Articles

Consumer Decisions and Revealed Preference

Reevaluating the Foundations of Samuelson's Economic Analysis

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I The "Red Text" of the U.S.S.R. Institute versus the "Blue Text" of Paul A. Samuelson

I have been doing research in economics more than sixty long years. In hindsight, we recall that in the 1960s when we were ambitious students in Japan, there existed two popular yet opposing textbooks in economics. Remarkably, the were nicknamed the "Red Text" and the "Blue text." The Red text was well-represented by the U.S.S.R. Academy Economics Institute (1958) *Economic Textbook : the third revised version* [Japanese translation, 1958], which was no doubt the most authoritarian text of the "Red Bloc" or the Soviet socialist bloc.[1)]

The issue of "socialism versus capitalism" constituted the central theme of the red text. The honorable coauthors of the big text took pride in reaching the following provocative conclusions:

> We have thus far discussed the whole processes of economic development of a society. As a result, we have reached the most important conclusion that from a historical viewpoint, capitalism is destined to collapse whereas socialism is marching for its final victory over socialism. There should be no other way possible. We are so confident of such historical inevitability. (U.S.S.R. Econ Institute, Japan ed. (1959), p. 1050).

In contrast to the powerful Red Text, the Blue Text, presumably being regarded as a

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1) For the rivalry between capitalism and socialism, see Tsuru (1961) and Galbraith (1977).

strong rival, seemed to be rather modest and even hesitant to the general public. Indeed, Paul A. Samuelson , one of the leading economists in the "Blue Block", wrote the world-popular textbook *Economics* (1955, 7th edition 1967), gently defending the American capitalist bloc in the following way:

> America leads Russia, but will the gap narrow? (Samuelson, 7th ed., 1967, p. 791.)

Comparing the "Red Text" and the "Blue Text", the majority of the people seemed to be more impressed by the powerful "Red Text" than the moderate "Blue Text." There existed the minority group including myself, however, who thought that the power of the "Red Wave" would be only temporary, and eventually overtaken by the "Blue Wave", which would act like the "Great Wave" of Japanese ukiyo-e Hokusai.

Let us recall the street demonstrations and campus boycotting which greatly hindered the Japanese students in their study. After some hesitation, I decided to go to the United States so that I could continue my graduate study without unduly political and psychological interruptions. I just wanted to get out of the Japanese university disturbance in the 1960's, thus daring to jump into the very core of the capitalist economy.

At appearance, it looked like an escape trip. However, in reality, it was an immense challenge or an enormous adventure to me. As the saying goes, god helps those who help themselves. In order to get financial and psychological help from the outside organizations, I had to do a lot of preparations for studies in advanced economics and higher mathematics.

II Revealed Preference Theory: Paul A. Samuelson's Analysis as a Starting Point

Paul A. Samuelson's *Foundations of Economic Analysis* (1947) served as a sort of bible for all the graduate students who already mastered his popular text *Economics* (1955, 7th edition 1967) and wanted to study at a more advanced level. The very first page per se was decorated with the following impressive passage:

> "Mathematics is a Language. "
> American mathematician. Willard Gibbs (1839-1903)

I certainly guess that the young and ambitious Samuelson was quite impressed by such a short yet powerful message by an influential mathematician when he decided to write his masterpiece *Foundations of Economics*. Not only that, Chapter 1 of Samuelson (1947) begins with the following impressive passage:

> *The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to those central features.* This fundamental principle of generalization by abstraction was enunciated by the eminent American mathematician E. H. Moore more than thirty years ago. It is the purpose of the pages that follow to work out its implications for theoretical and applied economics. (Samuelson 1947, p. 3.)

As was well-stated by E. H. Moore (1862-1932), although there exist a variety of *particular* theories, there is a *general* theory which underlies those particular theories and unifies them with respect to those central features. This generalization principle was clearly stated by the eminent mathematician Moore, and later confirmed by the young lion Samuelson. When I opened the first page of Samuelson's *Foundations*, I myself was greatly shocked like a thunderbolt from a clear sky, and firmly felt that this was the time to say good-bye to the infertile and unproductive controversy between the "Red Text" and the "Blue Text" mentioned above. Following Moor and Samuelson, I then felt that I must do everything in my power toward a "high ivory tower" of unifying particular theories.

