward LIBOR rates' volatilities in the extended LM models, and presents a test for the extended LM models. The result of the test using the Eurodollar future rates traded in the Chicago Mercantile Exchange rejected all of the extended LM models.

Keywords and Phrases: Diffusion process, Empirical analysis, Factor analysis, LIBOR market model, Specification, Test.

JEL Classification Numbers: C39, C51, C52, E43, G13.

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1. Introduction

The *LIBOR* (London InterBank Offered Rate)[1] *market model*, developed by Brace, Gatarek, and Musiela [7], Miltersen, Sandmann, Sondermann [22], and Jamshidian [14], is an interest rate version of the celebrated Black-Scholes model (Black and Scholes [6]) of stock price. In the Black-Scholes model, the change in stock price is subject to a lognormal distribution under the risk-neutral measure. In the LIBOR market (LM) model, the change in each forward LIBOR rate (resp. forward swap rate) is subject to a lognormal distribution (resp. an approximate lognormal distribution) under the associated equivalent martingale measure. Thus a Black-Scholes-like pricing formula (resp. approximate pricing formula) for each caplet (resp. swaption) can be derived. The LM model therefore can be calibrated using the formulas, and other interest rate derivatives can be speedily priced employing the calibrated model. Also, the bond markets in the LM model are arbitrage-free unlike the *Black model*[2] (Black [5]). These favorable properties make the LM model currently the most popular interest rate derivative pricing models among both practitioners and researchers. However, it is often observed that the *implied volatilities* of forward LIBOR rates, which are derived by substituting the market quoted prices of caps into the pricing formula, depend on the strike rates. This suggests that the LM model can not capture the dynamics of interest rates in real markets. Extended LM models – including the constant elasticity of volatility (CEV) model (Andersen and Andreasen [1]) and the affine volatility (AV) model (Zühlsdorff [31]) – have been proposed in order to account for the observation. In each extended LM model, a pricing formula (resp. an approximate pricing formula) for each caplet (resp. swaption) can be derived. An interesting question is whether or not the extended LM models can accurately price interest rate derivatives. To accurately price interest rate derivatives, the extended LM models should be statistically acceptable. While many authors (Brace, Gatarek, and Musiela [7], De Jong, Driessen, and Pelsser [11], Rebonato [24], Sidenius [27], *etc*) have calibrated the extended LM models from market quoted prices of caps and swaptions using the pricing formulas, no statistical test has been conducted for the extended LM models due to certain difficulties in estimating and testing them. The main purpose of this paper is to propose a statistical test of the extended LM models and to examine whether or not the extended LM models are statistically acceptable using the test.

The difficulties in estimating and testing the extended LM models is summarized as follows. Each extended LM model is expressed as a system of stochastic differential equations (SDEs) with a multidimensional Wiener process for forward LIBOR rates with different maturities. The first difficulties are that in the system of SDEs for forward LIBOR rates, neither the dimensionality of the Wiener process or the

[1]The LIBOR rate is the interest rate offered by banks on deposits from other banks in Eurocurrency markets and is frequently a reference rate of interest for loans in international financial markets. In the LIBOR market model, the dynamics of forward LIBOR rates are modeled. A representative real example of forward LIBOR rate is a Eurodollar future rate traded on the Chicago Mercantile Exchange. In the case of Eurodollar futures, the underlying instrument of Eurodollar future contracts is the 90-day LIBOR and future rates with 48 different times to maturity, *i.e.*, one month, two month, $\cdots$, one year, one year and three month, one year and six month, $\cdots$, ten years, are traded.

[2]Practitioners had used the Black model in which the change in each forward LIBOR rate and forward swap rate is subject to a lognormal distribution under the associated equivalent martingale measure. However, if the change in a forward LIBOR rate is subject to a log-normal distribution under the associated equivalent martingale measure, then a forward swap rate is not subject to a lognormal distribution under the associated equivalent martingale measure in arbitrage-free markets. Thus, there exists an arbitrage opportunity in the bond markets assumed in the Black model.

term structure of forward LIBOR rate volatilities is specified. Thus, some appropriate statistical methods of specifying the dimensionality of the Wiener process and the term structure of forward LIBOR rate volatilities are required. The next difficulty is that the likelihood function of forward LIBOR rates in the extended LM model is unavailable in analytic form. Hence, the system of SDEs are discretized. The discretized system of SDEs is a multivariate time series model with the discretized Wiener process *i.e.*, unobservable common factors that follow an identically and independently distributed multivariate normal distribution. However, the conditional likelihood function of forward LIBOR rates is still unavailable in analytic form in the time series model. Some approximations of the time series model are required to have an analytic conditional likelihood function. The final difficulty is that the traded future LIBOR rates are too many to compute the conditional likelihood function. Thus, certain appropriate statistical method of selecting the set of future LIBOR rates among all the traded future LIBOR rates is required.

This paper first presents estimable approximate extended LM models, which can be regarded as factor analysis models. Thus, methods for selecting explaining variables and choosing the number of common factors in factor analysis can be applied to the approximate extended LM models. The set of future LIBOR rates and the dimensionality of the Wiener process are chosen in this way. Moreover, a method for specifying the term structure of future LIBOR rate volatilities is presented using the factor analysis. Then the approximate extended LM models are specified by the factor analysis of data of Eurodollar future rates. Next, the specified approximate extended LM models are estimated using the maximum likelihood method, and some models are selected among them using the likelihood test and Schwartz-Bayesian information criterion. The result indicates that the approximate LM model is completely rejected while some of the approximate CEV and AV models are selected. Finally, a test for the selected models is conducted and the result shows that the distribution of factors' estimates for every selected model has a much fatter tail than the normal distribution.

The result suggests the following three promising extensions of the extended LM models. The first one is to replace the deterministic volatility in an extended LM model with a stochastic one. The second one is to introduce a jump process into an extended LM model. The third one is to conduct these two extensions together. Recently, a stochastic volatility LIBOR market model (Andersen and Ratcliffe [2]), jump-diffusion LIBOR market models (Glasserman and Kou [13], Kusuda [16]), and a stochastic volatility jump-diffusion LIBOR market model (Kusuda [18]) have been proposed.

The remainder of this paper is organized as follows. Section 2 reviews the extended LM models. Section 3 presents approximate extended LM models and the specification methods of the models. Section 4 specifies the approximate extended LM models based on the factor analysis of data of Eurodollar future rates. Section 5 estimates and and tests the approximate extended LM models. Section 6 offers a conclusion.

2. Extended LIBOR Market Models

In this section, the extended LM models are reviewed following Brace, Gatarek, and Musiela [7], Musiela and Rutkowski [23], Andersen and Andreasen [1], and Zühlsdorff [31].

Continuous-time frictionless security markets with time span $[0, T^\dagger]$ for a fixed horizon time $T^\dagger > 0$ are considered. investors' common subjective probability and information structure are modeled by a complete filtered probability space

$(\Omega, \mathcal{F}, \mathbb{F}, P)$ where the filtration $\mathbb{F} = (\mathcal{F}_t)$ is the natural filtration generated by a d-dimensional Wiener process W.

There are markets for securities at every time $t \in [0, T^\dagger]$. The traded securities are nominal-risk-free security (NOT the risk-free security) called the *money market account* and $K^\dagger$ zero-coupon bonds whose maturities are $T_1, T_2, \cdots, T_{K^\dagger}$ where $\delta \in (0,1]$ and $T_k = k\delta$ for $k \in \{1, 2, \cdots, K^\dagger\}$ and $T_{K^\dagger} = T^\dagger$, each of which pays one unit of cash at its maturity. Let B and $(B^k)_{k\in\{1,2,\cdots,K^\dagger\}}$ denote the nominal money market account price process and nominal bond price processes, respectively. The collection $(B, (B^k)_{k\in\{1,2,\cdots,K^\dagger\}})$ is abbreviated by $\mathbf{B}$ and called the *family of bond prices*.

