Analyzing Three-Way Playoffs in the Sumo World

Fairness and Excitement

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I Sumo as an old and popular sport in Japan

Sumo is one of the oldest and most popular sports in Japan. Two big sumo wrestlers clap their hands before they start fierce wrestling on the small circle with a diameter of less than five meters. It is really exciting to see how each of the wrestlers exerts all his energy defeats his opponent. Usually, he engages in one bout in a day, and one tournament consists of fifteen bouts. If he succeeds in wining all fifteen bouts, he will no doubt be declared the tournament victor. Suppose a special occasion in which there are three wrestlers, say Adam, Bill, and Carl, with 13 wins and two losses, so that each can claim a contender. Then, we have to invent a specific method to make three-way play-offs to determine the final victor. Although there might be several methods conceivable for such play-offs, it is a sort of small wonder that only one specific method, called a *tomoesen*, has long been adopted in the traditional sumo world.

According to this *tomoesen*, sumo fights are carried out in a step-by-step fashion. At the first step, only two wrestlers, say Adam and Bill, are picked up, with Carl being left for a short rest. Suppose that Adam defeats Bill on the first counter. Then, there appears the second stage, in which Adam has to encounter Carl, the third remaining wrestler. Adam may or may not defeat Carl. If Adam succeeds in beating Carl, then it is the end of the story: Adam is declared the final winner, receiving the victory trophy. If Carl defeats Adam, however, another set of sumo bouts would be necessary. It is now Carl's turn to have a sumo match with Bill. There are two possibilities, Carl may or

may not defeat Bill. If Carl defeats Bill, then Carl is declared the tournament winner because Carl is proud of winning two successive bouts, against both Adam and Bill. If Bill beats Carl, however, another possibility would appear. This would imply an extra complication. Then, sumo matches would continue until there appears a strong wrestler who successively downs the two opponents.

Theoretically speaking, sumo bouts would go on and on, with no end in sight. Fortunately or unfortunately, reality is different from theory. According to the long history of sumo records, three-way playoffs have been rarely seen. Besides, even if tomoesen happens in the sumo world, two or even more bouts would be needed to determine the final winner. The historical, physical or psychological reasons why this is so will be discussed later in greater details.

The outline of this paper is as follows. Section 1 serves as an introduction, discussing *sumo* as one of the oldest and most popular sports in Japan. The *tomoesen* or a specific form of three-way playoff is heuristically described with recent historical data. Section 2 turns attention to a more theoretical analysis with the support of mathematical probability theory. It is shown that mathematically, *tomoesen* is not a fair game even though it has been a very favorite sport from the good old days to the present time. Section 3 attempts to reply to the question why a mathematically unfair game such as sumo has been overwhelmingly loved by the Japanese people. Then, in Section 4, we come to the conclusion that sumo is a sort of human drama in which excitement sometimes tends to overwhelm rationality. After all, human being is a complex animal, always seeking a delicate balance between reason and emotion.

II A Historical Analysis of *Tomoesen* : Previous Records Outlined

Concerning the study of the sumo world, so many approaches are available. Among them, it would be very reasonable to think that historical and theoretical approaches play key roles in such study. Remarkably, *tomoesen* characterizes one of the cores of Japanese sumo world. The question of why and how it distinguishes from other sports seems to be quite intriguing, thus reserving careful investigation. While a compact historical approach to *tomoesen* will be taken in this section, a more detailed theoretical approach will be developed in the next session.

The first historical approach should be based on the accumulation of real sumo data over so many years. It is well-known that sumo has a very long history, being firmly embedded in the natural and cultural features of Japan. Although no man is 100 percent sure of the exact origin of sumo, several myths about the contests of physical strength, vividly written in *Kojiki* (Records of Ancient Matters) and *Nihonshoki* (Chronicles of Japan) in the 8th century, are believed by a number of researchers to be such origin. Whether this is truly correct or not, it is a historical fact that Classical Sumo contests have been carried out in agricultural villages as the formal and religious rituals celebrating rice harvest of a year in question. [1)]

Modern Sumo began in the Edo period, more than 400 years ago, and uninterruptedly continues until today. At every fight, two big wrestlers and a referee step up into the rather small ring with the diameter of around 4.5 me-

1) For details, see Ohno-Yasumaro (ed.) (712) and Fuhito et al. (720). Note that the records of those classical books are not perfectly reliable.

ters. The ring is regarded as a very sacred place on which each wrestler is asked to pitch a handful of pure salt before the sumo fight begins. The fight is speedy and exciting, usually being finished in a short time. If a wrestler either knocks down his opponent on the ring or pushes him out of the ring, then the successful player is declared by the referee a sure winner. All spectators at each sumo match are naturally very excited and applaud the excellent performance of the wrestlers.

