Fisher markets equilibrium in a linear exchange economy with an infinite dimensional commodity space

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Abstract

In this paper, we will investigate an existence problem of a general equilibrium of perfectly competitive Fisher markets in a linear exchange model with an infinite dimensional commodity space. We will indicate that finite dimensional approximation method in functional analysis guarantees the existence result in our settings.

Keywords: general equilibrium theory, existence problem, linear exchange model, Fisher markets, infinite dimensional commodity space

JEL Classification: C02, C62, D01, D41, D51

1 Introduction

In this paper, we will investigate an existence problem of a general equilibrium of perfectly competitive Fisher markets in a linear exchange model with an infinite dimensional commodity space.

Recently, Fisher markets model in general equilibrium analysis have been researched intensively in connection with computation of general equilibria[*1], and related to the the computaional aspect, Fisher markets model is usually constructed as a linear exchange model or a piecewise-linear exchange model[*2*3]. As with the positive and normative issue of economics, the existence of market equilibrium is an essential problem for the computation of equilibrium. Therefore, in the analysis of Fisher markets, the existence problem has been researched carefully and deeply. Our main concern in this paper is the extension of such existence issue to the infinite dimensional settings[*4].

Construction of Fisher markets model is the followings: There are (possibly, infinitely) many markets in an economy, which are assumed to be complete, and finitely many consumers and finitely many suppliers or merchants participate in the trades in each markets. Spending his/her wealth or income, each consumer purchases commodities in order to maximize his/her preference, and determines his/her demand schedule. On the other hand, suppliers' or merchant's supply schedule have been already deter-

[*1] For the very extensive details which include computaional aspects of Fisher markets model, see [15].
[*2] For the linear exchange model, see [11], [13], [15], and for the piecewise-linear exchange model, see [12], [15]
[*3] In the research area of Fisher markets model, linearlity is a crucial analityc tool.
[*4] Computational aspect in infinite dimensional settings is in itself very interesting issue.

mined before participating in the markets. Therefore, decision makers in Fisher markets are consumers only, and Fisher markets equilibrium is a pair of equilibrium price and equiribrium allocation which consists of equiribrium demand schedules, where equilibrium total demand schedule is basically equal to the total supply schedule. Fisher markets model in a linear exchange model or a piecewise-linear model is characterized by the linearlity or piecewise-linearlity of some kinds of economic primitives, in paticular, consumer's preference or utility[*5]. In this paper, we will treat a linear exchange model.

In infinite dimensional setting in our Fisher markets model, our commodity space is $\mathcal{L}^\infty$ and our price space is ba[*6]. For the detail, see next chapter. A commodity space $\mathcal{L}^\infty$ is known to be suitable for the analysis of dynamic economy or/and the analysis of uncertainty in an economy. In the following chapters, we will introduce our model and establish the existence result.

2 Fisher markets model in a linear exchange economy with an infinite dimensional commodity space

(S, Σ, μ) is a complete $\sigma-$finite measure space, where Σ is a σ-algebra on S and μ is a complete σ-finite positive measure on Σ. Let $\mathcal{L}^\infty$ be a set of all real valued essentially bounded measurable functions[*7] on (S, Σ, μ), which is equipped with a sup-norm $||\cdot||_\infty$[*8]. The norm dual of the Banach space $(\mathcal{L}^\infty, ||\cdot||_\infty)$ is a set of all bounded additive measures on the measurable space (S, Σ) absolutely continuous with respect to μ, which is denoted by ba. Then, the bilinear form of the dual pair $\langle \mathcal{L}^\infty, ba \rangle$ is defined by $\int x \, d\pi$ for any $\pi \in ba$ and $x \in \mathcal{L}^\infty$, which is denoted by $(\pi|x)$. A commodity space in our Fisher markets is $\mathcal{L}^\infty$ and a price space in our Fisher markets is ba. Then, the value form of a commodity $x \in \mathcal{L}^\infty$ under the given price $\pi \in ba$ is naturally defined by $(\pi|x)$.

The topology of a commodity space $\mathcal{L}^\infty$ is the Mackey topology $\tau(\mathcal{L}^\infty, \mathcal{L}^1)$[*9], where $\mathcal{L}^1$ is a set of all integrable functions on (S, Σ, μ)[*10]. The bilinear form of the dual pair $\left\langle \mathcal{L}^\infty, \mathcal{L}^1 \right\rangle$ is defined by $\int \alpha \cdot x \, d\mu$ for $x \in \mathcal{L}^\infty$ and $\alpha \in \mathcal{L}^1$, which will be denoted by $\langle \alpha | x \rangle$[*11]. On the other hand, the topology of a price space ba is induced by the bounded variation norm $|| \cdot ||_{ba}$[*12][*13].

Throughout this paper, Fisher markets are assumed to be perfectly competitive and complere ones.

[*5] In the former, his/her preference or utility is assumed to be linear. In the latter, his/her utility is assumed to be piecewise-linear.

[*6] For general equilibrium theory with infinite dimensional commodity space, see [3], [16]. We owe to the result of [3] particularly.

[*7] A measurable function f is called essentially bounded if there exists $c \in \mathbb{R}$ such that $|f(s)| < c$ for a.e. $s \in S$.

[*8] $|| \cdot ||_\infty$ is defined by $||f||_\infty = \sup\{|f(x)| : a.e. \, x \in S\}$ for any $f \in \mathcal{L}^\infty$. Under this norm, $\mathcal{L}^\infty$ becomes a complete normed space, in other words, a Banach space.

[*9] The Mackey topology $\tau(\mathcal{L}^\infty, \mathcal{L}^1)$ has a natural interpretation from the viewpoint of the consumer's preference. That is "myopic preference" [3].

[*10] $\mathcal{L}^1$ is a Banach space under the norm $|| \cdot ||_1$, which is defined by $||f||_1 = \int |x| \, d\mu$ for any $f \in \mathcal{L}^1$.

[*11] The Mackey topology $\tau(\mathcal{L}^\infty, \mathcal{L}^1)$ is the strongest toplogy of $\mathcal{L}^\infty$ under which the linear functional $\langle \alpha, \cdot \rangle : \mathcal{L}^\infty \longrightarrow \mathbb{R}$ is continuous for any $\alpha \in \mathcal{L}^1$.

[*12] ba is a Banach space under the norm $|| \cdot ||_{ba}$.

[*13] $|| \cdot ||_{ba}$ is defined by $||\pi||_{ba} = \sup_\delta\{\sum |\pi(A_k)| : A_k \in \delta\}$ for any $\pi \in ba$, where δ is a finite subset of Σ whose elements are mutually disjoint.