A family $\mathbf{B}$ of bond prices is said to be *viable* if and only if the following conditions hold:

(1) The dynamics of nominal money market account price process satisfy

$$\frac{dB_t}{B_t} = r_t^B \, dt \qquad \forall t \in [0, T^\dagger) \tag{2.1}$$

with $B_0 = 1$ where r^B is an absolutely integrable nonnegative adapted process.

(2) For every $K \in \{1, 2, \cdots, K^\dagger\}$, the dynamics of nominal bond price process B^K satisfy the following SDE

$$\frac{dB_t^K}{B_t^K} = r_t^K dt + v_t^K \cdot dW_t \qquad \forall t \in [0, T_K) \tag{2.2}$$

with $B_{T_K}^K = 1$ where $r_t^K = r_t^B + v_t^{\mathbf{B}} \cdot v_t^K$ where $v^{\mathbf{B}}$ and v^K are square integrable adapted processes.

The process $v^{\mathbf{B}}$ is called the *market price of risk*. It immediately follows from Ito's formula (see Appendix A.1) and Girsanov's Theorem (see Appendix A.2) that there is no *arbitrage* (for definitions of arbitrage, see Appendix B) in the bond markets with any viable family of bond prices are arbitrage-free.

For every $K \in \{1, 2, \cdots, K^\dagger - 1\}$, the T_K*-forward LIBOR rate process* L^K over the future time period $[T_K, T_{K+1}]$ is defined by

$$L_t^K = \frac{1}{\delta}\left(\frac{B_t^K}{B_t^{K+1}} - 1\right) \qquad \forall t \in [0, T_K].$$

Let $K_t = \lceil K - \frac{t}{\delta} \rceil - 1$. The *extended LM models* are specified by the following assumption.

Assumption 1. *The family* $\mathbf{B}$ *of bond prices is viable and such that for every* $K \in \{1, 2, \cdots, K^\dagger - 1\}$ *and* $t \in [0, T_K)$*, there exist functions* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *and* $b : [0, T^\dagger] \times \{1, 2, \cdots, K^\dagger\} \to \mathbb{R}^d$ *satisfying*

$$v_t^K = \begin{cases} -\sum_{k=1}^{K_t} \frac{\delta\varphi(L_t^{K-k})}{1+\delta L_t^{K-k}} b_t^{K-k} & \forall t \in [0, T_{K-1}) \\ 0 & \forall t \in [T_{K-1}, T_K). \end{cases} \tag{2.3}$$

Here the notion of *forward martingale measure* is introduced.

Definition 1. Let $K \in \{1, 2, \cdots, K^\dagger\}$. A probability measure P^K on $(\Omega, \mathcal{F})$ is a T_K*-forward martingale measure* if and only if P^K is equivalent to P, and $\frac{B}{B^K}$ and $(\frac{B^k}{B^K})_{k\in\{1,2,\cdots,K^\dagger\}}$ are local martingales under P^K.

The following lemma shows the dynamics of T_K-forward LIBOR rate process under the investors' subjective probability P and the T_{K+1}-forward martingale measure P^{K+1} in the extended LM models.

Lemma 1. *Under Assumption 1, it follows that for every $K \in \{1, 2, \cdots, K^\dagger - 1\}$ the dynamics of T_K-forward LIBOR rate process satisfy for every $t \in [0, T_K)$,*

$$dL_t^K = \varphi(L_t^K)\, b_t^K \cdot \Big\{(v_t^{\mathbf{B}} - v_t^{K+1})\, dt + dW_t\Big\}, \tag{2.4}$$

or equivalently

$$dL_t^K = \varphi(L_t^K)\, b_t^K \cdot dW_t^{K+1} \tag{2.5}$$

where

$$W_t^{K+1} = W_t + \int_0^t (v_s^{\mathbf{B}} - v_s^{K+1})\, ds, \tag{2.6}$$

and W_t^{K+1} is a Wiener process under P^{K+1}.

Proof. Note that the following follows from (2.3)

$$b_t^K = \frac{1 + \delta L_t^K}{\delta \varphi(L_t^K)} (v_t^K - v_t^{K+1}). \tag{2.7}$$

Applying Ito's formula to the definition of L^K yields

$$\begin{aligned} dL_t^K &= \frac{1 + \delta L_t^K}{\delta} \Big[\Big\{ r_t^K - r_t^{K+1} - v_t^{K+1} \cdot (v_t^K - v_t^{K+1}) \Big\} dt + (v_t^K - v_t^{K+1}) \cdot dW_t \Big] \\ &= \frac{1 + \delta L_t^K}{\delta} \Big[(v_t^{\mathbf{B}} - v_t^{K+1}) \cdot (v_t^K - v_t^{K+1})\, dt + (v_t^K - v_t^{K+1}) \cdot dW_t \Big] \end{aligned} \tag{2.8}$$

for every $t \in [0, T_K)$. Substituting $r_t^K = r_t^B + v_t^{\mathbf{B}} \cdot v_t^K$ and (2.7) into (2.8) yields (2.4), and substituting (2.6) into (2.4) yields (2.5). It follows from Girsanov's Theorem that W_t^{K+1} is a Wiener process under P^{K+1}. □

Brace, Gatarek, and Musiela [7] and Miltersen, Sandmann, and Sondermann [22] specify the function φ in the extended LM models by

$$\varphi(L) = L.$$

Then the model is called the *LIBOR market model*.

However, it is often observed that the implied volatilities depend on the strike rates of caps. This suggests that the LM model cannot appropriately capture the dynamics of forward LIBOR rates. To explain the observation, Andersen and Andreasen [1] and Zühlsdorff [31] have proposed constant elasticity of volatility (CEV) models and affine volatility (AV) models, respectively. The *CEV models* are specified by

$$\varphi(L) = L^\alpha$$

for $\alpha \in \mathbb{R}_+$. The *AV models* are specified by

$$\varphi(L) = L + \beta$$

for $\beta \in \mathbb{R}$. The CEV and AV models are called the *extended LM models*. It is shown that in each extended LM model, a pricing formula (resp. an approximate pricing formula) for each caplet (resp. swaption) can be derived.[3]

Hereafter, the notations CEV(α_0) and AV(β_0) are used as the CEV model with $\alpha = \alpha_0$ and as the AV model with $\beta = \beta_0$, respectively. Note that CEV(1) = AV(0) = LM and CEV(0) = AV(∞). The CEV(0) (= AV(∞)) model is called the *LIBOR Gaussian model*, and is written as the LG model.

[3]For the pricing formulas, see Andersen and Andreasen [1] and Zühlsdorff [31].

3. Approximate Extended LM Models

The extended LM models needs to be estimated and tested to examine whether or not the models can accurately price interest rate derivatives. However, it is difficult to estimate and test the extended LM models themselves since the dynamics of forward LIBOR rates are analytically intractable under the probability measure P. In this section, estimable approximate extended LM models and specification methods of the extended LM models are presented.