At present, there are six sumo performances a year: they are carried out in January, March, May, July, September, and November. Each performance has its own feature, reflecting the change of place and season. Besides, the final victor is awarded the Emperor Cup together with many commemorative items. In short, all sumo wrestlers make strenuous efforts to become the Cup winner at every performance.

Every healthy wrestler is asked to take part in all 6 performances, and every performance consists of 15 bouts. If a certain wrestler, say Adam, wins all of those bouts in a tournament, then he is naturally declared to be the victor, winning the Emperor Cup. If Adam cannot win all bouts, the determination of a victor is not an easy job. The most difficult job to decide a victor occurs when there exist three wrestlers, Adam, Bill and Carl, who has won 13 bouts but lost 2 bouts at the end of the tournament. On this occasion, Historical Tradition plays a critical factor in seeking a reasonable way out, teaching us that the *tomoesen* which sounds classic yet romantic is indeed an effective and exciting method to find the solution.

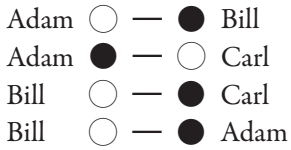
In what follows, let us scrutinize the essence of *tomoesen*. According to this *tomoesen*, sumo fights are carried out in a step-by-step fashion. At the first step, only two wrestlers, say Adam and Bill, engage in their fight, with the third wrestler, say Carl, being left for a short rest. At the second stage, the winner, say Adam, of the first bout has to engage in his second bout against Carl, a resting third wrestler. Adam may be a winner again, demonstrating the two successive wins, and he is declared a tournament victor. In short, such two bouts with win○-lost● results will be shown as follows.

Adam ○ — ● Bill
Adam ○ — ● Carl

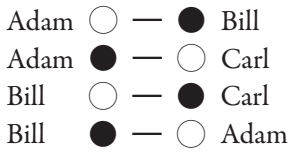
This is merely one possibility, however. Another possibility is that at the second bout, Adam may be unfortunately defeated by Carl. Then, Carl can continue his sumo fight against Bill. Carl may be either a winner or a loser. If Carl defeats Bill, then Carl has beaten the two successive bouts, winning the Emperor Cup. Such successive fighting results will be illustrated as follows.

Adam ○ — ● Bill
Adam ● — ○ Carl
Bill ● — ○ Carl

However, this is not the end of the story. We have to consider another possibility that Carl may unfortunately be a loser against Bill. Then, Bill, the man who has lost bout against Adam before, can get a second chance to clinch the crown. If Bill is lucky enough to beat Adam, Bill is a trophy winner in spite of the loser of the first bout. Summing up, the four bouts are played as shown below.



At this point, another question might occur to our mind. Let us instead assume that at the final contest, Adam is strong enough to beat Bill. Then, we will face the opposite situation to be depicted below.

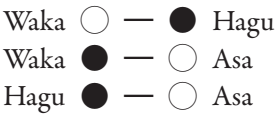


As is easily seen, Adam's record is "two wins and one loss". Bill's record is "one win and two losses" whereas Carl's record is "one win and one loss". None can claim to be two successive winners among those three players. We wonder who will be the final trophy winner. Under such complex situations, we have no choice but stick to the original rule by which any wrestler *winning two successive matches* must be celebrated as the glorious Cup winner. Perhaps, it may take time and pains to find a final reasonable solution. We do not care for it at all. In our opinion, this is no more than the essence of *tomoesen* of Japanese sumo world.

Now, we are in a position to document the real records of *tomoesen* in recent times. As far as our memory tells us, the March Tournament in 1956 is the first recorded *tomoesen* after the Pacific War. Let us pay attention to it in details below.

(1) 1956, March

The following wrestlers were played in *tomoesen*: Wakanohana (Waka for short), Wakahaguro (Hagu for short), and Asashio (Asa for short).

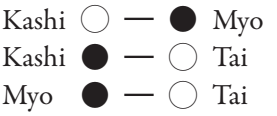


Asa was a young and ambitious wrestler. He sat down out of ring first, closely watching the match between Waka and Hagu. After knowing that Waka beaten Hagu, Asa succeeded in defeating Waka and Hagu successively. This demonstrated Asa's impressive victory in *tomoesen*. Even now, we can easily guess that all sumo fans got very much excited by such fantastic achievement by Asa.

In what follows, we will briefly outline other *tomoesen* results and make their characteristics.