Pratically, an economy $\mathcal{E}$ consists of a set of economic primitives, $\{(X_i, \succsim_i, m_i, (\mathcal{N}_i^{k_i})_{k_i \in \mathcal{K}_i})_{i \in I}, (\mathcal{S}_j)_{j \in J}\}$:

- $I = \{1, \cdots, i, \cdots, m\}$, $m < \infty$, is an index set of consumers,
  - $(X_i, \succsim_i, m_i, (\mathcal{N}_i^{k_i})_{k_i \in \mathcal{K}_i})_{i \in I}$ is a set of consumer i's characteristics,
    - $X_i \subset \mathcal{L}_+^\infty$ is a consumption set of i, where $\mathcal{L}_+^\infty$ is a positive cone of $\mathcal{L}^\infty$ [*14].
    - $\succsim_i \subset X_i \times X_i$ is a rational preference relation of consumer i,
    - m_i, $m_i > 0$, is an income of consumer i,
    - $(\mathcal{N}_{k_i})_{k_i \in \mathcal{K}_i}$ is a family of napsack constraint functionals of consumer i,
      - $\mathcal{K}_i = \{1, \cdots, k_i, \cdots, K_i\}, K_i < \infty$, is an index set of i's napsack constraint functionals,
      - napsack constraint functional $\mathcal{N}_{k_i} : X_i \longrightarrow \mathbb{R}$ represents some kind of additional indivisually specific constraint of consumer i in the decision making under his/her market budget constraint[*15],
- $J = \{1, \cdots, j, \cdots, n\}$, $n < \infty$, is an index set of suppliers or merchants,
  - $S_j \in \mathcal{L}_+^\infty$ is a given supply schedule of supplier or merchant j.

In the following, we will require several kinds of linearity assumptions for economic primitives. In Fisher markets, consumer i's budget constraint under given price $\pi \in ba$ is

$$\mathcal{B}_i^F(\pi) \equiv \{x \in X_i : (\pi|x) \leqq m_i \wedge \mathcal{N}_{k_i}(x) \leqq 0 \, for \, any \, k_i \in \mathcal{K}_i\}.$$

Assumption 1. (*linearity of napsack constraint*[*16]) *: For any $k_i \in \mathcal{K}_i$, there exist unique $\pi^{k_i} \in \mathcal{L}^1$ and a constant $M^{k_i} \in \mathbb{R}_+$ such that $\mathcal{K}_{k_i}(x) = \langle \pi^{k_i} | x \rangle - M^{k_i}$.*

Under Assumption1, restatement of consumer i's budget constraint under given price $\pi \in ba$ is

$$\begin{aligned}\mathcal{B}_i^F(\pi) &\equiv \{x \in X_i : (\pi|x) \leqq m_i, \wedge \langle \pi^{k_i}|x \rangle \leqq M^{k_i} \, for \, any \, k_i \in \mathcal{K}_i\} \\ &= \{x \in X_i : (\pi|x) \leqq m_i\} \cap \{x \in \mathcal{L}^\infty : \langle \pi^{k_i}|x \rangle \leqq M^{k_i} \, for \, any \, k_i \in \mathcal{K}_i\} \\ &= \{x \in X_i : (\pi|x) \leqq m_i\} \cap \bigcap_{k_i \in \mathcal{K}_i} \{x \in \mathcal{L}^\infty : \langle \pi^{k_i}|x \rangle \leqq M^{k_i}\}\end{aligned}$$

In the following, we will denote $\{x \in X_i : (\pi|x) \leqq m_i\}$, $\{x \in \mathcal{L}^\infty : \langle \pi^{k_i}|x \rangle \leqq M^{k_i} \, for \, any \, k_i \in \mathcal{K}_i\} \, and \, \{x \in \mathcal{L}^\infty : \langle \pi^{k_i}|x \rangle \leqq M^{k_i}\}$ by $\mathcal{B}_i(\pi)$, Ξ_i, $and \, \Xi_{k_i}$ respectively.

Consumer i chooses the maximal element for his/her preference relation $\succsim_i$ under his/her budget constraint $\mathcal{B}_i^F(\pi)$. Thus, consumer i's decision making under Fisher markets is represented by his/her demand relation under given price $\pi \in ba$

$$D_i^F(\pi) \equiv \{x \in \mathcal{B}_i^F(\pi) : (x, z) \in \succsim_i \, for \, any \, z \in \mathcal{B}_i^F(\pi)\}.$$

Then, we will make the following asuumption for i's preference relation.

[*14] A natural ordering of $\mathcal{L}^p$, where $p = 1$, $or \, \infty$ is defuned by: $x \geqq y \iff x(s) \geqq y(s) \, a.e. \, s \in S$.
[*15] For the detail, see [15].
[*16] More correctly, affine property of napsack constraint.

Assumption 2. (*linear representation of i's preference relation*) : *Consumer i's preference relation* $\succsim_i$ *is represented by a linear utility function* $\mathcal{U}_i : X_i \longrightarrow \mathbb{R}$, *which is defined by* $x \in X_i \longmapsto \mathcal{U}_i(x) = \langle u_i | x \rangle$ *for some unique* $u_i \in \mathcal{L}^1$.

Let $\succ_i$ be i's strict preference relation defined by $\succsim_i$, and $\sim_i$ be i's indifference relation defined by $\succsim_i$. Then, note that for any x, $y \in X_i$,

$$(x, y) \in \succ_i \iff \mathcal{U}_i(x) > \mathcal{U}_i(y),\ and\ (x, y) \in \sim_i \iff \mathcal{U}_i(x) = \mathcal{U}_i(y).$$

Moreover, $\mathcal{U}_i$ is a concave κ-continuous function on X_i, where κ is a topology on $\mathcal{L}^\infty$ which is weaker than $\tau(\mathcal{L}^\infty, \mathcal{L}^1)$ and stronger than weak topology $\sigma(\mathcal{L}^\infty, \mathcal{L}^1)$*17.

Under the above economic circumstaces , Fisher markets quasi-equilibrium is defined in the following way.

Definition 1. $(\pi, (x_i)_{i\in I}) \in ba \setminus \{0\} \times \prod_{i\in I} X_i$ *is said to be a Fisher markets quasi-equilibrium of an economy* $\mathcal{E}$ *if it satisfies the following conditions*

$$(i)\ x_i \in D_i^F(\pi)\ for\ any\ i \in I,\ \ (ii) \sum_{i\in I} x_i \leqq \sum_{j\in J} \mathcal{S}_j,\ a.e.\ s \in S.$$

Additionally, we will assume:

Assumption 3. $(i)\ X_i = \mathcal{L}^\infty_+$, $(ii)\ \mathcal{S}_j \in \mathcal{L}^\infty_{++}$, *where* $\mathcal{L}^\infty_{++} = \{f \in \mathcal{L}^\infty : a.e.\ s \in S,\ f(s) > 0\}$*18.

Assumption 4. (*existence of some kinds of desirable directions not binded by napsack constraints*) *For any* $i \in I$, *there exists measurable set* $\Omega_i \in \Sigma$ *with* $\mu(\Omega_i) > 0$ *such that* $(x + \alpha\chi_i^{\Omega_i}, x) \in \succ_i$ *for any* $x \in X_i$ *and* $\alpha > 0$, *and such that* $z + \beta\chi_i^{\Omega_i} \in \Xi_i$ *for any* $z \in \Xi_i$ *and* $\beta > 0$, *where* $\chi_i^{\Omega_i}$ *is an indicator function of* Ω_i.

Under these assumptions, our main result is the following theorem.