3.1. **Approximate Extended LM Models.** Since the likelihood function of forward LIBOR rates are unavailable in analytic form, the SDE (2.4) is discretized by the Euler-Maruyama discretization scheme[4]. Let $N^* \in \mathbb{N}$ and $\Delta = \frac{\delta}{N^*}$. Then the Euler-Maruyama discretization of the SDE on the T^k-forward LIBOR rate is

$$L^k_{t+\Delta} - L^k_t \approx \varphi(L^k_t)\Big\{ b^k_t \cdot (v^{\mathbf{B}}_t - v^{k+1}_t)\Delta + b^k_t \cdot (W_{t+\Delta} - W_t) \Big\}. \tag{3.1}$$

Suppose that the estimation period is $[0, T_{I^\dagger M}]$ where $I^\dagger, M \in \mathbb{N}$. The estimation period $[T_0, T_{I^\dagger M}]$ is divided into the following $I^\dagger$ subperiods $[T_0, T_M), [T_M, T_{2M}), \cdots, [T_{(I^\dagger-1)M}, T_{I^\dagger M})$. The estimation subperiod $[T_{(i-1)M}, T_{iM})$ is called the *i-th estimation subperiod* for every $i \in \{1, 2, \cdots, I^\dagger\}$, and the period $[T_{m-1}, T_m)$ is called the *m-th unit period* for every $m \in \{1, 2, \cdots, I^\dagger M\}$. Note that every estimation subperiod consists of M unit periods.

Suppose that during any unit period $[T_{m-1}, T_m)$, K future LIBOR rates with maturities $T_m, T_{m+1}, \cdots, T_{m+K-1}$ are traded. Write $t_n = n\Delta$. It is assumed that the volatility of T_m-forward LIBOR rate is a parameterized function of the time to maturity during each estimation subperiod, *i.e.*

$$b^m_{t_n} = \sum_{i=1}^{I^\dagger} 1_{\{t_n \in [T_{(i-1)M}, T_{iM})\}} \tilde{b}^m_{i,n} \tag{3.2}$$

where $\tilde{b}^m_{i,n}$ is a function of the time $(T_m - t_n)$ to maturity for every $i \in \{1, 2, \cdots, I^\dagger\}$. The market price $v^{\mathbf{B}}$ of risk is assumed to be constant during each estimation subperiod, *i.e.*

$$v^{\mathbf{B}}_{t_n} = \sum_{i=1}^{I^\dagger} 1_{\{t_n \in [T_{(i-1)M}, T_{iM})\}} v_i \tag{3.3}$$

where $v_i \in \mathbb{R}^d$ for every $i \in \{1, 2, \cdots, I^\dagger\}$. Under the above assumptions, the model can be estimated subperiod by subperiod. Let $i \in \{1, 2, \cdots, I^\dagger\}$ and consider the estimation for i-th estimation subperiod $(T_{(i-1)M}, T_{iM}]$. For convenience, the suffix i is omitted, hereafter.

It is computationally infeasible and unnecessary to compute the likelihood on all the K traded future rates. Therefore, $K'(< K)$ future rates are selected among all the K traded future rates for computing the likelihood in each unit period. Let the index set of future rates denote $\mathbf{K'} = \{k_1, k_2, \cdots, k_{K'}\}$ where $k_l \in \{1, 2, \cdots, K\}$ for every $l \in \{1, 2, \cdots, K'\}$ and $k_1 < k_2 < \cdots < k_{K'}$. Here note that the term v^{K+1}_t in the right-hand side of (3.1) includes all the future rates with maturities between t and T_{K+1}. Thus, some suitable interpolation is conducted for the future rates that do not belong to $\mathbf{K'}$. Let $\tilde{L}^{m-k}_{t_n}$ be such that $\tilde{L}^{m-k}_{t_n} = L^{m-k}_{t_n}$ if the future rate

[4]The Euler-Maruyama discretization scheme is the simplest one, but it is shown that under Lipschitz and linear growth conditions on the drift and diffusion coefficients, the Euler-Maruyama discretization scheme has the 0.5 order of strong convergence (see Theorem 10.2.2 in Kloeden and Platen [19]).

belongs to $\mathbf{K}'$ and $\tilde{L}_{t_n}^{m-k}$ is some appropriate interpolation if the future rate does not belong to $\mathbf{K}'$. Let

$$\tilde{v}_n^m = \begin{cases} -\sum_{k=1}^{m_{t_n}} \frac{\delta\varphi(\tilde{L}_{t_n}^{m-k})}{1+\delta\tilde{L}_{t_n}^{m-k}} b_s^{m-k} & \forall t_n \in [0, T_m - \delta) \\ 0 & \forall t_n \in [T_m - \delta, T_m). \end{cases} \tag{3.4}$$

where $m_{t_n} = \lceil m - \frac{t_n}{\delta} \rceil - 1$.

Next, consider the method of setting the number d of common factors. Suppose $d = K'$. Then the parameters of the d-dimensional parameterized volatility function would become too many to estimate. However, if $d < K'$, then the likelihood on the set $\mathbf{K}'$ of future rates is not defined since the variance-covariance matrix becomes singular. Therefore, $K'' < K'$ future rates are selected among the set $\mathbf{K}'$ of future rates, and error terms are introduced into the discretized equations for the set $\mathbf{K}''$ of future rates. Then the number of common factors is set such that $d = K' - K''$ where $K'' = \#\mathbf{K}''$. Let $\mathbf{K}'' = \{k_{l_1}, k_{l_2}, \cdots, k_{l_{K''}}\}$ where $k_{l_m} \in \{k_1, k_2, \cdots, k'_K\}$ for $m \in \{1, 2, \cdots, K''\}$ and $k_{l_1} < k_{l_2} < \cdots < k_{l_{K''}}$.

Let m_n denote $t_n \in [T_{m_n}, T_{m_n+1})$, and write $\mathbf{y}_n^m = \frac{\tilde{L}_{t_n}^m - \tilde{L}_{t_{n-1}}^m}{\varphi(\tilde{L}_{t_{n-1}}^m)}$. Now the following approximate extended LM models is obtained.

$$\begin{aligned} \mathbf{y}_n^{m_n+k_1} &= \Delta\tilde{b}_n^{m_n+k_1} \cdot (v - \tilde{v}_n^{m_n+1+k_1}) + \sqrt{\Delta}\,\tilde{b}_{n-1}^{m_n+k_1} \cdot \mathbf{w}_n + \epsilon_{1n} \\ \mathbf{y}_n^{m_n+k_2} &= \Delta\tilde{b}_n^{m_n+k_2} \cdot (v - \tilde{v}_n^{m_n+1+k_2}) + \sqrt{\Delta}\,\tilde{b}_{n-1}^{m_n+k_2} \cdot \mathbf{w}_n + \epsilon_{2n} \\ &\cdots\cdots\cdots\cdots \\ \mathbf{y}_n^{m_n+k_{K'}} &= \Delta\tilde{b}_n^{m_n+k_{K'}} \cdot (v - \tilde{v}_n^{m_n+1+k_{K'}}) + \sqrt{\Delta}\,\tilde{b}_{n-1}^{m_n+k_{K'}} \cdot \mathbf{w}_n + \epsilon_{K'n} \end{aligned} \tag{3.5}$$

where $\mathbf{w}_n \overset{\text{IID}}{\sim} N(0_d, \mathrm{I}_d)$, and

$$\epsilon_{kn} \begin{cases} \overset{\text{IID}}{\sim} N(0, \psi_{m_n}) & \forall k \in \mathbf{K}'' \\ = 0 & \forall k \notin \mathbf{K}'', \end{cases}$$

and ϵ_{kn} and $\epsilon_{k'n}$ are also independent for every $k, k' \in \mathbf{K}''$.