(2) 1961, September

The following wrestlers are played in *tomoesen*: Kashiwado (Kashi for short), Myobudani (Myo for short), and Taiho (Tai for short).



Tai was the final victor of the tournament. His winning pattern was as the same as (1) above: namely, he sat out of the ring and successively defeated his two opponents. This must excite all sumo fans in Japan. No doubt, it symbolized the arrival of the Golden Age of Grand Champion Taiho.

(3) 1965, September

The following wrestlers were played in *tomoesen*: Kashiwado (Kashi for short), Myobudani (Myo for short), and Sadanoumi (Sada for short).

Kashi ○ — ● Myo
Kashi ○ — ● Sada

Kashi defeated his two opponents easily and impressively. This was undoubtedly the simplest result of *tomoesen*, indicating Kashi as a dominant winner.

(4) 1990, March

The following wrestlers were played in *tomoesen*: Konishiki (Koni for short), Kirishima (Kiri for short), and Hokutoumi (Hoku for short).

Koni ○ — ● Hoku
Koni ● — ○ Kiri
Hoku ○ — ● Kiri
Hoku ○ — ● Koni

This *tomoesen* was quite unique in that four matches were needed to decide the final victor. Such complication was "historically first" (and probably "historically last").

(5) 1993, July

The following wrestlers were played in *tomoesen* : Akebono (Ake for short), Wakanohana (Waka for short), and Takanohana (Taka for short).

Ake ○ — ● Waka
Ake ○ — ● Taka

Ake was the first grand champion from Hawaii. It was quite impressive to see that Ake defeated the popular brothers Waka and Taka in this *tomoesen*. No doubt, the "Golden Age of the Sumo World" began with the brilliant arrival of those three wrestlers.

(6) 1994, March

The following wrestlers were played in *tomoesen*: Takanonami (Nami for short), Takatouriki (Riki for short), and Akebono (Ake for short).

Nami ○ — ● Riki
Nami ● — ○ Ake
Riki ● — ○ Ake

Ake was again the final winner of *tomoesen*. This showed his explosive power over all his opponents.

(7) 2022, November

The following wrestlers were played in *tomoesen*: Takayasu (Yasu for short), Ahbi (Abi for short), and Takakeisho (Kei for short).

Yasu ● — ○ Abi
Kei ● — ○ Abi

This demonstrated how exciting sumo was as a sport. Abi was physically a rather small wrestler, but technically a very skillful player. In the sumo world, power was, and is, not everything: namely, skill was, and is, sometimes more important than mere power. This eloquently taught us another attraction of *tomoesen*. [2)]

2) Remarkably, among those seven cases of *tomoe-sen*, three exciting cases (namely, (1), (2) and (6)) represent what we may call the Lucky Cases, in which the third man is lost in the first fight but finally wins the crown of victory. We think that the probability of "three out of seven" is amazingly a large number. This clearly shows how exciting *tomoe-sen* is in the sumo world.

III A Theoretical Analysis of *Tomoesen* : Rationality versus Excitement

We now turn our attention to a theoretical analysis of *tomoesen* in the sumo world. The problem of greatest interest is whether and to what extent *tomoesen* is a "fair game" from a theoretical and mathematical viewpoint. [3)]

It is not a plain and easy job to determine the "degree of fairness" in the sumo world. Suppose that three players are engaged in *tomoesen*. Since three persons are involved, with each person seeking the winning trophy, it seems that *tomoesen* should be a "fair game" if the probability of winning the trophy is just the same, i.e. one third, among the three players. Reality may entirely be different from appearance, however. The central question to ask is whether and in which way *tomoesen* can be regarded as a "fair game".

Imagine the scene that three big wrestlers step up into the rather small ring with the diameter of around 4.5 meter. Is it practically possible that those three wrestlers together start sumo wrestling simultaneously? Of course, it is completely out of our imagination. On reality, only two wrestlers are allowed to be on the ring, a third wrestler being idle out of the ring. And then, the winner of the first match turns to the next match against a third resting wrestler. This is clearly nothing but the essence of *tomoesen*.

Following the previous example we used before, let us continue to assume that there are three wrestlers: They are Adam, Bill and Carl. Practically, let us consider the following two successive games. At the first step, Adam and Bill are supposed to fight together on the ring. And at the next step, the winner of the game, say Adam for instance, is destined to fight against Carl, a third and resting wrestler. Adam may or may not defeat Carl. Conceivably, several extra fights will be followed until the dominant winner of two successive fights appears. Theoretically speaking , even an infinite number of fights would be required to settle the issue. But, practically speaking, it is quite convenient for us to separately scrutinize two different patterns toward the final victory. The first pattern is a sort of "Straight Pattern" in which Adam beats Bill in the first bout, and somehow wins the victory trophy. The second pattern is distinct from the first pattern in the sense that although Adam is lost in the first fight against Bill, he somehow wins the trophy in a roundabout fashion. This may be named "Roundabout Pattern" toward final victory. In what follows, both Straight and Roundabout Patterns will theoretically be discussed in more details.