Theorem 1. *There exists a Fisher markets quasi-equilibrium* $(\pi, (x_i)_{i\in I}) \in ba \setminus \{0\} \times \prod_{i\in I} X_i$ *under all Assumptions in the above.*

3 Existence result in the finite dimensional truncation of economy $\mathcal{E}$

Let $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ be a family of all finite dimensional subspaces of $\mathcal{L}^\infty$ which include all elements of $(\mathcal{S}_j)_{j\in J}$ and $(\chi_i^{\Omega_i})_{i\in I}$ *19, and define a subeconomy $\mathcal{E}^{\mathcal{F}_\lambda} = \{(X_i^{\mathcal{F}_\lambda}, \succsim_i^{\mathcal{F}_\lambda}, m_i^{\mathcal{F}_\lambda}, (\mathcal{N}_{k_i}^{\mathcal{F}_\lambda})_{k_i\in\mathcal{K}_i})_{i\in I}, (\mathcal{S}_j^{\mathcal{F}_\lambda})_{j\in J}\}$ of economy $\mathcal{E}$ for any $\lambda \in \Lambda$ in the following way.

*17 The weak topology $\sigma(\mathcal{L}^\infty, \mathcal{L}^1)$ is the weakest toplogy of $\mathcal{L}^\infty$ under which the linear functional $\langle \alpha, \cdot \rangle : \mathcal{L}^\infty \longrightarrow \mathbb{R}$ is continuous for any $\alpha \in \mathcal{L}^1$.

*18 $\mathcal{B}_i^F(\pi) \neq \emptyset$ for any non-zero $\pi \in ba$ under the Assumption 3 (i).

*19 Note that $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ constitutes a directed set by set inclusion relations. For the details about a directed set (and related notion of net), see [9].

- For any $i \in I$, $X_i^{\mathcal{F}_\lambda} = X_i \cap \mathcal{F}_\lambda$, $\succsim_i^{\mathcal{F}_\lambda} = \succsim_i \cap (\mathcal{F}_\lambda \times \mathcal{F}_\lambda)$[*20], $m_i^{\mathcal{F}_\lambda} = m_i$, $\mathcal{N}_{k_i}^{\mathcal{F}_\lambda} = \mathcal{N}_{k_i} \circ inc^{\mathcal{F}_\lambda}$, where $inc^{\mathcal{F}_\lambda} : \mathcal{F}_\lambda \longrightarrow \mathcal{L}^\infty$ is an inclusion mapping.
- For any $j \in J$, $\mathcal{S}_j^{\mathcal{F}_\lambda} = \mathcal{S}_j$

Note that any finite dimensional topological vector subspace is homeomorphic to the same dimensional Euclidean space. Also, by Assumption 1, $\mathcal{K}_{k_i}^{\mathcal{F}_\lambda}(x) = \langle \pi^{k_i} | x \rangle - M^{k_i}$ for any $x \in \mathcal{F}_\lambda$ is an affine mapping on a finite dimensional subspace $\mathcal{F}_\lambda$, and hence, we can regard $\{x \in \mathcal{F}_\lambda : \langle \pi^{k_i} | x \rangle \leqq M^{k_i}\}$ as a closed half space in $\mathcal{F}_\lambda$.

Let $\mathcal{F}_\lambda^*$ be a dual space of $\mathcal{F}_\lambda$. Since $\mathcal{F}_\lambda$ is a finite dimensional topological vector space, $\mathcal{F}_\lambda^*$ coinsides with $\mathcal{F}_\lambda$ and the bilinear form on $\mathcal{F}_\lambda^* \times \mathcal{F}_\lambda$ can be regarded as an inner product between them. Abusing notations, we will denote the bilinear form by $(p|x)$ for any $p \in \mathcal{F}_\lambda^*$ and $x \in \mathcal{F}_\lambda$. We can define $\mathcal{B}_i^{\mathcal{F}_\lambda}(p)$, $\Xi_i^{\mathcal{F}_\lambda}$, $and\, \Xi_{k_i}^{\mathcal{F}_\lambda}$ respectively by $\{x \in X_i{}^{\mathcal{F}_\lambda} : (p|x) \leqq m_i\}$, $\{x \in \mathcal{F}_\lambda : \langle \pi^{k_i} | x \rangle \leqq M^{k_i}\, for\, any\, k_i \in \mathcal{K}_i\}$ and $\{x \in \mathcal{F}_\lambda : \langle \pi^{k_i} | x \rangle \leqq M^{k_i}\}$. And, consumer i's budget constraint $\mathcal{B}_i^{\mathcal{F}_\lambda}(p)$ and his/her demand relationin $D_i^{\mathcal{F}_\lambda}(p)$ in the subeconomy $\mathcal{E}^{\mathcal{F}_\lambda}$ under given price $p \in \mathcal{F}_\lambda^* \setminus \{0\}$ is respectively

$$\mathcal{B}_i^{\mathcal{F}_\lambda}(p) \equiv \{x \in X_i{}^{\mathcal{F}_\lambda} : (p|x) \leqq m_i, \wedge \langle \pi^{k_i} | x \rangle \leqq M^{k_i}\, for\, any\, k_i \in \mathcal{K}_i\}$$
$$D_i^{\mathcal{F}_\lambda}(p) \equiv \{x \in \mathcal{B}_i^{\mathcal{F}_\lambda}(\pi) : (x, z) \in, \succsim_i^{\mathcal{F}_\lambda}\ \ for\, any\, z \in \mathcal{B}_i^{\mathcal{F}_\lambda}(\pi)\}.$$

In the following arguments, we will restrict a price set to a normalized price set $\Delta^{\mathcal{F}_\lambda} \equiv \{p \in \mathcal{F}_\lambda^* : (p|\mathcal{S}) = M \wedge (p|d) \geqq 0\, for\, any\, d \in \mathcal{F}_{\lambda+}\}$, where $\mathcal{S} = \sum_{j\in J} \mathcal{S}_j\ and\ M = \sum_{i \in I} m_i > 0$[*21]. Clearly, $\Delta^{\mathcal{F}_\lambda}$ is a nonempty, convex and compact subset of $\mathcal{F}_{\lambda+}$[*22].

Under the above descriptions, Fisher markets quasi-equilibrium of the subeconomy $\mathcal{E}^{\mathcal{F}_\lambda}$ is defined in the following way, and existing result holds in the subeconomy $\mathcal{E}^{\mathcal{F}_\lambda}$.

Definition 2. $(p, (x_i)_{i\in I}) \in \Delta^{\mathcal{F}_\lambda} \times \prod_{i \in I} X_i^{\mathcal{F}_\lambda}$ *is said to be a Fisher markets quasi-equilibrium of the subeconomy* $\mathcal{E}^{\mathcal{F}_\lambda}$ *if it satisfies the following conditions*

$$(i)\, x_i \in D_i^{\mathcal{F}_\lambda}(p)\, for\, any\, i \in I,\ (ii) \sum_{i\in I} x_i \leqq \sum_{j \in J} \mathcal{S}_j,\ a.e.\, s \in S.$$

Proposition 1. *For a fixed* $\mathcal{F}_\lambda$*, there exists a Fisher markets quasi-equilibrium* $(p, (x_i)_{i\in I}) \in \Delta^{\mathcal{F}_\lambda} \times \prod_{i\in I} X_i^{\mathcal{F}_\lambda}$ *of subeconomy* $\mathcal{E}^{\mathcal{F}_\lambda}$ *under all Assumptions in the previous chapter.*

In the followings, based on [6] [7], we will give a proof of the Proposition 1[*23]. For the purpose, we will truncate a fixed subeconomy $\mathcal{E}^{\mathcal{F}_\lambda}$ again by a convex nonempty compact cube $C_\theta = \{x \in \mathcal{F}_{\lambda+} : ||x||_\infty \leqq \theta\}$, where $\theta \in \mathbb{R}_{++}$ is a constant such that $\max\{\sum_{j\in J} ||\mathcal{S}_j||_\infty, 1\} < \theta$[*24], and we can define a subeconomy

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[*20] Based on $\succsim_i^{\mathcal{F}_\lambda}$, $\succ_i^{\mathcal{F}_\lambda}$ is defined by $\succ_i \cap(\mathcal{F}_\lambda \times \mathcal{F}_\lambda)$, and $\sim_i^{\mathcal{F}_\lambda}$ is defined by $\sim_i \cap(\mathcal{F}_\lambda \times \mathcal{F}_\lambda)$.