To accurately construct the approximate model (3.5), one needs to select the set $\mathbf{K}'$ of future rates among the set $\mathbf{K}$ of future rates, and the set $\mathbf{K}''$ of future rates among the $\mathbf{K}'$ set of future rates based on some appropriate statistical methods. Here it should be noted that the approximate model (3.5) can be regarded as an approximate factor analysis model because the terms $\tilde{b}$ and $\tilde{v}$ in (3.5) can be approximated to constants during every estimation subperiod. This suggests that various methods of factor analysis can be exploited to select $\mathbf{K}'$ and $\mathbf{K}''$.

First, certain appropriate information criterion in the factor analysis with the maximum likelihood (ML) method can be employed to decide the number K' of future rates, and some appropriate variable selection method in the factor analysis can be exploited to select the set $\mathbf{K}'$ of future rates. Next, an appropriate information criterion in the factor analysis with ML method can be used to decide the number $d = K' - K''$ of common factors, and the set $\mathbf{K}''$ of future rates can be selected based on the communality estimates in the factor analysis. Moreover, the functional form on forward LIBOR rate volatilities can be specified based on the factor loading matrix estimates of the factor analysis.

The procedure for specifying the approximate extended LM models is summarized as follows.

(1) Conduct the factor analysis of the set $\mathbf{K}$ of future rates using the ML method.

(2) Decide the number K' of factors based on an appropriate information criterion, and select the set $\mathbf{K}'$ of future rates using some variable selection method such as the method of Tanaka and Kodake [29] or that of Yanai [30].
(3) Conduct the factor analysis of the set $\mathbf{K}'$ of future rates using the ML method.
(4) Decide the number $d = K' - K''$ of common factors based on some information criterion, and select $\mathbf{K}''$ referring to the communality estimates.
(5) Specify the functional form on forward LIBOR rate volatilities based on the factor loading matrix estimates.

4. Specification based on Factor Analysis

In this section, the extended LM models are specified based on factor analysis.

The cases of the LM model, *i.e.* CEV(1)=AV(0), and the LG (LIBOR Gaussian) model, *i.e* CEV(0)=AV(∞) were analyzed since the assumed range of α in CEV models is 0 to 1 and that of β is 0 to ∞. The daily data of Eurodollar future rates (mid rate) traded on the Chicago Mercantile Exchange were employed. The estimation period is December 14, 1998 to December 18, 2000. The underlying tenor of Eurodollar future rates is 90-day, *i.e.* 0.25 year, and the observations are 509. We therefore set $\delta = 0.25$ and $\Delta = 2/509$.

Considering that many empirical analyses show that volatilities of interest rates often change, the subperiod was set as three months, *i.e.* $M = 1$. The estimation period were divided into the following subperiods.

1st subperiod:	12/14/1998 – 03/15/1999
2nd subperiod:	03/15/1999 – 06/14/1999
3rd subperiod:	06/14/1999 – 09/13/1999
4th subperiod:	09/13/1999 – 12/13/1999
5th subperiod:	12/13/1999 – 03/13/2000
6th subperiod:	03/13/2000 – 06/19/2000
7th subperiod:	06/19/2000 – 09/18/2000
8th subperiod:	09/18/2000 – 12/18/2000

4.1. **Selection of Set of Future Rates.** Using the ML method, factor analyses of the set of 39 future rates with the following maturity dates were conducted: 3-month (3M hereafter), 6M, 9M, 1 year (1Y hereafter), 1Y3M, $\cdots$, 9Y6M, 9Y9M. To determine the number K' of factors, the Schwartz-Bayesian information criterion (SBC) of Schwartz [25] was used. The results showed that the model with 10 factors was the best in both cases of the LM model and the LG model, and therefore the number of factors were set as $K' = 10$.

Due to time restrictions, the Tanaka-Kodake or Yanai methods were not implemented. A set of 10 future rates were chosen with the following times to maturity of 6M, 1Y, 1Y6M, 2Y, 3Y, 4Y, 5Y, 6Y6M, 8Y, 9Y9M because the estimated factor loading matrix for all the 39 future rates and that for the 10 future rates were close to each other.

4.2. **Selection of Number of Common Factors.** The factor analysis of the selected set of 10 future rates with one common factor to six common factors were conducted using the ML method,[5] and the six models were compared using the SBC. The results are as shown in Tables 4.1 and 4.2. The figures with asterisks shown in these tables present Heywood cases (refer to Lawley and Maxwell [20]).

[5]The squared multiple correlation was used as initial estimates of communality.

Table 4.1. The SBC of LG model.

d	Subperiod								Whole
	1st	2nd	3rd	4th	5th	6th	7th	8th	Period
1	*608.5	732.9	899.0	*786.0	*1170.3	*1139.0	923.5	*1160.4	*6837.1
2	*206.8	270.0	*434.9	*306.6	*502.3	*477.5	245.7	182.7	*2190.6
3	*−11.4	*29.7	*72.1	−24.5	*71.1	*41.3	*21.9	*−34.9	*−134.2
4	*−34.9	*−9.2	*2.0	*−32.0	*−24.3	*−20.1	*−18.8	*−34.6	*−354.9
5	*−18.8	*−11.2	*−6.7	*−15.0	*−13.1	*−18.3	*−12.7	*−16.5	*−195.4
6	—	*3.0	*4.0	—	—	—	*0.8	*0.3	—

Table 4.2. The SBC of LM model.

d	Subperiod								Whole
	1st	2nd	3rd	4th	5th	6th	7th	8th	Period
1	*616.5	725.7	892.1	*791.0	*1165.8	*1145.1	924.0	*1158.5	*6836.3
2	*212.5	256.8	*438.4	*313.2	*443.9	*481.3	242.0	185.1	*2140.6
3	*−8.2	26.2	*73.5	−24.7	*107.7	*40.9	*21.2	*−36.1	*−101.2
4	*−34.8	*−11.5	*2.4	*−32.3	*−25.1	*−18.6	*−19.9	*−36.4	*−359.3
5	*−18.4	*−11.3	*−6.2	*−15.1	*−13.8	*−18.7	*−13.0	*−15.0	*−194.6
6	—	*2.9	*4.5	—	—	—	*0.7	*0.2	—

In Heywood case, the computed ML estimates (MLEs) are not real MLEs, but can be regarded as quasi-MLEs. Also, dashes shown in these tables indicate that the corresponding numbers of factors were not retained. One can see that the four factor model is the best for both the LG and LM models. In the four factor model, the estimates of factor loading matrices were fairly stable both cases of the LG and LM models, but the fourth factor's contribution to the variances of explained variables were too low to appropriately specify the volatility function on the fourth factor. Here attention was paid to the following two properties of an orthogonal rotation:

- Any matrix obtain by any orthogonal rotation is also an estimated factor loading matrix.
- An orthogonal rotation called the *Orthomax rotation* can equalize the factors' contributions to the variances of explained variables.