3-1. The First Straight Pattern: Adam wins the first battle and proceeds to capture to the cup

There are three wrestlers, namely Adam, Bill and Carl. At the first bout, Adam beats Bill, with Carl taking a short rest. When Adam defeats Carl at the second bout, Adam will surely win the victory trophy. The data of such two battles will be illustrated as follows.

Adam ○ — ● Bill
Adam ○ — ● Carl

This indicates possibly the shortest straight pattern case for the determination of the victor of *tomoesen*. Note that such Adam's sumo data

3) To our knowledge, only a few papers on *tomoe-sen* have been published so far. We believe that this paper distinguishes from them, sheding a new light on it.

is recorded as Case 1A: ○○ in Table 1. The winning probability is $(0.5)^2$ as is written in the right of Table 1.

There are some other ways for Adam to win the final victory. Even if Adam is defeated by Carl in the second battle, Adam should not be discouraged so easily. If Carl happens to be beaten by Bill as indicated by the striking sharp sign (#), Adam may have a second chance to win the trophy. As is carefully shown below, if Adam succeeds in successively defeating Bill and Carl, Adam is declared the final victor Note that such sharp sign (#) indicates the fight in which Adam is not directly participated.

Adam ○ — ● Bill
Adam ● — ○ Carl
Bill ○ — ● Carl (#)
Bill ● — ○ Adam
Carl ● — ○ Adam

If we focus on Adam's records, his sumo records can be shown in Table 1 as Case 1B: ○● #○○, together with the probability of $(0.5)^5$. Compared with the previous case 1A, Adam's effort to clinch the trophy is very great here: Indeed, he has to take part in four painful battles.

Now, let us gaze with peering eyes at the following score sheet.

Adam ○ — ● Bill
Adam ● — ○ Carl
Bill ○ — ● Carl (#)
Bill ● — ○ Adam
Carl ○ — ● Adam
Carl ● — ○ Bill (#)
Adam ○ — ● Bill
Adam ○ — ● Carl

Here, there are as many as eight bouts including the two bouts which are not directly related to Adam. This clearly shows Adam's suffering history toward the final goal. Summing up, Adam's records are shown in Table 1 as Case 1C: ○●#○●#○○ , together with the probability of $(0.5)^8$.

Fortunately or unfortunately, this not the end of story. If Adam happens to be lost against Carl at the eighth boat, the relevant *tomoesen* must continue until the final glorious victor comes on the stage. Then, as Table 1 teaches us, we will have to take care of Case 1D or Case 1E, or ... until the very final victor go up to the special commendation ring.

Now, suppose that Adam got lost in the fifth fight against Carl. Adam nevertheless should not be out of hope, however. Indeed, if Carl is a loser against Bill, Adam's hope is back again. If the reviving fighter Adam successively defeats Bill and Carl, then Adam can be the final victor after struggling. Then, Adam's long fighting records can be illustrated below.

Adam ○ — ● Bill
Adam ● — ○ Carl
Bill ○ — ● Carl (#)
Bill ● — ○ Adam
Carl ○ — ● Adam
Carl ● — ○ Bill (#)
Adam ○ — ● Bill
Adam ● — ○ Carl
Bill ○ — ● Carl
Bill ● — ○ Adam
Carl ● — ○ Adam

Note that there are as many as eleven bouts including the three bouts which are not direct-

Table 1 Requirement for a sumo wrestler to win the tomoesen

Case 1 (Rightful Case): Won the first bout, and also the crown of victory			
		Win○*-Loss*● *Data*	*Probability*
Case 1A	:	○○	$(0.5)^2$
Case 1B	:	○●#○○	$(0.5)^5$
Case 1C	:	○●#○●#○○	$(0.5)^8$
Case 1D	:	○●#○●#○●#○○	$(0.5)^{11}$
Case 1E	:	○●#○●#○●#○●#○○	$(0.5)^{14}$
・		・ ・ ・	・
・		・ ・ ・	・