[*21] Note that $\mathcal{F}_{\lambda+} = \mathcal{L}_+^\infty \cap \mathcal{F}_\lambda$, and that $\mathcal{S} \in int_{||\cdot||_\infty} \mathcal{L}_+^\infty$ by Assumption3 (ii).

[*22] For the compactness of $\Delta^{\mathcal{F}_\lambda}$, it follows from Banach-Alaoglu theorem [17], [18]. See Lemma5 and the related discussion in the later section, too.

[*23] D. M. Jalota, et al. have already given a proof of a existing result of a Fisher markets equilibrium with a finite dimensinal commodity space, which is an Euclidean, in linear exchange model [15]. Our proof is a little bit different from theirs reflecting the difference of a model-constructions, particurely, topological and normalization settings.

[*24] Therefore, C_θ includes all the elements of $(\mathcal{S}_j)_{j\in J}$ and $(\chi_i^{\Omega_i})_{i \in I}$. Note that $||\chi_i^{\Omega_i}||_\infty = 1$ for any $i \in I$. Moreover, $X_i{}^{\mathcal{C}_\theta} = \mathcal{C}_\theta \cap X_i^{\mathcal{F}_\lambda} = \mathcal{C}_\theta$ in the below.

$\mathcal{E}^{\mathcal{C}_\theta} = \{(X_i^{\mathcal{C}_\theta}, \succsim_i^{\mathcal{C}_\theta}, m_i^{\mathcal{C}_\theta}, (\mathcal{N}_{k_i}^{\mathcal{C}_\theta})_{k_i \in \mathcal{K}_i})_{i \in I}, (\mathcal{S}_j^{\mathcal{C}_\theta})_{j \in J}\}$ of $\mathcal{E}^{\mathcal{F}_\lambda}$ in the same way as the definition of $\mathcal{E}^{\mathcal{F}_\lambda}$*25. Then, consumer i's budget constraint $\mathcal{B}_i^{\mathcal{C}_\theta}(p)$ and his/her demand relationin $D_i^{\mathcal{C}_\theta}(p)$ in the a subeconomy $\mathcal{E}^{\mathcal{C}_\theta}$ under given price $p \in \Delta^{\mathcal{F}_\lambda}$ is respectively

$$\mathcal{B}_i^{\mathcal{C}_\theta}(p) \equiv \mathcal{B}_i^{\mathcal{F}_\lambda}(p) \cap \mathcal{C}_\theta,$$
$$D_i^{\mathcal{C}_\theta}(p) \equiv \{x \in \mathcal{B}_i^{\mathcal{C}_\theta}(p) : (x, z) \in \succsim_i^{\mathcal{C}_\theta} \ for\, any\, z \in \mathcal{B}_i^{\mathcal{C}_\theta}(p)\}.$$

Definition 3. *$(p, (x_i)_{i \in I}) \in \Delta^{\mathcal{F}_\lambda} \times \prod_{i \in I} X_i^{\mathcal{C}_\theta}$ is said to be Fisher markets quasi-equilibrium of the subeconomy $\mathcal{E}^{\mathcal{C}_\theta}$ if it satisfies the following conditions*

$$(i)\, x_i \in D_i^{\mathcal{C}_\theta}(p)\, for\, any\, i \in I,\ (ii) \sum_{i \in I} x_i \leqq \sum_{j \in J} \mathcal{S}_j,\ a.e.\, s \in S.$$

Proposition 2. *For the $\mathcal{C}_\theta$, there exists a Fisher markets quasi-equilibrium $(p, (x_i)_{i \in I}) \in \Delta^{\mathcal{F}_\lambda} \times \prod_{i \in I} X_i^{\mathcal{C}_\theta}$ of subeconomy $\mathcal{E}^{\mathcal{C}_\theta}$ under all Assumptions in the previous chapter.*

First, we will give a proof of Propsition 2, and show that this result implies Proposition 1.

Lemma 1. *A budget relation $\mathcal{B}_i^{\mathcal{C}_\theta} : \Delta^{\mathcal{F}_\lambda} \longrightarrow X_i^{\mathcal{C}_\theta}$ is a continuous relation.*

Proof. $\mathcal{B}_i^{\mathcal{C}_\theta}$ is nonempty-, convex- and compact-valued since $0 \in \mathcal{B}_i^{\mathcal{C}_\theta}(p)$ for any $p \in \Delta^{\mathcal{F}_\lambda}$ and $\mathcal{C}^\theta$ is a convex and compact subset of $\mathcal{F}_\lambda$.

Upper semicontinuity of $\mathcal{B}_i^{\mathcal{C}_\theta}$: It suffices to show that $\mathcal{B}_i^{\mathcal{C}_\theta}$ has a closed graph. Take a sequence $\{(p^k, x^k)\} \subset \Delta^{\mathcal{F}_\lambda} \times \mathcal{B}_i^{\mathcal{C}_\theta}(p^k)$ such that $(p^k, x^k) \longrightarrow (p, x) \in \Delta^{\mathcal{F}_\lambda} \times \mathcal{F}_\lambda$ in order to show that $x \in \mathcal{B}_i^{\mathcal{C}_\theta}(p)$. Suppose that $x \notin \mathcal{B}_i^{\mathcal{C}_\theta}(p)$. Then, $(p|x) > m_i$*26 and, hence, $(p^k|x^k) > m_i$ for all k large enough, which leads to a contradiction.

Lower semicontinuity of $\mathcal{B}_i^{\mathcal{C}_\theta}$: Take a sequence $\{p^k\} \subset \Delta^{\mathcal{F}_\lambda}$ such that $p^k \longrightarrow p \in \Delta^{\mathcal{F}_\lambda}$, and $x \in \mathcal{B}_i^{\mathcal{C}_\theta}(p)$ in order to show that there exists a sequence $\{x^k\} \subset \mathcal{B}_i^{\mathcal{C}_\theta}(p^k)$ such that $x^k \longrightarrow x$. If $(p|x) < m_i$, $(p^k|x) < m_i$ for all k large enough. Fix such a k_0 among ones and define $\{x^k\}$ by

$$x^k = \begin{cases} x & : k \geqq k_0 \\ a^k \in \mathcal{B}_i^{\mathcal{C}_\theta}(p^k) & : k < k_0 \end{cases}$$

, where $a^k \in \mathcal{B}_i^{\mathcal{C}_\theta}(p^k)$ is some arbitralily fixed element of $\mathcal{B}_i^{\mathcal{C}_\theta}(p^k)$. This sequence is desired one. If $(p|x) = m_i (> 0)$, note thar $(1-t)x + t0 = (1-t)x \in \mathcal{B}_i^{\mathcal{C}_\theta}(p)$ for any $t \in [0, 1)$ and that $(p|(1-t)x) < m_i$. Therefore, $(p^k|(1-t)x) < m_i$ for any $t \in [0, 1)$ and for all k large enough. Fix such a k_0 among ones and define $\{x^k\}$ by

$$x^k = \begin{cases} (1 - \frac{1}{k})x & : k \geqq k_0 \\ a^k \in \mathcal{B}_i^{\mathcal{C}_\theta}(p^k) & : k < k_0 \end{cases}$$

, where $a^k \in \mathcal{B}_i^{\mathcal{C}_\theta}(p^k)$ is some arbitralily fixed element of $\mathcal{B}_i^{\mathcal{C}_\theta}(p^k)$*27. This sequence is desired one. □

*25 Based on $\succsim_i^{\mathcal{C}_\theta}$, $\succ_i^{\mathcal{C}_\theta}$ is defined by $\succ_i^{\mathcal{F}_\lambda} \cap (C_\theta \times C_\theta)$, and $\sim_i^{\mathcal{C}_\theta}$ is defined by $\sim_i^{\mathcal{F}_\lambda} \cap (C_\theta \times C_\theta)$.