To explain the above properties, consider the following factor analysis model:

$$\mathbf{y}_n = \mu + A\mathbf{w}_n + \epsilon_n$$

where $\mathbf{y}_n$ is a K-dimensional process, ϵ is a K-dimensional constant vector, A is a $K \times d$ constant matrix,

$$\mathbf{w}_n \overset{\text{IID}}{\sim} N(0, I_d), \qquad \epsilon_n \overset{\text{IID}}{\sim} N(0, \Psi).$$

and $\mathbf{w}_n$ and ϵ_n are independent. Let O denote an $d \times d$ orthogonal matrix. Write $\mathbf{w}'_n = O'\mathbf{w}_n$ and $A' = AO$. It follows from $O'O = OO' = I_d$ that

$$\mathbf{y}_n = \mu + A'\mathbf{w}'_n + \epsilon_n,$$

where $\mathbf{w}'_n \overset{\text{IID}}{\sim} N(0, I_d)$ and $\mathbf{w}'_n$ and ϵ_n are independent. This implies that any matrix obtained by any orthogonal rotation is also an estimated factor loading matrix. Next, let $A' = (a'_{ij})$. An orthogonal rotation $O(\eta)$ defined by

$$\text{argmax}_{O(\eta)} \sum_{i=1}^{d} \sum_{j=1}^{K} {a'}_{ij}^{4} - \frac{\eta}{K} \sum_{i=1}^{d} \left(\sum_{j=1}^{K} a'_{ij} \right)^2$$

is called an *Orthomax rotation*. It is known that differences among factors' contributions become smaller as the parameter η becomes larger. Thus, Orthomax rotation $O(\eta)$ with sufficiently large parameter η can make the fourth factor's contribution sufficiently large.

Orthomax rotations $O(\eta)$ with various ηs were applied to the factor loading matrix estimates. In the result, a tendency was found that the shape of the estimated volatility function becomes unstable over subperiods as η increases. Therefore, the initial value of η was set sufficiently small and then gradually increased until the fourth factor's contribution became sufficiently high. The result showed that the shape of the volatility function become unstable over subperiods when the fourth factor's contribution becomes high enough. Hence, the number of common factors was reduced from four to three. Since the third factor's contribution seemed too low to appropriately specify the volatility functions on the third factor, Orthomax rotations were applied in the similar way. The results showed that by the Orthomax rotation $O(1.125)$, the third factor's contribution were high enough and the shapes of the estimated volatility functions were fairly stable as shown in Tables 4.3-4.7. Therefore, the number of factors was decided to be three.

Table 4.3. The 1st factor loading estimates for the LG model.

Time to	Subperiod							
maturity	1st	2nd	3rd	4th	5th	6th	7th	8th
0Y6M	.00165	.00185	.00188	.00192	.00149	.00158	.00194	.00115
1Y0M	.00328	.00372	.00452	.00351	.00200	.00328	.00299	.00195
1Y6M	.00370	.00415	.00618	.00391	.00266	.00393	.00333	.00228
2Y0M	.00416	.00458	.00726	.00441	.00340	.00475	.00368	.00289
3Y0M	.00520	.00556	.00865	.00534	.00453	.00636	.00452	.00423
4Y0M	.00609	.00651	.00983	.00597	.00560	.00752	.00533	.00544
5Y0M	.00673	.00714	.01041	.00644	.00669	.00817	.00596	.00642
6Y6M	.00767	.00774	.01083	.00680	.00821	.00940	.00657	.00740
8Y0M	.00822	.00782	.01111	.00694	.00965	.01031	.00709	.00783
9Y9M	.00861	.00813	.01148	.00700	.01131	.01128	.00756	.00821

4.3. **Specification of Volatility Function.** First, Tables 4.3 and 4.4 show that the first factor loading estimates are positive at $\tau = 0$, increase as τ increases, and seem to converge as τ tends to infinity. Thus, the volatility function for the first factor was specified as

$$b_1(t, \tau) = b_{11t} - b_{12t} e^{-\lambda_{1t}\tau} \tag{4.1}$$

where $b_{1t}, b_{2t}, \lambda_{1t} \geq 0$, and $b_{1t} \geq b_{2t}$.

Second, Tables 4.5 and 4.6 show that the second factor loading estimates once increase and then turn to decrease at around $\tau = 1$ as τ increases, and seem to converge as τ tends to infinity. Hence, the volatility function for the second factor was specified as

$$b_2(t, \tau) = b_{21t}(\tau - \tau_{2t}) e^{-\lambda_{2t}(\tau - \tau_{2t})} + b_{22t} \tag{4.2}$$

where $b_{21t}, \lambda_{2t} \geq 0$.

Third, Tables 4.7 and 4.8 show that the third factor loading estimates once increase and then turn to decrease at around $\tau = 4$ as τ increases, and seem to

Table 4.4. The 1st factor loading estimates for the LM model.

Time to maturity	Subperiod 1st	2nd	3rd	4th	5th	6th	7th	8th
0Y6M	.0327	.0366	.0322	.0324	.0228	.0223	.0277	.0180
1Y0M	.0607	.0694	.0743	.0563	.0287	.0444	.0427	.0325
1Y6M	.0696	.0737	.0950	.0608	.0375	.0527	.0472	.0377
2Y0M	.0757	.0788	.1100	.0668	.0467	.0638	.0525	.0469
3Y0M	.0933	.0926	.1275	.0791	.0615	.0856	.0645	.0666
4Y0M	.1074	.1060	.1418	.0865	.0749	.1004	.0753	.0833
5Y0M	.1155	.1137	.1466	.0908	.0880	.1079	.0830	.0959
6Y6M	.1278	.1192	.1465	.0925	.1063	.1221	.0887	.1069
8Y0M	.1294	.1164	.1466	.0915	.1213	.1317	.0942	.1096
9Y9M	.1910	.1155	.1463	.0886	.1394	.1404	.0979	.1114

Table 4.5. The 2nd factor loading estimates for the LG model.

Time to maturity	Subperiod 1st	2nd	3rd	4th	5th	6th	7th	8th
0Y6M	.00545	.00534	.00687	.00476	.00521	.00749	.00450	.00473
1Y0M	.00782	.00735	.00947	.00723	.00824	.01012	.00658	.00647
1Y6M	.00763	.00720	.00982	.00672	.00866	.00949	.00649	.00619
2Y0M	.00681	.00649	.00891	.00578	.00851	.00812	.00567	.00525
3Y0M	.00526	.00549	.00723	.00465	.00737	.00640	.00469	.00444
4Y0M	.00434	.00471	.00592	.00397	.00611	.00508	.00416	.00373
5Y0M	.00391	.00426	.00529	.00360	.00505	.00445	.00372	.00277
6Y6M	.00363	.00401	.00511	.00359	.00392	.00379	.00325	.00226
8Y0M	.00333	.00382	.00505	.00361	.00296	.00330	.00327	.00230
9Y9M	.00298	.00379	.00501	.00355	.00177	.00282	.00289	.00172

Table 4.6. The 2nd factor loading estimates for the LM model.

Time to maturity	Subperiod 1st	2nd	3rd	4th	5th	6th	7th	8th
0Y6M	.1117	.1036	.1175	.0811	.0802	.1073	.0639	.0739
1Y0M	.1453	.1359	.1549	.1159	.1175	.1399	.0950	.1043
1Y6M	.1459	.1273	.1508	.1041	.1208	.1296	.0929	.0991
2Y0M	.1248	.1117	.1349	.0874	.1165	.1109	.0817	.0828
3Y0M	.0942	.0919	.1064	.0686	.1006	.0874	.0671	.0685
4Y0M	.0757	.0771	.0851	.0573	.0827	.0687	.0586	.0567
5Y0M	.0665	.0684	.0747	.0506	.0675	.0591	.0514	.0418
6Y6M	.0598	.0619	.0695	.0487	.0514	.0489	.0435	.0334
8Y0M	.0516	.0567	.0670	.0473	.0375	.0416	.0431	.0330
9Y9M	.0442	.0538	.0646	.0447	.0217	.0344	.0371	.0245

converge as τ tends to infinity. Therefore, the volatility function for the second factor was specified as

$$b_3(t,\tau) = b_{31t}(\tau - \tau_{3t})e^{-\lambda_{3t}(\tau-\tau_{3t})} + b_{32t} \tag{4.3}$$

where $b_{31t}, \lambda_{3t} \geq 0$.