Case 2 (Lucky Case): Lost in the first bout, but won the crown of victory			
		Win○*-Loss*● *Data*	*Probability*
Case 2A	:	●#○○	$(0.5)^4$
Case 2B	:	●#○●#○○	$(0.5)^7$
Case 2C	:	●#○●#○●#○○	$(0.5)^{10}$
Case 2D	:	●#○●#○●#○●#○○	$(0.5)^{13}$
Case 2E	:	●#○●#○●#○●#○●#○○	$(0.5)^{16}$
・		・ ・ ・	・
・		・ ・ ・	・

Remark. The sharp mark "#" indicates the occurrence of the situation under which the wrestler, who won the previous bout, is beaten himself in the present bout.

ly related to Adam. This clearly shows Adam's suffering history toward the final goal. Summing up, Adam's records are shown in Table 1 as Case 1D: ○●#○●#○●#○○ , together with the probability of $(0.5)^8$.

Fortunately or unfortunately, this should not be the end of story. If Adam happens to be lost against Carl at the eighth boat, the relevant *tomoesen* must continue until the final glorious victor comes on the stage. Then, as Table 1 teaches us, we will have to take care of Case 1D or Case 1E, or ... until the very final victor go up to the special commendation ring.

Summarizing those cases of victory or defeat, we obtain the upper half of Table 1. If we focus on Case 1, the total probability for Adam to win the victory cup, written as P (Case 1), should be calculated as the sum of the probabilities of Cases 1A, 1B, 1C, 1D, 1E and other possible cases. Therefore, it is simply computed as follows.

$$\begin{aligned} \mathrm{P}(\mathrm{Case1}) &= (0.5)^2 + (0.5)^5 + (0.5)^8 + (0.5)^{11} \\ &\quad + (0.5)^{14} + \cdots \\ &= (0.5)^2 \times \left[1 + (0.5)^3 + (0.5)^6 + \right. \\ &\quad \left. (0.5)^9 + (0.5)^{12} + \cdots\right] \qquad (1) \end{aligned}$$

Note that the term in the square bracket 【…】means the geometric series of the first term 1 and the common ratio $(0.5)^3$. Clearly, such series must converge, and the limit is $1 / [1 - (0.5)^3]$, namely 8/7. Hence, in the light of Eq. (1), we obtain the following result.

$$P(\text{Case1}) = (1/4) \times (8/7) = 2/7. \qquad (2)$$

In plain English, the probability that Adam wins the first boat and proceeds to win the crown of victory is just 2/7. Needless to say, Bill's probability of winning the victory should be the same as Adam's probability.

3-2. The Second Roundabout Pattern: Adam loses the first battle yet manages to capture to the cup

In the above, we have scrutinized the "First Straightforward Pattern", showing how the respectable wrestler Adam, after winning the first battle, proceeds to win the victory trophy. In what follows, we will carefully examine the "Second Roundabout Pattern", demonstrating how in spite of the loss of the first battle, Adam somehow manages to become the final victor.

We start our analysis with the unlucky situation under which Adam got lost against Bill in the first match. The urgent problem is now how Adam deals with this adversity toward the final victory. Presumably, the easiest way out for him is that after patiently waiting out of the ring and watching Carl defeating Bill, he makes a swift recovery by successively beating the two wrestlers, Carl and Boll. Such a come-from-behind victory will be demonstrated below.

Adam ● — ○ Bill
Carl ○ — ● Bill (#)
Carl ● — ○ Adam
Bill ● — ○ Adam

Adam's fighting records are also confirmed in Table 1 as Case 2A : ●#○○. The winning probability is as small as $(0.5)^4$.

Note that there remain some other ways for the indomitable wrestler Adam to win the final trophy. When Adam got lost against Bill at his fourth battle, he may take a short break out of the ring and wait for Bill's defeat by Carl at the next match. In fact, if Adam is a lucky person, he will still have a chance to become the final victor as indicated below.

Adam ● — ○ Bill
Carl ○ — ● Bill (#)
Carl ● — ○ Adam
Bill ○ — ● Adam
Bill ● — ○ Carl (#)
Adam ○ — ● Carl
Adam ○ — ● Bill

Note that after the loss-win-loss records in his earlier bouts, Adam have to successively defeat Carl and Bill in his later bouts. This is really a nice achievement for the tireless man Adam. His fighting records are confirmed in Table 1 as Case 2B :

●#○●#○○ , the winning probability being as small as $(0.5)^7$.