Revealed preference theory was first pioneered by the young Samuelson in his short article in 1938, and later fully developed in his main work *Foundations* 1947. Before Samuelson came on the stage, the existing theories of consumer demand were based on the traditional idea that consumers made consumption decisions to maximize their utility levels. For details for this point, see R, G. D. Allen (1936), J.R. Hicks (1st ed. 1930, 2nd ed. 1948), and others.

Revealed preference theory initiated by Samuelson requires us to make a drastic change in our way of thinking. Instead of stating with utility functions or ordinal preferences to derive demand functions, we first pay our attention to consumer's demand behavior, and later infer consumer's preferences on the basis of a certain set of "rational behavior assumptions." The question to ask is what the "rational behavior assumption" are all about. Although there may be several assumptions conceivable, Samuelson's choice was as simple and lucid as what we now call "the weak axiom of revealed preference."

The fundamental idea of Samuelson's analysis is that "consumers reveal their preferences through their market behavior" if people's judgment is " rational and consistent." As is seen Fig. 1, let us suppose that there are two goods in the market, x_1 and x_2, and two bundles of goods, $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Let us also suppose that the two budget sets, $B(p, m)$ and $B(p', m')$, and the two isoquants or utility curves, u and u', are drawn as nicely as in Fig. 1.

If the commodity bundles x and y are respectively chosen at the budget sets $B(p, m)$ and $B(p', m')$ by the consumer, we may conveniently obtain the following relations:

$$x = h(p, m) \in B(p, m)\ ; \qquad (1)$$

$$y = h(p', m') \in B(p', m')\ . \qquad (2)$$

It is clear in Fig. 1 that the two bundles, $x = (x_1, x_2)$ and $y = (y_1, x_2)$, belong to the same budget set, $B(p, m) = \{ (x_1, x_2) : p_1 x_1 + p_2 x_2 \leqq m\}$. Although these two bundles can be bought at (p, m), it is only x that is actually chosen .

Under such circumstances, Samuelson characteristically believes that x is revealed preferred to y . Someone might think that this is a too strong statement to accept. Later developments of consumer demand theory, however, have more or less taken Samuelson's side, with a certain qualification. In fact, we now know that the *weak axio*m of revealed preference a la Samuelson is not strong enough to guarantee

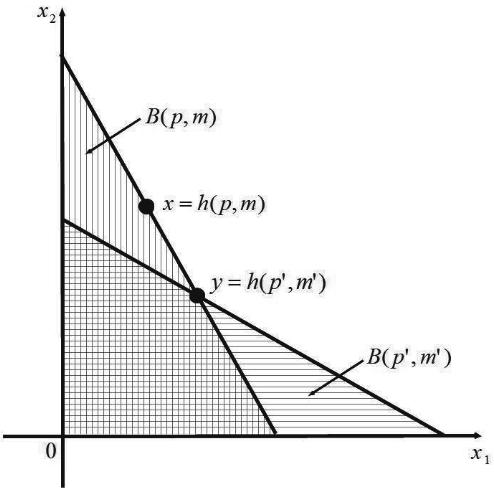

Fig. 1 Samuelson claims that x is "*revealed preferred*" to y. $B(p, m)$ and $B(p', m')$ are respectively indicated by the vertically-shaded triangle and the horizontally-shaded triangle.

the derivation of utility functions. We need to have something more than the *weak axiom* : Actually, the *strong axiom* of revealed preference that was later proposed by the cool-headed Houtakker should be adopted to successfully derive utility functions..

In short, this is the brief essence of revealed preference theory, showing the starting point of Samuelson's unique analysis. More detailed and technical discussions will gradually be done in the following sections.

III Some Technical Preparations

We are concerned with a consumer faced with the problem of choosing a commodity bundle subject to given prices and income. Let us assume that there are n commodities, labeled $i = 1, \ldots, n$. Therefore, we will work within the n - dimensional vector space R^n.