*26 Note the closedness of related primitives.

*27 In both cases, note the closedness and convexities of related primitives.

Lemma 2. *A demand relation* $\mathcal{D}_i^{\mathcal{C}_\theta} : \Delta^{\mathcal{F}_\lambda} \longrightarrow X_i^{\mathcal{C}_\theta}$ *is a nonempty-, convex- and compact-valued upper semicontinuous relation.*

Proof. $\mathcal{D}_i^{\mathcal{C}_\theta}$ is clearly nonempty-, convex- and compact-valued by assumptions in the previous chapter and Lemm 1. Take a sequence $\{(p^k, d^k)\} \subset \Delta^{\mathcal{F}_\lambda} \times \mathcal{D}_i^{\mathcal{C}_\theta}(p^k)$ such that $(p^k, d^k) \longrightarrow (p, d) \in \Delta^{\mathcal{F}_\lambda} \times \mathcal{F}_\lambda$ in order to show that $d \in \mathcal{D}_i^{\mathcal{C}_\theta}(p)$. Note that $d \in \mathcal{B}_i^{\mathcal{C}_\theta}(p)$ by Lemma 1. Suppose that $d \notin \mathcal{D}_i^{\mathcal{C}_\theta}(p)$. Then, there exists $x \in \mathcal{B}_i^{\mathcal{C}_\theta}(p)$ such that $(x, d) \in \succ_i^{\mathcal{C}_\theta}$. By the continuity of preferences and similar argument of Lemma 1, $(1 - \frac{1}{k})x \in \mathcal{B}_i^{\mathcal{C}_\theta}(p^k)$ and $((1 - \frac{1}{k})x, d^k) \in \succ_i^{\mathcal{C}_\theta}$ for all k large enough, which leads to a contradiction. □

Define an excess demand relation $\zeta^{\mathcal{C}_\theta} : \Delta^{\mathcal{F}_\lambda} \longrightarrow \mathcal{C}_\theta$ by $\zeta^{\mathcal{C}_\theta}(p) = \sum_{i\in I} \mathcal{D}_i^{\mathcal{C}_\theta}(p) - \sum_{j\in J} \mathcal{S}_j$ [*28]. By Lemma 1, $\zeta^{\mathcal{C}_\theta}$ is a nonempty- convex- and compact- valued upper semicontinuous relation. Next, define a price adjustment relation $\mu^{\mathcal{C}_\theta} : \mathcal{C}_\theta \longrightarrow \Delta^{\mathcal{F}_\lambda}$ by $\mu^{\mathcal{C}_\theta}(z) = \{p \in \Delta^{\mathcal{F}_\lambda} : (p|z) = \max_{q\in\Delta_{\mathcal{F}_\lambda}} (q|z)\}$ for any $z \in \mathcal{C}_\theta$. Since $(\cdot|z) : \Delta^{\mathcal{F}_\lambda} \longrightarrow \mathbb{R}$ is continuous, clearly, $\mu^{\mathcal{C}_\theta}$ is a nonempty-, convex- and compact-valued upper semicontinuous relation. Therefore, the relation $\psi^{\mathcal{C}_\theta} : \mathcal{C}_\theta \times \Delta^{\mathcal{F}_\lambda} \longrightarrow \mathcal{C}_\theta \times \Delta^{\mathcal{F}_\lambda}$, which is defined by $\psi^{\mathcal{C}_\theta}(z, p) = \zeta^{\mathcal{C}_\theta}(p) \times \mu^{\mathcal{C}_\theta}(z)$ for any $(z, p) \in \mathcal{C}_\theta \times \Delta^{\mathcal{F}_\lambda}$, satisfies all of the conditions in Kakutani's fixed point theorem[*29].

Lemma 3. *A fixed point* $(z^*, p^*) \in \psi^{\mathcal{C}_\theta}(z^*, p^*)$ *constituties a Fisher markets quasi-equilibrium of the subeconomies* $\mathcal{E}^{\mathcal{C}_\theta}$*, and* $\mathcal{E}^{\mathcal{F}_\lambda}$*.*

Proof. $(z^*, p^*) \in \psi^{\mathcal{C}_\theta}(z^*, p^*) = \zeta^{\mathcal{C}_\theta}(p^*) \times \mu^{\mathcal{C}_\theta}(z^*)$ implies that there exists $d_i^* \in D_i^{\mathcal{C}_\theta}(p^*)$ for any $i \in I$ such that $z^* = \sum_{i\in I} d_i^* - \sum_{j\in J} \mathcal{S}_j$ and that $(p^*| \sum_{i\in I} d_i^* - \sum_{j\in J} \mathcal{S}_j) \geqq (q| \sum_{i\in I} d_i^* - \sum_{j\in J} \mathcal{S}_j)$ for any $q \in \Delta^{\mathcal{F}_\lambda}$. By Assumption 4, $(p^*|d_i^*) = m_i$, and hence, $(p^*| \sum_{i\in I} d_i^* - \sum_{j\in J} \mathcal{S}_j) = 0$. Therefore, $0 \geqq (q| \sum_{i\in I} d_i^* - \sum_{j\in J} \mathcal{S}_j)$ for any $q \in \Delta^{\mathcal{F}_\lambda}$. Suppose that there would exist a measurable set $E \in \Sigma$ with $\mu(E) > 0$ such that $\sum_{i\in I} d_i^* > \sum_{j\in J} \mathcal{S}_j$ a.e. $s \in E$. Define $\pi_E \in ba_+$ by $\pi_E(A) = \int_A \chi_E \, d\mu$ for any $A \in \Sigma$[*30], and put $\gamma = (\pi_E|\mathcal{S}) > 0$ and $\Pi_E \in ba_+$ by $\dfrac{M}{\gamma}\pi_E$. Then, we can regard Π_E as an element of $\Delta^{\mathcal{F}_\lambda}$. Then, $(\Pi_E| \sum_{i\in I} d_i^* - \sum_{j\in J} \mathcal{S}_j) > 0$, which leads to a contradiction. Thus, $\sum_{i\in I} d_i^* \leqq \sum_{j\in J} \mathcal{S}_j$ a.e. $s \in S$.