Adam ● — ○ Bill
Carl ○ — ● Bill (#)
Carl ● — ○ Adam
Bill ○ — ● Adam
Bill ● — ○ Carl (#)
Adam ○ — ● Carl
Adam ● — ○ Bill
Carl ○ — ● Bill (#)
Carl ● — ○ Adam
Bill ● — ○ Adam

Amazingly, from a pure theoretical viewpoint, Adam can be beaten a couple of times

before winning the crown. As the very long series of battles seen above tells us, however, this should really be a very tiresome ordeal. According to Case 2 in Table2 , as far as the accumulation of exhaustion on the side of wrestlers can entirely be ignored, such extremely long battles will continue so that even Case 2D or Case 2E may be realized. In fact, Case 2D can be illustrated below, with focus on Adam's win-loss table.

Adam ● — ○ Bill
Carl ○ — ● Bill (#)
Carl ● — ○ Adam
Bill ○ — ● Adam
Bill ● — ○ Carl (#)
Adam ○ — ● Carl
Adam ● — ○ Bill
Carl ○ — ● Bill (#)
Carl ● — ○ Adam
Bill ○ — ● Adam
Bill ● — ○ Carl (#)
Adam ○ — ● Carl
Adam ○ — ● Bill

As is shown below, the demonstration of Case 2E will be more complicated than that of Case 2D. Take a close watch on the fierce alternation of win ○ and loss ●.

Adam ● — ○ Bill
Carl ○ — ● Bill (#)
Carl ● — ○ Adam
Bill ○ — ● Adam
Bill ● — ○ Carl (#)
Adam ○ — ● Carl
Adam ● — ○ Bill
Carl ○ — ● Bill (#)
Carl ● — ○ Adam
Bill ○ — ● Adam
Bill ● — ○ Carl (#)
Adam ○ — ● Carl
Adam ● — ○ Bill
Carl ○ — ● Bill (#)
Carl ● — ○ Adam
Bill ● — ○ Adam

No doubt, it would be a sort of miracle that Adam comes out as the very final winner of such complicated battles. Mathematically speaking, no matter how slight it seems to be, it should be counted as a mere possibility .

When we summarize those cases of victory or defeat, we can obtain the lower half of Table 1. If we focus on Case 2, then the total probability for Adam to win the victory cup, namely P (Case 2), should be computed as the sum of the probabilities of Cases 2A, 2B, 2C, 2D, 2E and other possible cases. Overall, it is computed as follows.

$$\begin{aligned} P(\text{Case 2}) &= (0.5)^4 + (0.5)^7 + (0.5)^{10} + \\ &\quad (0.5)^{13} + (0.5)^{16} + \cdots \\ &= (0.5)^4 \times \left[1 + (0.5)^3 + (0.5)^6 + \right. \\ &\quad \left. (0.5)^9 + (0.5)^{12} + \cdots \right] \end{aligned} \qquad (3)$$

As was already stated above (see Eq. (1)), the term in the square bracket【⋯】means the geometric series of the first term 1 and the common ratio $(0.5)^3$. Clearly, such series must converge, and the limit is $1 / [1 - (0.5)^3]$, namely 8/7. Therefore, in the light of Eq. (3), we are nicely led to the following result.

$$P(\text{Case 2}) = (1/16) \times (8/7) = 1/14 \qquad (4)$$

3-3. Analyzing Adam's probability of winning the victory cup

We are ready to mathematically compute the probability for Adam to win the victory cup. In the light of meticulous preparations we have made so far, this is a relatively easy job. As the saying goes, a danger is not as great as it seems in advance.

Frankly speaking, Adam is a lucky person in the sense that his possibility never vanishes even if he is lost in the first match against Bill. Generally speaking, there are two patterns toward the final victory —— the "First Straight Pattern" and the "Second Roundabout Pattern". If Adam wins the fist battle, the First Straight Pattern occurs. Then, he may proceed to finally capture the victory cup. If he loses the first battle, however, the Second Roundabout Pattern begins to take place. Then, in spite of his unhappy start, he may nevertheless manage to finally capture to the cup.

Adam's victory probability, denoted by P (Adam wins the cup), should be calculated as the sum of the probabilities of those two patterns. Hence, in the light of Eqs. (2) and (4), we have the following result.

$$\begin{aligned} P(\text{Adam wins the cup}) &= P(\text{Case 1}) + P(\text{Case 2}) \\ &= (2/7) + (1/14) \\ &= 5/14 \end{aligned} \tag{5}$$

A little reflection would tell us that Bill's chance of winning the cup is the same as Adam's chance. Hence, Bill's probability of winning the cup can be computed as follows.

$$\begin{aligned} P(\text{Bill wins the cup}) &= (2/7) + (1/14) \\ &= 5/14 \end{aligned} \tag{5}$$

3-4. The third man Carl 's probability of winning the cup

Tomoesen is a special form of three-way playoffs in which at the first battle, only two wrestlers, say Adam and Bill, are allowed to be on the ring, with the third man, say Carl, must stay out of the ring and patiently wait for the result of the battle between Adam and Bill. Such a non-symmetric treatment of the three men may suggest that *tomoesen* is not a perfectly fair game: in fact, Carl may be in a disadvantageous position in comparison with Adam and Bill. We will show that this guessing turns out to be true. We also have to understand, however, that *mathematical* disadvantage might not directly lead to *psychological* disadvantage.