The commodity space Ω , or the set of all conceivable bundles, is defined as follows:

$$\Omega = \{x = (x_1 , \ldots , x_n) : x \geqq 0\}. \tag{3}$$

Let P and M be the price space and the income space, respectively. We make the following assumptions:

$$P = \{p = (p_1 , \ldots , p_n) : p \geqq 0\}, \tag{4}$$

$$M = \{m : m \geqq 0\}. \tag{5}$$

Now, we denote by B the family of all competitive budgets, where for each $(p, m) \in P \times M$, each budget is defined as follows.

$$b(p, m) = \{x \in \Omega : px \leqq m\}. \tag{6}$$

Let h be a single-valued demand function on B, that is, a function which to each $b(p, m) \in M$, assigns a commodity bundle $x = h(b (p, m))$. It is supposed here that $h(b(p, m)) = 0$ for $m = 0$. We denote by X the range of the demand function h :

$$X = \cup \{h(b(p ,m)) : b(p, m) \in M\}. \tag{7}$$

In what follows, we make the two assumptions:

(D1) X is a convex set in Ω .

(D2) h satisfies the budget equation $ph(p, m) = m$ for any $(p, m) \in P \times M$.

For convenience, for $h(b(p, m))$ and $b(p, m) \in B$, we will also simply write $h(p, m)$ and $(p, m) \in B$.

We are now in a position to define "preference relations in terms of h" as follows. To this end, suppose the following relations hold:

(SRP) There is $b(p, m) \in B$ such that $x = h(p, m)$, $y \in b(p, m)$, and $x \neq y$.

Then, following Samuelson (1947), we say that x is *directly revealed preferred to y*, and we simply write xSy. If we look back at Fig. 1 above, we may easily understand the graphical meaning of the relation (SRP). Indeed, seeing is believing !

Fortunately or unfortunately, (SRP) is not only one of revealed preference relations conceivable; Indeed, Houthakker (1950) succeeded in extending the Samuelson revealed relation (SRP) in a very natural way. More specifically, let us consider a new preference relation specified as a long chain of (SRP):

(HRP) For some finite sequence $u^1, \ldots, u^m$, we have $xSu^1S \ldots u^mSy$.

Then, in line with Houthakker (1950), we say that x is *indirectly revealed preferred to y*, conveniently writing xHy. In other words, H is the transitive closure of S on Ω. Clearly, xSy implies xHy, but not the other way around.

Samuelson's weak axiom (W) and Houthakker's strong axiom (S) of revealed preference are formulated as follows:

(W) For any $x, y \in \Omega$, xSy implies $\sim ySx$.

(S) For any $x, y \in \Omega$, xHy implies $\sim yHx$.

In other words, the weak axiom (W) argues that the relation S is not symmetric, whereas the strong axiom (S) states that the relation H is not symmetric. Each axiom clearly shows the consistency of consumer's choice behavior in a direct or indirect way.

Let $E(p)$ be the well-known Engel curve associated with a price vector p:

$$E(p) = \cup \{h(p, m) : m \in M\}, p \in \mathrm{P}. \quad (8)$$

Denoting by $Cls(X)$ the closure of X, let us define the following sets of commodity bundles:

$$\underline{A}(x^0) = \{x \in Cls(X) : \sim xHx^0\}, x^0 \in X; \quad (9)$$

$$\bar{A}(x^0) = \{x \in Cls(X) : \sim x^0Hx\}, x^0 \in \mathrm{X}. \quad (10)$$

On the one hand, the under-bar set $\underline{A}(x^0)$ stands for the set of commodity bundles which are not indirectly revealed preferred to a given commodity bundle x^0. On the other hand, the upper-bar set $\bar{A}(x^0)$ denotes the set of commodity bundles to which a given commodity bundle x^0 is not indirectly revealed preferred.