Table 4.7. The 3rd factor loading estimates for the LG model.

Time to maturity	Subperiod 1st	2nd	3rd	4th	5th	6th	7th	8th
0Y6M	.00095	.00145	.00130	.00124	.00067	.00097	.00105	.00033
1Y0M	.00150	.00264	.00315	.00282	.00113	.00172	.00198	.00018
1Y6M	.00174	.00320	.00409	.00360	.00117	.00219	.00237	.00040
2Y0M	.00211	.00385	.00513	.00412	.00163	.00273	.00258	.00074
3Y0M	.00288	.00456	.00608	.00416	.00239	.00381	.00311	.00124
4Y0M	.00315	.00470	.00610	.00393	.00308	.00432	.00356	.00157
5Y0M	.00313	.00455	.00532	.00355	.00330	.00416	.00338	.00145
6Y6M	.00204	.00361	.00434	.00292	.00246	.00282	.00267	.00088
8Y0M	.00138	.00289	.00346	.00249	.00124	.00177	.00201	.00037
9Y9M	.00108	.00233	.00261	.00238	.00000	.00066	.00151	−.00032

Table 4.8. The 3rd factor loading estimates for the LM model.

Time to maturity	Subperiod 1st	2nd	3rd	4th	5th	6th	7th	8th
0Y6M	.0192	.0281	.0226	.0212	.0105	.0144	.0145	.0053
1Y0M	.0256	.0486	.0521	.0450	.0165	.0240	.0276	.0029
1Y6M	.0302	.0554	.0629	.0558	.0170	.0306	.0328	.0066
2Y0M	.0361	.0648	.0774	.0628	.0230	.0382	.0359	.0119
3Y0M	.0493	.0745	.0902	.0620	.0331	.0532	.0434	.0195
4Y0M	.0537	.0753	.0890	.0572	.0421	.0599	.0492	.0237
5Y0M	.0521	.0712	.0760	.0503	.0444	.0571	.0460	.0216
6Y6M	.0322	.0547	.0597	.0398	.0326	.0379	.0350	.0126
8Y0M	.0199	.0426	.0463	.0329	.0163	.0237	.0258	.0053
9Y9M	.0141	.0330	.0337	.0302	.0006	.0089	.0186	−.0041

Remark 1. There are many studies[6] on principal component analysis applied to interest rate term structures. They have shown that components are usually found to have particular shapes. The first component is roughly flat, the second component is downward sloping, and the third component is hump-shaped. These results have been interpreted as follows. The first component is causes a parallel shift in the term structure, the second component causes the term structure to fit, and the third component causes the term structure to flex. However, our results based on factor analysis are quite different from those based on principal component analysis.

5. Estimation, Model Selection, and Diagnosis

In this section, the approximate extended LM models specified in the last section are tested using the data of Eurodollar future rates. First, they are estimated with the ML method, and then several models are selected based on the likelihood ratio (LR) test and the SBC. Finally, the selected models are tested.

The same data of Eurodollar future rates employed in the factor analysis were used. The object models of estimation were CEV(α) models with $\alpha = 0.0, 0.1, \cdots, 1.0$ and AV(β) models with $\beta = \infty, 0.9, 0.8, \cdots, 0.0$. To test the maintained hypothesis that α (resp. β) is constant in the CEV (resp. AV) models, *period-wise CEV* (resp. *period-wise AV*) *models* was introduced in which α (resp. β) is constant in each estimation subperiod and could change over subperiods. The period-wise

[6]Knez, Litterman, and Scheinkman [15], Litterman and Scheinkman [21], Steeley [28], *etc.*

CEV (resp. period-wise AV) model chosen by the ML method is called the *best period-wise CEV* (resp. *best period-wise AV*) *model*, and it is denoted by BPC (resp. BPA).

A set of initial parameters' estimates was chosen such that it fits the set of estimates obtained by the factor analysis. Also, a quasi-Newton method called the *Broyden-Fletcher-Goldfarb-Shanno method* (see Bertsekas [4]) was exploited as a method of computation to find the maximum of the log-likelihood function.

5.1. **Estimation Results.** The estimation results were obtained as shown in Tables 5.1 and 5.2. Among both of the classes of CEV and AV models, the maximum log-likelihood estimate of the LG model is the greatest in the 1st, 2nd, 4th, 7th, and 8th subperiods. In the results, the maximum log-likelihood estimate of the LG model is the greatest in the whole periods among both classes of CEV and AV models. Also, the maximum log-likelihood estimate of the BPC model is almost the same as that of the BPA model.

Table 5.1. Maximum log-likelihood estimates for CEV models.

α	Subperiod								Whole
	1st	2nd	3rd	4th	5th	6th	7th	8th	Period
0.0	4662.7	4926.3	4804.2	5282.0	4823.4	5200.1	4983.6	5233.5	39915.8
0.1	4662.3	4926.2	4804.7	5281.0	4823.7	5200.1	4983.5	5232.7	39914.1
0.2	4661.9	4926.1	4805.0	5279.5	4823.8	5200.1	4983.4	5231.7	39911.5
0.3	4661.4	4925.8	4805.1	5277.8	4823.9	5200.0	4983.2	5230.7	39908.0
0.4	4660.8	4925.5	4805.2	5275.9	4824.0	5199.9	4983.1	5229.6	39903.9
0.5	4660.2	4925.1	4805.1	5273.6	4823.9	5199.8	4982.9	5228.4	39899.0
0.6	4659.5	4924.7	4804.9	5271.4	4823.9	5199.6	4982.7	5227.2	39893.8
0.7	4658.7	4924.2	4804.6	5268.9	4823.8	5199.5	4982.5	5225.9	39888.2
0.8	4657.9	4923.7	4804.4	5266.3	4823.5	5199.2	4982.3	5224.6	39881.9
0.9	4657.1	4923.1	4804.0	5263.7	4823.4	5199.0	4982.0	5223.2	39875.4
1.0	4656.2	4922.5	4803.5	5261.0	4823.1	5198.7	4981.8	5221.7	39868.5
BPC	4662.7	4926.3	4805.2	5282.0	4824.0	5200.1	4983.6	5233.5	39917.3

Table 5.2. Maximum log-likelihood estimates for AV models.

β	Subperiod								Whole
	1st	2nd	3rd	4th	5th	6th	7th	8th	Period
∞	4662.7	4926.3	4804.2	5282.0	4823.4	5200.1	4983.6	5233.5	39915.8
0.9	4662.5	4926.3	4804.6	5281.2	4823.6	5200.1	4983.5	5232.9	39914.5
0.8	4662.4	4926.3	4804.6	5281.1	4823.6	5200.1	4983.5	5232.9	39914.5
0.7	4662.4	4926.3	4804.6	5281.0	4823.6	5200.1	4983.5	5232.8	39914.2
0.6	4662.3	4926.2	4804.7	5281.0	4823.6	5199.1	4983.5	5232.7	39913.9
0.5	4662.3	4926.2	4804.8	5280.5	4823.7	5199.1	4983.5	5232.5	39913.5
0.4	4662.2	4926.2	4804.8	5280.1	4823.7	5199.1	4983.4	5232.3	39912.8
0.3	4662.0	4926.2	4805.0	5279.5	4823.7	5199.1	4983.4	5231.9	39911.6
0.2	4661.7	4926.0	4805.1	5278.2	4823.8	5199.0	4983.3	5231.2	39909.3
0.1	4660.9	4925.6	4805.1	5275.0	4823.8	5198.8	4983.0	5229.6	39903.0
0.0	4656.2	4922.5	4803.5	5261.0	4823.1	5198.7	4981.8	5221.7	39868.5
BPA	4662.7	4926.3	4805.2	5282.0	4823.8	5200.1	4983.6	5233.5	39917.1

5.2. **Model Selection.** The procedure for model selection is as follows:

Step 1: Rank each model based on the SBC. Proceed to Step 2(1) if the BPC or BPA model is the best model, otherwise proceed to Step 2(2).