Let us more carefully analyze the fundamental difference between the first two men and the third man. All three wrestlers aim at winning the tournament. Then, either Adam or Bill may be lost in the first fight. The defeat at the first battle looks unlucky, but not quite so. This is because, as the "Lucky Case" in Table 1 tells us, either man is guaranteed to possibly come back and finally win the final victory. In short, things will eventually start going his way.

Now, let us compare the destiny of the third man Carl and the one of the other two men. Alas, Carl must stick to only the "Second Lucky Case", not being allowed to have the "First Rightful Case". Let us explain why and how Carl is destined to be in such an unfair position. On the one hand, suppose that Adam defeats Bill at the first feat. Then, if Carl is lost against Adam, then Adam should be the victor of the tournament. On the other hand, suppose that Bill is instead a winner. Then, if

Carl is defeated by Bill, then it is Bill who is the final victor. In those cases of "sudden death", the third man Carl is in a miserable position, having no chance at all to win back. In other words, in contrast to Adam and Bill, if Carl is lost in the first feat, then he will never be smiled by the Goddess of Fortune.

Summing up, Carl is destined to deal with only the "First Rightful Case", having nothing to do with the "Second Lucky Case". Therefore, the probability for Carl to win the cup is expressed as follows.

$$\begin{aligned} P(\text{Carl wins the cup}) &= P(\text{Case 1}) \\ &= 2/7 \end{aligned} \qquad (6)$$

Interestingly enough, this is definitely less than 1/3. Compared with this, the probabilities for both Adam's victory and Bill's victory should be the same, being expressed as 5/14, which is clearly greater than 1/3.

3-5. Considering the rationality of *tomoesen*

We are in a position to discuss the rationality of *tomoesen* from many angles. Let us carefully argue the problem of what "rationality" is all about in the sumo world.

Tomoesen is a specific form of three-way playoffs in which the three wrestlers, Adam, Bill and Carl, are fighting on the ring. If it is a strictly rational game, each player's probability of winning the victory cup must be the same. As a result, the victory probabilities should equally be distributed among the three persons in the following fashion.

$$1/3,\ 1/3,\ 1/3 \qquad (7)$$

As we discussed above in great detail, however, the victory probabilities for the three persons are not equal at all, showing a sort of "distributive bias" as follows.

$$5/14,\ 5/14,\ 4/14 \qquad (8)$$

Such bias clearly exists in the sense that it is inclining a bit toward the first two wrestlers, Adam and Bill. As a result, the third man Carl is forced to be in a slightly disadvantageous position. Exactly speaking, the size of such bias is computed as 1/14 or 0.0714286... . The question of much interest is how an ordinary person evaluates this fraction. It depends on the character and feeling of a person. While some people might think that it is large enough and cannot be neglected, other people might not think so. Perhaps, it is a psychological problem rather than a mathematical one. In human sport, "warm heart" may work strongly than "cool head".

We have thus far extensively discussed how a three-way playoff in the sumo world shoud not be regarded as "a fair game" in a narrow sense. Such a conclusion, however, may be compromised if we are allowed to introduce a sort of "third factor" such as "decision by drawing lots" into the playoff. In reality, before the playoff is conducted, the order in which three wrestlers face one another is decided by drawing lots. As a result, the first wrestler has a sumo bout with the second wrestler, ant the third wrestler sits out of the sumo ring and waits for the chance to have another bout with the winner of the previous bout.

Then, the probability of each wrestler to win the victory trophy seems to be calculated as follows.

$$(5/14)(1/3)+(5/14)(1/3)+(4/14)(1/3)$$
$$=(14/14)(1/3)=1/3$$

We have to keep in mind that such a conclusion is no more than the result of the strange combination of the two different items, namely a three-way playoff in pure form and drawing lots in practice. Besides, if we completely rely on drawing lots per se, playoffs on the ring would be regarded as unnecessary sport shows.

We believe that theory is just theory. No practiocal compromise such as drawing lots or tarot reading should be desirable.