Now, in order to postulate a "regularity condition" under which the weak and strong axioms are to be established, it is quite useful for us to introduce two different kinds of *expenditure compensation functions as follows* : [2]

$$\bar{a}(p, x^0) = sup\ \{px : x \in \underline{A}(x^0) \cap E(p)\}, \quad (p, x^0) \in P \times X, \quad (11)$$

$$\underline{a}(p, x^0) = inf\ \{px : x \in \bar{A}(x^0) \cap E(p)\}, \quad (p, x^0) \in P \times X. \quad (12)$$

2) The expenditure compensation function was first introduced and effectively employed by McKenzie (1957).

On the one hand, the newly defined function $\tilde{a}(p, x^0)$ stands for the supremum of those expenditures px ($= m$) at price p such that the commodity bundle x = $h(p, m)$ is not indirectly revealed preferred to a given commodity bundle x^0. On the other hand, the function $\underline{a}(p, x^0)$ indicates the infimum of those expenditures $px(= \mathrm{m})$ at price p such that a given commodity bundle x^0 is not indirectly revealed preferred to the commodity bundle x = $h(p, m)$. It will be seen that under assumptions (D1), (D2), and (S), the functions $\tilde{a}(p, x^0)$ and $\underline{a}(p, x^0)$ are well-defined for all $(p, x^0) \in P \times X$.

At first glance, the meanings of those functions may appear to be complicated and difficult for the reader to comprehend. As the saying goes, however, seeing is really believing. Graphical representations will help us to easily understand what is going on here. For instance, in Fig. 2, the whole commodity set X is the union of the upper-shaded set {x : xHx^0} and the lower-shaded set {x : ~ xHx^0}, where the base commodity x^0 belongs to the lower-shaded set, but not the upper-shaded set. Depending on the condition of the demand curve $h(p, m)$, the Engel curve $E(p)$ may, or may not, be continuous. If it happens to be continuous, it can be drawn as a continuous curve like $E(p)$ in Fig. 2. So, the expenditure compensation function $\tilde{a}(p, x^0)$ can roughly be depicted as a downward-sloped line there.

We are now in a position to carefully introduce the *regularity condition* which has to play a very important role as a bridge between the weak and strong axioms of revealed preference. Specifically, such a *regularity condition* is formulated as follows:

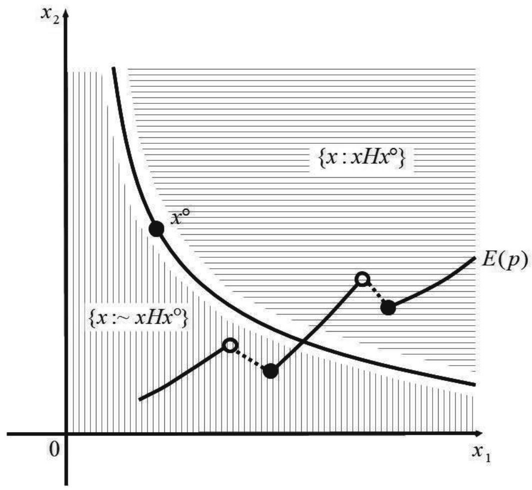

Fig. 2 The upper-shaded set $\{x : xHx^0\}$ and the lower-shaded set $\{x : \sim xHx^0\}$.
The Engel curve $E(p)$ may, or may not, be continuous. It breaks twice here.

(R) For all $(p, x^0) \in P \times X$, $\tilde{a}(p, x^0) \geqq \underline{a}(p\ , x^0)$.

It would not be an easy job to describe the exact meaning of the condition (R). Broadly speaking, the condition (R) says that for a given price vector p and a given commodity bundle x^0, the maximum of "worse-than-or-as-good-as- x^0" expenditures is greater that or equal to the minimum of "better-than-or-as-good-as x^0" expenditures. Once again, graphical illustrations would help us to understand what is going on here. Fig. 3 shows the situation under which the condition holds, while Fig. 4 shows the situation under which it is violated. As is clear from a comparison between those two figures, the regularity condition (R) requires that there be no "*~H gap*" on the Engel curve $E(p)$ for any p, yet allowing for "*~H overlap*" on it. It should be noted that there may exist "jumps" on the Engel curve since the continuity of the demand function is not guaranteed here.