Step 2: (1) Conduct the following test using the LR test:

H_0: CEV(0) (resp. AV(∞)) model.

H_1: Period-wise CEV (resp. Period-wise AV) models.

Select only the BPC and BPA models if the LG model is rejected, otherwise select the LG model and proceed to Step 2.2.

(2) Execute the following test for each CEV(α_0) (resp. AV(β_0)) model using the LR test:

H_0: CEV(α_0) (resp. AV(β_0)) model.

H_1: CEV(α) (resp. AV(β)) model for $\alpha \neq \alpha_0$.

Select the CEV(α_0) (resp. AV(β_0)) model if it is not rejected, otherwise reject it. Also, select both of the BPC and BPA models if their ranks are higher than any other rejected model, otherwise reject them.

First, the Step 1 was executed, and the results shown in Table 5.3 were obtained. Thus, the following ranking was obtained:

Table 5.3. Models ranking.

α	SBC	β	SBC
0.0	−78784.6	∞	−78784.6
0.1	−78781.2	0.9	−78782.3
0.2	−78775.9	0.8	−78781.9
0.3	−78768.9	0.7	−78781.4
0.4	−78760.7	0.6	−78780.8
0.5	−78751.0	0.5	−78779.9
0.6	−78740.6	0.4	−78778.5
0.7	−78729.3	0.3	−78776.2
0.8	−78716.7	0.2	−78771.5
0.9	−78703.8	0.1	−78758.5
1.0	−78689.9	0.0	−78689.9

$$\begin{aligned}&\mathrm{LG} \succ \mathrm{AV}(0.9) \succ \mathrm{AV}(0.8) \succ \mathrm{AV}(0.7) \succ \mathrm{CEV}(0.1)\\ &\qquad \succ \mathrm{AV}(0.6) \succ \mathrm{AV}(0.5) \succ \mathrm{AV}(0.4) \succ \mathrm{AV}(0.3) \succ \mathrm{AV}(0.2) \succ \mathrm{BPC} \approx \mathrm{BPA}.\end{aligned}$$

Next, the Step 2.2 was proceeded since neither the BPC or the BPA model was the best model, and the following test was conducted for each CEV(α) (resp. AV(β)) model using the LR test:

H_0: CEV(0) (resp. AV(∞)) model.

H_1: CEV(α) (resp. AV(β)) model for $\alpha \neq \alpha_0$.

The LM model is rejected with a 0.1% significance level. The LM model is widely used among practitioners for pricing interest rate derivatives, but this result urges them to reconsider using the LM model. Also, CEV(α) models for $\alpha \geq 0.2$ and AV(β) models for $\beta \leq 0.5$ are rejected with a 5% significance level. Moreover, Since BPC $\approx$ BPA $<$ CEV(0.2) both of BPC and BPA models were rejected. In the end, the following six models were selected:

$$\mathrm{LG} \succ \mathrm{AV}(0.9) \succ \mathrm{AV}(0.8) \succ \mathrm{AV}(0.7) \succ \mathrm{CEV}(0.1) \succ \mathrm{AV}(0.6).$$

Here the parameters' estimates of the LG model which was selected as the best model are shown in Table 5.4. The unique factor estimates are sufficiently small as estimated in the preliminary factor analysis. In every model, the unique factor estimates were sufficiently small as estimated in the preliminary factor analysis, which suggests that the approximate extended LM models are good approximations of the extended LM models.

TABLE 5.4. Parameter estimates of the LG model.

	Subperiod							
	1st	2nd	3rd	4th	5th	6th	7th	8th
v_1	−0.56	−2.61	−2.03	2.17	−0.10	1.14	−2.60	3.72
v_2	−2.08	−0.67	0.19	−1.11	−2.67	−0.38	3.79	4.45
v_3	−1.99	−4.68	3.27	−1.41	−1.76	1.66	2.01	−2.45
b_{11}	0.00946	0.00795	0.01173	0.00729	0.05171	0.02436	0.00803	0.00858
b_{12}	0.00801	0.00703	0.01142	0.00610	0.05145	0.02366	0.00676	0.00545
b_{21}	0.01646	0.01141	0.01756	0.01347	0.01180	0.02295	0.00814	0.01050
b_{22}	0.00182	0.00344	0.00401	0.00323	0.00226	0.00023	0.00273	−.00107
b_{31}	0.00444	0.00404	0.00629	0.00422	0.00624	0.00488	0.00414	0.00621
b_{32}	−.00195	0.00113	0.00193	0.00139	−.00445	0.00087	0.00011	−.00241
τ_2	−0.190	−0.206	−0.220	−0.278	−0.216	−0.024	−0.224	0.183
τ_3	−1.190	−0.111	0.179	0.304	−1.483	−0.975	−0.486	−1.301
λ_1	0.255	0.327	0.662	0.524	0.024	0.052	0.248	0.327
λ_2	1.190	1.246	1.113	1.337	0.928	1.130	1.070	0.879
λ_3	0.262	0.315	0.405	0.536	0.211	0.248	0.308	0.345
ψ_1	0.00337	0.00213	0.00389	0.00252	0.00202	0.00319	0.00162	0.00182
ψ_2	0.00336	0.00148	0.00178	0.00160	0.00249	0.00198	0.00116	0.00108
ψ_3	0.00178	—	—	—	0.00129	—	—	—
ψ_4	—	0.00097	0.00104	0.00081	—	0.00124	0.00071	0.00063
ψ_5	0.00043	0.00074	0.00088	0.00049	0.00058	0.00072	0.00060	0.00053
ψ_6	—	—	—	—	—	—	—	—
ψ_7	0.00072	0.00065	0.00085	0.00043	0.00067	0.00068	0.00052	0.00049
ψ_8	0.00090	0.00071	0.00065	0.00029	0.00059	0.00058	0.00053	0.00060
ψ_9	—	—	—	—	—	—	—	—
ψ_{10}	0.00120	0.00107	0.00075	0.00039	0.00064	0.00094	0.00077	0.00074

5.3. **Test for Selected Models.** Finally, the assumption was tested that the common factors $\mathbf{w}_n$ in the approximate extended LM model are IID (Identically and Independently Distributed) with $N(0, I_3)$ for the selected six models. Note that the estimates of common factors $\mathbf{w_n}$ are computed from the estimation results. Let $\mathbf{x}_n$ be such that $\mathbf{x}_1 = \mathbf{w}_{11}, \mathbf{x}_2 = \mathbf{w}_{12}, \mathbf{x}_3 = \mathbf{w}_{13}, \mathbf{x}_4 = \mathbf{w}_{21}, \mathbf{x}_5 = \mathbf{w}_{22}$, and so on. Then $\mathbf{x}_n$ is IID with $N(0, 1)$. Let $\hat{\mathbf{x}}_n$ denote the estimates of $\mathbf{x}_n$.

The assumption of $\mathbf{x}_n \overset{\text{IID}}{\sim} N(0, 1)$ is tested by the following procedure.

Step 1: Conduct the following normality test under the assumption that $\hat{\mathbf{x}}$ is IID:

H$_0$: $\hat{\mathbf{x}}_n$ is normal.