Next, suppose that there would exist $i \in I$ such that $d_i^* \notin D_i^{\mathcal{F}_\lambda}(p^*)$[*31]. Then, there would exist $z \in \mathcal{B}_i^{\mathcal{F}_\lambda}(p^*)$ such that $(z, d_i^*) \in \succ^{\mathcal{F}_\lambda}$. $(tz + (1-t)d_i^*, d_i^*) \in \succ^{\mathcal{F}_\lambda}$ for any $t \in (0, 1]$ because of convexity (or linearlity) of preference, and $tz + (1-t)d_i^* \in B_i^{\mathcal{C}_\theta}(p^*)$ for t sufficiently close to 1 because of $d_i^* \in \operatorname{int} \mathcal{C}_\theta$, which leads to a contrudiction.

Thus, $(p^*, (d_i^*)_{i\in I}) \in \Delta^{\mathcal{F}_\lambda} \times \prod_{i\in I} X_i^{\mathcal{F}_\lambda}$ is a Fisher markets quasi-equilibrium of both of the subeconomy $\mathcal{E}^{\mathcal{C}_\theta}$ and the subeconomy $\mathcal{E}^{\mathcal{F}_\lambda}$, which completes the proofs of Proposition 1 and 2. □

[*28] Recall that ${X_i}^{\mathcal{C}_\theta} = \mathcal{C}_\theta$.

[*29] Kakutani's fixed point theorem asserts that a nonempty-, convex- and compact-valued upper semicontinuous relation from a nonempty, convex and compact subset of a finite dimensional topological vector space to the same subset has a fixed point [2], [4].

[*30] A positive measure π_E is regarded as an elment of $\mathcal{L}^1$ which is suitably embeded in ba.

[*31] Recall that $\mathcal{B}_i^{\mathcal{C}_\theta}(p) = \mathcal{B}_i^{\mathcal{F}_\lambda}(p) \cap \mathcal{C}_\theta$ for any $p \in \Delta^{\mathcal{F}_\lambda}$, and hence, $d_i^* \in \mathcal{B}_i^{\mathcal{F}_\lambda}(p^*)$.

4 A proof of Theorem 1

In this chapter, we will give a proof of Theorem 1. For the purpose, we will first make the price-extension argument. Let (L, κ) be a Hausdorff locally convex space*32, where κ is a vector space topology of L, $D \subset L$ be a convex cone with vertex 0, and L° be a topological dual of $(L.\kappa)$. Abusing notation, we will denote the bilinear form of the dual pair $\langle L, L^{\circ}\rangle$ by $(\,\cdot\,|\,\cdot\,) : L^{\circ} \times L \longrightarrow \mathbb{R}$. In this more general settings, the next two lemmata hold.

Lemma 4. *In addition to the above conditions, assume that M is a finite dimensional vector subspace of L and that $M \cap int_{\kappa}D \neq \emptyset$. Then, if a linear functional $\rho : M \longrightarrow \mathbb{R}$ is continuous in the relative κ-topology, which hence is homeomorphic with the same dimensional Eucledean topology as M, and if $(\rho|d) \geqq 0$ for any $d \in D \cap M$, there exists a κ-continuous linear functional $\pi : L \longrightarrow \mathbb{R}$ such that $\pi \circ inc_M = \rho$*33 *and $(\pi|d) \geqq 0$ for any $d \in D$.*

Lemma 5. *In addition to the above conditions, assume that $int_{\kappa}D \neq \emptyset$. Then, $\{\pi \in L^{\circ} : (\pi|d_0) = \alpha \wedge (\pi|d) \geqq 0\, for\, any\, d \in D\}$ is a $\sigma(L^{\circ}, L)$-compact subset of L° for any constant $\alpha > 0$ and $d_0 \in int_{\kappa}L$, where $\sigma(L^{\circ}, L)$ is a weak* topology of L°**34.

We will give a proof of Lemma 4 and 5 later in Appendix.

Let $\{p^{\mathcal{F}_{\lambda}}\}_{\lambda\in\Lambda}$ be a family of Fisher market quasi-equilibrium price of subeconomy $\mathcal{E}^{\mathcal{F}_{\lambda}}$, which constitutes net. Take $||\cdot||_{\infty}$-topology as κ, $\mathcal{F}_{\lambda}$ as M, and $\mathcal{L}^{\infty}_{+}$ as D in Lemma 4. Then, $p^{\mathcal{F}_{\lambda}}$ satisfies all of the conditions of Lemma 4 for any $\lambda \in \Lambda$. Therefore, $p^{\mathcal{F}_{\lambda}}$ has a price-extension $\pi^{\mathcal{F}_{\lambda}} \in ba_{+}$ with $(\pi|\mathcal{S}) = M$. Next, define Δ by $\{\pi \in ba : (\pi|\mathcal{S}) = M \,\wedge\, (\pi|x) \geqq 0\, for\, any\, x \in \mathcal{L}_{+}{}^{\infty}\}$. Take $\mathcal{S}$ as d_0, and α as $M > 0$. Then, the above Δ is $\sigma(ba, \mathcal{L}^{\infty})$-compact subset of ba by Lemma 5. Note that $\pi^{\mathcal{F}_{\lambda}} \in \Delta$ by the definition of Δ.

Next, define $\mathbb{F}_{\mathcal{E}}$ by $\{(x_i)_{i\in I} \in \prod_{i\in I} X_i : \sum_{i\in I} x_i \leqq \mathcal{S}\}$, and $\mathcal{C}_{\mathcal{S}}$ by $\{x \in \mathcal{L}^{\infty} : ||x||_{\infty} \leqq ||\mathcal{S}||_{\infty}\}$. $\mathbb{F}_{\mathcal{E}}$ is a feasible set of an economy $\mathcal{E}$, and it is a subset of m-hold product of $\mathcal{C}_{\mathcal{S}}$. Clearly, $\mathcal{C}_{\mathcal{S}}$ is convex, and $||\cdot||_{\infty}$-bounded, and m-fold product $\mathcal{C} \equiv \mathcal{C}_{\mathcal{S}} \times \cdot \times \mathcal{C}_{\mathcal{S}}$ of $\mathcal{C}_{\mathcal{S}}$ includes $\mathbb{F}_{\mathcal{E}}$. Since $\mathcal{C}_{\mathcal{S}}$ is a $||\cdot||_{\infty}$-closed ball at a center 0 is a $\sigma(\mathcal{L}^{\infty}, \mathcal{L}^{1})$-compact subset of $\mathcal{L}^{\infty}$ by Banach-Alaoglu theorem and Minkowski's inequality*35, $\mathcal{C}$ is compact with a product topology of $\sigma(\mathcal{L}^{\infty}, \mathcal{L}^{1})$.

Let $d_i^{\mathcal{F}_{\lambda}}$ be a i's equilibrium demand schedule of a truncated subeconomy $\mathcal{E}^{\mathcal{F}_{\lambda}}$. According to the above argument, a net $\{(\pi^{\mathcal{F}_{\lambda}}, (d_i^{\mathcal{F}_{\lambda}})_{i\in I})\}_{\lambda\in\Lambda}$ has a convergent subnet with a product topology $\sigma(ba, \mathcal{L}^{\infty}) \times \sigma(\mathcal{L}^{\infty}, \mathcal{L}^{1}) \times \cdots \times \sigma(\mathcal{L}^{\infty}, \mathcal{L}^{1})$. Suppose that $(\pi^{**}, (d_i^{**})_{i\in I}) \in \Delta \times \mathcal{C}$ is a limit of a convergent subnet of $\{(\pi^{\mathcal{F}_{\lambda}}, (d_i^{\mathcal{F}_{\lambda}})_{i\in I})\}_{\lambda\in\Lambda}$.