IV The charm of tomoesen : the spiritual climate of Japan

In the above, we have carefully discussed *tomoesen* in both historical and theoretical perspectives. It is a specific form of three-way playoff that is observed only in Japan in the world. Historically speaking, while almost 80 years have passed after the end of the Pacific War, *tomoesen* has been played only seven times. Therefore, *tomoesen* is neither a commonly seen event nor a very rare event: it can be seen almost once in ten years.

We have also done a detailed theoretical analysis with the support of mathematical probability theory. As has been seen above, from a mathematical point of view, *tomoesen* cannot be said to be a fair game, even though it has been a very favorite sport from the old good days to the present time. Then, the interesting question which might naturally come to our mind is why such a mathematically unfair game such as *tomoesen* of sumo has been overwhelmingly loved by the Japanese people for more than a thousand years.

Perhaps, there would be a certain group of people who say that excitement sometimes tends to overwhelm rationality. After all, human being is a complex animal, always seeking a delicate balance between reason and emotion. Famous economics professors George A. Akerlof and Robert J. Shiller have jointly written a very exciting yet a bit controversial book in order to show how human psychology tends to derive the economy and why it matters for global capitalism. For details, see Akerlof and Shiller (2009).

In order to understand the charm of *tomoesen*, it is quite useful to discuss the problem of whether and to what extent "mathematics in Europe" is similar to, or different from, "mathematics in Japan". Although mathematics seems to be an highly abstract and global entity, we believe that it cannot escape from a delicate product of local cultures and traditions.

First of all, it should be noticed that the following three mathematicians were outstanding contemporaries in the seventeenth century, making great contributions to the development of differential and integral analysis. [4)]

Isaac Newton (1642 - 1727), Great Britain
Gottfried W. Leibniz (1646 - 1716), Germany
Takakazu Seki (1649? - 1708), Japan

Although the exact birth date of the Japanese genius Seki remains unknown to us, it should be a sort of miracle that in almost similar periods yet in different countries, those three geniuses were actively involved in abstract mathematics, especially in the difficult subject of differential and integral calculus.

4) For the men of mathematics, see Bell (1937) and Yano (1966).

A bit earlier than those calculus geniuses, another branch of mathematics, named the theory of risk and probability, was created and largely developed by the following two mathematicians. [5)]

Pierre Fermat (1601 - 1666), France
Blaise Pascal (1621 - 1662), France

French mathematicians seem to distinguish themselves from other people in the sense that they devoted their lives to developing unique subjects such as risk and probability, which are closely related to society and common sense. For example, Pierre Simon Laplace (1749 - 1827), a successor of the French tradition of mathematical science, remarked as follows. [6)]

> "We see that the theory of probability is at bottom only common sense reduced to calculation; it makes us appreciate with exactitude what reasonable minds feel by a sort of instinct, often without being able to account for it. It is remarkable that this science, which originated in the consideration of games of chance, should have become the most important object of human knowledge."
>
> (Laplace (1812))

According to the superman Laplace, the theory of probability is a common sense reduced to calculation, being originated in the consideration of games of chance. Looking back to *tomoesen*, a unique way of three-way playoffs in Japan, it has been originated in the long tradition of sumo, i.e. unequaled sport game of chance. It is really a kind of mystery why the mathematical consideration of Japanese *tomoesen* has been rather neglected so far. We sincerely hope that this paper will shed a new light on such a problem of great importance.

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5) For the history of the theory of risk and probability, see Gillies (2000) and Laplace (1812).

6) For this point, see Akerlof and Shiller (2009), Sakai (2019) and Sen (1977). All of those works indicate that a human being is neither a "mere economic animal" nor "a rational fool". We should keep in mind that there should be a reasonable balance between fairness and excitement.

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Analyzing Three-Way Playoffs in the Sumo World

Fairness and Excitement

Yasuhiro Sakai

This paper is concerned with *tomoesen*, a special method of three-way playoffs in the Japanese sumo world. In the *tomoesen*, only two wrestlers fight each other, with the third man resting for a while. Then, the winner of the first fight has to fight against the resting man. The former may be either a winner or a loser. This kind of fight will continue until there appears some one who bravely wins the two successive fights. Since the Pacific War ended, while more than 450 fights have been carried out, we have witnessed special *tomoesen* merely seven times. From a theoretical viewpoint, we can show that each of the men who come up to the ring has the victory probability of 5/14, and the third man, the probability of 4/14. The difference between those two numbers is 1/14 or 7.14286 „„ . Although *tomoesen* is not a fair game in a strict sense, it has nevertheless been accepted by the public as an exciting sport game. It seems that there exists a reasonable balance between fairness and excitement.

Key words: *tomoesen*, three-way playoffs, fairness versus excitement