H$_1$: $\hat{\mathbf{x}}_n$ is non-normal.

If H_0 is rejected, then reject the assumption of $\mathbf{x}_n \overset{\text{IID}}{\sim} N(0, 1)$, otherwise proceed to Step 2.

Step 2: Execute the following IID test using BDS test (Brock *et al.* [8]):

H$_0$: $\hat{\mathbf{x}}_n$ is IID.

H$_1$: $\hat{\mathbf{x}}_n$ is not IID.

If H_0 is rejected, then reject the assumption, otherwise proceed to Step 3.

Step 3: Conduct the following test:

H$_0$: $\hat{\mathbf{x}}_n$ is IID $N(0,1)$.

H$_1$: $\hat{\mathbf{x}}_n$ is IID $N(\mu,\sigma)$ where $(\mu,\sigma) \neq (0,1)$.

If H_0 is rejected, then reject the assumption, otherwise accept it.

There are many normality tests such as Geary's G test (Geary [12]), the Anderson-Darling test (Anderson and Darling [3]), Shapiro-Wilk test (Shapiro and Wilk [26]), and D'Agostino's D test (D'Agostino [9]). However, there does not any uniformly powerful test against any estrangement from normality. There are two well known measures on estrangements from normality, that is, the *skewness* m_3 which shows the skew of the distribution and the *kurtosis* m_4 which shows the thickness of the distribution's tail. The skewness and the kurtosis are defined by

$$m_3 = \frac{E[(\mathbf{x}-\mu)^3]}{\sigma^3}, \qquad m_4 = \frac{E[(\mathbf{x}-\mu)^4]}{\sigma^4}.$$

For any normal distribution, $m_3 = 0$ and $m_4 = 3$. A Monte Carlo simulation conducted in D'Agostino and Stephens [10] showed that the Shapiro-Wilk test is most powerful for the test $H_0 : m_3 = 0$ against $H_1 : m_3 \neq 0$, and that the D'Agostino's D test is most powerful for the test $H_0 : m_3 = 0, m_4 = 3$ against $H_1 : m_3 = 0, m_4 > 3$. The kurtoses of common factor estimates in all of the selected models were much greater than three. Therefore, the Shapiro-Wilk test and the D'Agostino's D test were employed. The results are as shown in Table 5.5. Both of the test results present that the normality of common factors is rejected for every model with a 0.01% significance level. Considering that the kurtoses of common factor estimates in all of the selected models are much greater than three, it is thought to be the main cause of these rejections that the distributions of common factors have much fatter tails than normal distribution.

Table 5.5. The result of normality tests for the selected models.

Model	Shapiro-Wilk test		D'Agostino's D test	
	W	p-value	D	p-value
LG	0.98787	< 0.0001	0.274701	< 0.0001
AV(0.9)	0.98793	< 0.0001	0.274743	< 0.0001
AV(0.8)	0.98791	< 0.0001	0.274738	< 0.0001
AV(0.7)	0.98790	< 0.0001	0.274737	< 0.0001
CEV(0.1)	0.98794	< 0.0001	0.274743	< 0.0001
AV(0.6)	0.98790	< 0.0001	0.274737	< 0.0001

6. Conclusion

The results of this paper presented that the distribution of change in each future LIBOR rate has fatter tails than normal distribution does. To make the distribution of change in each future LIBOR rate have fatter tails while keeping the favorable properties of the extended LM models, the following three extensions are thought to be promising. The first one is to replace the deterministic volatility in an extended LM model with a stochastic one. The second one is to introduce a jump process into an extended LM model. The third one is to conduct these two extensions together. Recently, a stochastic volatility LIBOR market model (Andersen and Ratcliffe [2]), jump-diffusion LIBOR market models (Glasserman and Kou [13], Kusuda [16]), and a stochastic volatility jump-diffusion LIBOR market model (Kusuda [18]) have been proposed.

Appendix A. Ito's Formula and Girsanov's Theorem

A.1. **Ito's Formula.** Let $X = (X^1, ..., X^d)'$ be a d-dimensional Ito process, and g be a real-valued C^2-function on $\mathbb{R}^d$. Then $g(X)$ is an Ito process in the form

$$g(X_t) = g(X_0) + \sum_{i=1}^{d} \int_0^t \frac{\partial}{\partial x_i} g(X_{s-})\, dX_s^i + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} g(X_{s-})\, d\langle X^i, X^j \rangle$$

where $\langle X^i, X^j \rangle$ is the quadratic covariation of X^i and X^j.

A.2. **Girsanov's Theorem.**

(1) Assume that there exists a $\mathcal{P}$-measurable process v satisfying the integrability condition

$$\int_0^t \|v_s\|^2\, ds < \infty$$

Define a probability measure $\tilde{P}$ on $(\Omega, \mathcal{F}, \mathbb{F})$ by $d\tilde{P} = \Lambda_{T^\dagger}\, dP$ where the stochastic process Λ is defined by the dynamics

$$\frac{d\Lambda_t}{\Lambda_{t-}} = -v_t \cdot dW_t \qquad \forall t \in [0, T^\dagger)$$

with $\Lambda_0 = 1$. If $E[\Lambda_t] = 1$ for every $t \in [0, T^\dagger]$, then it follows that:

   (a) The measure $\tilde{P}$ is equivalent to P.
   (b) The stochastic process given by $\tilde{W}_t = W_t + \int_0^t v_s\, ds$ is a Wiener process under $\tilde{P}$.

(2) Every probability measure equivalent to P has the structure above.

Appendix B. Definitions of Arbitrage

A *portfolio* $\theta = (\theta^0, \theta^1, \cdots, \theta^{K^\dagger})$ is a $(K^\dagger + 1)$-dimensional adapted process. The value process $V_t(\theta)$ of a portfolio θ is defined by

$$V_t(\theta) = B_t \theta^0 + \sum_{k=1}^{K^\dagger} B_t^k \theta_t^k.$$

Then definitions of *feasible portfolio* and *admissible portfolio* are given below.

(1) A *feasible portfolio at* $\mathbf{B}$ is an adapted process θ such that

$$\int_0^{T^\dagger} |B_s r_s^B|\, |\theta_s^0|\, ds < \infty, \quad \sum_{k=1}^{K^\dagger} \int_0^{T^\dagger} |B_s^k r_s^k|\, |\theta_s^k|\, ds < \infty, \quad \sum_{k=1}^{K^\dagger} \int_0^{T^\dagger} \left(\|B_s^k v_s^k\|\, |\theta_s^k| \right)^2 ds < \infty.$$

(2) An *admissible portfolio at* $\mathbf{B}$ is a feasible portfolio θ at $\mathbf{B}$ such that its discounted value process $\frac{V_t(\theta)}{B_t}$ is bounded below.

Definitions of *self-financing portfolio* and *arbitrage portfolio* are given in the following.

(1) A *self-financing portfolio at* $\mathbf{B}$ is an admissible portfolio at $\mathbf{B}$ such that the value process satisfies

$$V_t(\theta) = V_0(\theta) + \int_0^t \theta_s^0\, dB_s + \sum_{k=1}^{K^\dagger} \int_0^t \theta_s^k dB_s^k \qquad \forall t \in [0, T^\dagger].$$

(2) An *arbitrage portfolio* is an admissible self-financing portfolio θ such that there exists $T \in (0, T^\dagger]$ such that $V_0(\theta) \leq 0$ and $V_T(\theta) > 0$, or $V_0(\theta) < 0$ and $V_T(\theta) \geq 0$.

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