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*32 A locally convex space is a topological vetor space whose topology is genarated by a separating family of seminorms. Note that all of spaces in this paper are locally convex ones. For the details, see [17], [18]

*33 $inc_M : M \longrightarrow L$ is an inclusion mapping.

34 The weak topology $\sigma(L^{\circ}, L)$ is the weakest toplogy of L° under which the linear functional $(\cdot|x) : L^{\circ} \longrightarrow \mathbb{R}$ is continuous for any $x \in L$.

*35 See [1], [17], [18].

Lemma 6. $(\pi^{**}, (d_i^{**})_{i\in I}) \in \Delta \times \prod_{i\in I} X_i$ *is a Fisher markets quasi-equilibrium of an economy* $\mathcal{E}$.

Proof. Abusing notations, let $\{(\pi^{\mathcal{F}_\lambda}, (d_i^{\mathcal{F}_\lambda})_{i\in I})\}_{\lambda\in\Lambda}$ be a convergent subnet with a limit $(\pi^{**}, (d_i^{**})_{i\in I})$ in the above. As already mentioned, $\pi^{**} \in \Delta$ and hence $\pi^{**} \neq 0$. Since $X_i = \mathcal{L}_+^\infty$ is $\sigma(\mathcal{L}^\infty, \mathcal{L}^1)$-closed convex subset of $\mathcal{L}^\infty$, $d_i^{**} \in X_i$ for any $i \in I$. Note that we can regard a convergence of net $\{d_i^{\mathcal{F}_\lambda}\}_{\lambda\in\Lambda}$ to the limit d_i^{**} in the topology $\sigma(\mathcal{L}^\infty, \mathcal{L}^1)$ as a convergence in the topology $\tau(\mathcal{L}^\infty, \mathcal{L}^1)$ since $\sigma(\mathcal{L}^\infty, \mathcal{L}^1)$-closed convex subset of $\mathcal{L}^\infty$ is $\tau(\mathcal{L}^\infty, \mathcal{L}^1)$-closed convex subset of $\mathcal{L}^\infty$ [*36].

By the definition of the ordering on $\mathcal{L}^\infty$, $\sum_{i\in I} d_i^{**} \leqq \sum_{j\in J} \mathcal{S}_j$, *a.e.* $s \in S$ since $\sum_{i\in I} d_i^{\mathcal{F}_\lambda} \leqq \sum_{j\in J} \mathcal{S}_j$, *a.e.* $s \in S$ holds for any $\lambda \in \Lambda$. Therefore, $(d_i^{**})_{i\in I} \in \mathbb{F}_{\mathcal{E}}$.

Recall that $\Xi_i = \{x \in \mathcal{L}^\infty : \langle \pi^{k_i} | x\rangle \leqq M^{k_i}\ for\ any\ k_i \in \mathcal{K}_i\}$ is a set of commodity bundles which are attainable to i under i's all of the napsack constraints and that $d_i^{\mathcal{F}_\lambda} \in \mathcal{B}_i^{\mathcal{F}_\lambda}(p^{\mathcal{F}_\lambda})$. By the constructions or Assumption 1, $d_i^{**} \in \Xi_i$. Take $x \in X_i \cap \Xi_i$ such that $(x, d_i^{**}) \in \succsim_i$ in order to show that $(\pi^{**}|x) \geqq m_i$. By Assumption 4, there exists $y \in X_i \cap \Xi_i \cap \mathbb{B}(x:\epsilon)$ such that $(y, x) \in \succ_i$, where $\mathbb{B}(x:\epsilon)$ is a $||\cdot||_\infty$-closed ball at the center x with a radius $\epsilon > 0$. Then, by Assumption 2, the strict lower contour set $\{z \in X_i : (y, z) \in \succ_i\}$ is $\tau(\mathcal{L}^\infty, \mathcal{L}^1)$- or $\sigma(\mathcal{L}^\infty, \mathcal{L}^1)$-open subset of X_i. Let $\lambda_1 \in \Lambda$ be such that $y \in \mathcal{F}_{\lambda_1}$. Then, $(y, d_i^{\mathcal{F}_\lambda}) \in \succ_i^{\mathcal{F}_\lambda}$ for any successor $\mathcal{F}_\lambda$ of $\mathcal{F}_{\lambda_1}$ since $d_i^{\mathcal{F}_\lambda} \longrightarrow d_i^{**}$. Therefore, $(\pi^{\mathcal{F}_\lambda}|y) > (\pi^{\mathcal{F}_\lambda}|d_i^{\mathcal{F}_\lambda}) = m_i$ for any successor $\mathcal{F}_\lambda$ of $\mathcal{F}_{\lambda_1}$, and, hence, $(\pi^{**}|y) \geqq m_i$. Thus, there exists a $||\cdot||_\infty$-convergent net $\{y^\epsilon\}$ with a limit x such that $y^\epsilon \in X_i \cap \Xi_i \cap \mathbb{B}(x:\epsilon)$ and $(\pi^{**}|y^\epsilon) \geqq m_i$, and, hence, $(\pi^{**}|x) \geqq m_i$, which implies $(\pi^{**}|d_i^{**}) \geqq m_i$. Suppose that there would exist $i \in I$ such that $(\pi^{**}|d_i^{**}) > m_i$. Summing over these inequalities, we could get $(\pi^{**}| \sum_{i\in I} d_i^{**} - \mathcal{S}) > 0$, which leads to a contradiction since π^{**} is a positive linear functional. Therefore, $d_i^{**} \in \mathcal{B}_i^F(\pi^{**})$.

Finally, we would like to show that $d_i^{**} \in \mathcal{D}_i^F(\pi^{**})$. By the above argument, it suffices to show that $x \in X_i \cap \Xi_i$ and $(\pi^{**}|x) = m_i$ implies $(d_i^{**}, x) \in \succsim_i$. Suppose that $(x, d_i^{**}) \in \succ_i$. Since $0 \in \mathcal{B}_i^F(\pi^{**})$ and the continuity and the convexity of preference, there would exist $t \in [0,1)$ sufficiently close to 1 such that $(tx, d_i^{**}) \in \succ_i$. However, $(\pi^{**}|tx) < m_i$, which leads to a contradiction in the above argument.

Thus, $(\pi^{**}, (d_i^{**})_{i\in I}) \in \Delta \times \prod_{i\in I} X_i$ is a Fisher markets quasi-equilibrium of an economy $\mathcal{E}$. □

Finally, we will argue about the feasibility of a Fisher markets quasi-equilibriun allocation of an economy $\mathcal{E}$. Let $(\pi^{**}, (d_i^{**})_{i\in I}) \in \Delta \times \prod_{i\in I} X_i$ be a Fisher markets quasi-equilibrium.

Proposition 3. *For a Fisher markets quasi-equilibrium allocation* $(d_i^{**})_{i\in I} \in \prod_{i\in I} X_i$ *of an economy* $\mathcal{E}$, $\sum_{i\in I} d_i^{**} = \sum_{j\in J} \mathcal{S}_j$, *a.e.* $s \in E$ *for any* $E \in \Sigma$ *with* $\pi^{**}(E) > 0$.

Proof. Note that $(d_i^{**})_{i\in I}$ satisfies $\sum_{i\in I} d_i^{**} \leqq \sum_{j\in J} \mathcal{S}_j$, *a.e.* $s \in S$. Suppose that there would exist a measurable set $E \in \Sigma$ with $\pi^{**}(E) > 0$ such that $\sum_{i\in I} d_i^{**} \neq \sum_{j\in J} \mathcal{S}_j$, *a.e.* $s \in E$. Since π^{**} is a positive bounded

[*36] It is known that the converse is true. In general, if $L^\circ$is a topological dual of a locally convex space (L, κ), then a convex subset of L is κ-closed if and only if it is $\sigma(L, L^\circ)$-closed [17], [18].

additive non-zero measure,

$$
\begin{aligned}
(\pi^{**}|\sum_{j\in J}\mathcal{S}_j-\sum_{i\in I}d_i^{**}) &= \int(\sum_{j\in J}\mathcal{S}_j-\sum_{i\in I}d_i^{**})\,d\pi^{**} \\
=\int_E(\sum_{j\in J}\mathcal{S}_j-\sum_{i\in I}d_i^{**})\,d\pi^{**} &+ \int_{S\backslash E}(\sum_{j\in J}\mathcal{S}_j-\sum_{i\in I}d_i^{**})\,d\pi^{**}>0.
\end{aligned}
$$

On the otherhand, by linearlity of linear functional,

$$
(\pi^{**}|\sum_{j\in J}\mathcal{S}_j-\sum_{i\in I}d_i^{**})=(\pi^{**}|\mathcal{S})-\sum_{i\in I}(\pi^{**}|d_i^{**})=M-M=0
$$

,which leads to a contradicrion. □

Appendix

In this appendix, we will give proofs of Lemma 4 and 5. Recall the conditions about Lemma 4 and 5:

Let (L,κ) be a Hausdorff locally convex space, where κ is a vector space topology of L, $D\subset L$ be a convex cone with vertex 0, and L° be a topological dual of (L,κ). We will denote the bilinear form of the dual pair $\langle L,L^\circ\rangle$ by $(\,\cdot\,|\,\cdot\,):L^\circ\times L\longrightarrow\mathbb{R}$.

Lemma 4. *In addition to the above conditions, assume that M is a finite dimensional vector subspace of L and that $M\cap int_\kappa D\neq\emptyset$. Then, if a linear functional $\rho:M\longrightarrow\mathbb{R}$ is continuous in the relative κ-topology, which hence is homeomorphic with the same dimensional Eucledean topology as M, and if $(\rho|d)\geqq 0$ for any $d\in D\cap M$, there exists a κ-continuous linear functional $\pi:L\longrightarrow\mathbb{R}$ such that $\pi\circ inc_M=\rho$ and $(\pi|d)\geqq 0$ for any $d\in D$.*

Proof. There exists $d_0\in M\cap int_\kappa D$ and a balanced κ-open neighborhood of $\{0\}$ such that $\{d_0\}+V\subset D$*37. Take $x\in M\cap(V-D)$. Then, there exists $v\in V$ and $y\in D$ such that $x=v-y$, and , hence, $x=d_0-\{(d_0-v)+y\}\in M\cap(\{d_0\}-D)$ since V is balanced and D is a convex cone with vertex $\{0\}$. Putting $x=d_0-d$ for some $d\in D$, $(\rho|x)=(\rho|d_0)-(\rho|d)\leqq(\rho|d_0)$. Therefore, there exists $\alpha>0$ such that $(\rho|x)<\alpha$ for any $x\in M\cap(V-D)$ since $\{0\}\in M\cap(V-D)$. Define $N\subset M$ by $\{z\in M:(\rho|z)=\alpha\}$, which is an affine subspace of M and satisfies $N\cap(V-D)=\emptyset$ by its definition. Note that $int_\kappa(V-D)\neq\emptyset$. Therefore, by Hahn-Banach extension theorem*38, there exists a hyperplane $H\subset L$ such that $N\subset H$ and $H\cap(V-D)=\emptyset$, and hence, there exists a linear functional $\pi:L\longrightarrow\mathbb{R}$ such that $H=\{z\in L:(\pi|z)=\alpha\}$.

$(\pi|x)<\alpha$ for any $x\in(V-D)$ since $0\in V-D$ and $V-D$ is convex, and hence, $(\pi|x)>-\alpha$ for any $z\in D$ since $0\in V$. Suppose that there would exist $x\in D$ such that $(\pi|x)<0$ in order to show that $(\pi|x)\geqq 0$ for any $x\in D$. Then, there exists $\beta>0$ such that $(\pi|\beta x)<-\alpha$. Since D is a convex cone with a vertex $\{0\}$, $\beta x\in D$, which leads to a contradiction.

*37 The balancedness of an open neighborhood V of $\{0\}$ means that $\alpha V\subset V$ holds for any $\alpha\in[-1,1]$.
*38 See [17], [18]. Note that this version is sometimes called Geometric Hahn-Banach extension theorem [18].

Finally, we will show $\pi \in L^\circ$. Take $v \in V$ arbitrarily. Then, $(\pi|v) \geqq -(\pi|d_0)$ since $(\pi|v) = (\pi|v + d_0) - (\pi|d_0)$ and $v + d_0 \in D$. Since $v \in V$, $-v \in V$. Therefore, $(\pi| - v) \geqq -(\pi|d_0)$. These imply $|(\pi|v)| \leqq (\pi|d_0)$ for any $v \in V$. Since a bounded linear functional on κ-open neighborhood of $\{0\}$ is continuous, $\pi \in L^\circ$. □

Lemma 5. *In addition to the above conditions, assume that* $int_\kappa D \neq \emptyset$. *Then,* $\{\pi \in L^\circ : (\pi|d_0) = \alpha \wedge (\pi|d) \geqq 0 \, for \, any \, d \in D\}$ *is a* $\sigma(L^\circ, L)$*-compact subset of* L° *for any constant* $\alpha > 0$ *and* $d_0 \in int_\kappa D$, *where* $\sigma(L^\circ, L)$ *is a weak* topology of* L°.

Proof. Take a balanced κ-open neighborhood of $\{0\}$ such that $\{d_0\} + V \subset D$. Then, by Banach-Alaoglu Theorem [17] [18], $V^\circ \equiv \{\pi \in L^\circ : |(\pi|v)| \leqq \alpha \, for \, any \, v \in V\}$ is a $\sigma(L^\circ, L)$-compact subset of L°. It suffices to show that $P \equiv \{\pi \in L^\circ : (\pi|d_0) = \alpha \wedge (\pi|d) \geqq 0 \, for \, any \, d \in D\}$ is a $\sigma(L^\circ, L)$-closed subset of V°. Take $\pi \in P$. Since $d_0 \pm v \in D$ for any $v \in V$, $(\pi|d_0 \pm v) \geqq 0$. Therefore, $|(\pi|v)| \leqq \alpha$ for any $v \in V$, and hence, $P \subset V^\circ$. On the other hand, $P = \{\pi \in L^\circ : (\pi|d_0) = \alpha\} \cap \left[\bigcap_{d \in D} \{\pi \in L^\circ : (\pi|d) \geqq 0\}\right]$, which is an intersection of $\sigma(L^\circ, L)$-closed subsets of L°. □

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