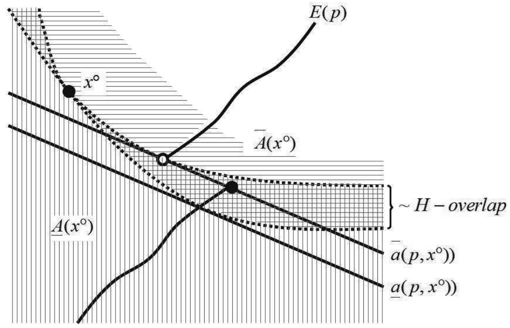

Fig. 3 (R) is satisfied here. $E(p)$ is not continuous, jumping once.

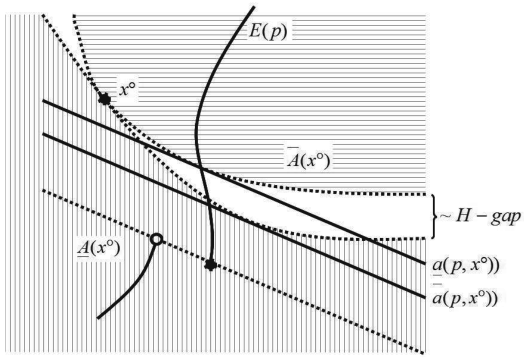

Fig. 4 (R) is violated here. $E(p)$ is not continuous, jumping once.

IV Equivalence of the Weak and Strong Axioms

In this section, we would like to establish an important equivalence theorem of the weak and strong axioms of revealed preference on $Cls(X)$, the closure of the range of the demand function. More exactly, we wish to demonstrate that, for the demand function satisfying (D1) and (D2), the strong axiom (S) holds on $Cls(X)$ if and only if the weak axiom (W) together with the regularity condition (R) holds on $Cls(X)$. This is no doubt the core of our research, requiring a careful, step-by-step inquiry.

We need several preliminary results in preparation for the equivalence theorem. To this end, we first note the following lemma.

LEMMA 4.1 ($\bar{A}$ as a closed set).

Let demand function h satisfy (D1, 2) and the weak axiom (W). Then, for any $x^0 \in X$, the upper-bar set $\bar{A}(x^0)$ is a closed set in $Cls(X)$.

Proof. It suffices to show that the set $\{x \in Cls(X) : x^0 H x\}$, $x^0 \in X$, is an open set in $Cls(X)$.

First, if $x^0 = 0$, then this set is evidently empty, so that it is open. Next, if $x^0 \neq 0$, then let us choose any $x^1 \in X$ such that $x^0 H x^1$. Then, by the definition of H, there exists an $x^2 \in X$ such that

$$x^0 = x^2 \quad \text{or} \quad x^0 H x^2 \tag{13}$$

and $p^2 x^2 \geqq p^2 x^1, x^2 \neq x^1$.

Let $x^t = (1 - t)x^1 + tx^2$, $0 \leqq t \leqq 1$. Then, we find $x^t \in X$ since X is convex by (D1). So, we should have the following:

$$p^2 x^2 \geqq p^2 x^t, x^t \neq x^1 . \tag{14}$$

This relation together with (W) must yields the following results:

$$p^t x^2 > p^t x^t. \qquad (15)$$

Because of definition of x^t, we must have the following equation:

$$(1-t)\,(p^t x^1 - p^t x^t) + t(p^t x^2 - p^t x^t) = 0,$$

whence we find $p^t\,x^t > p^t x^1$ by (15.). Here, we can choose a neighborhood $V(x^1)$ of x^1 such that

$$p^t x^t > p^t y \text{ for any } y \in V(x^1) \cap Cls(X). \qquad (16)$$

If we combine Eqs. (13), (14) and (16), we obtain $x^0 Hy$ for any $y \in V(x1) \cap Cls(X)$. This ensures the desired result that the set $\{x \in Cls(X) : x^0 Hx\}$, $x^0 \in X$, is an open set in $Cls(X)$.

Q.E.D.

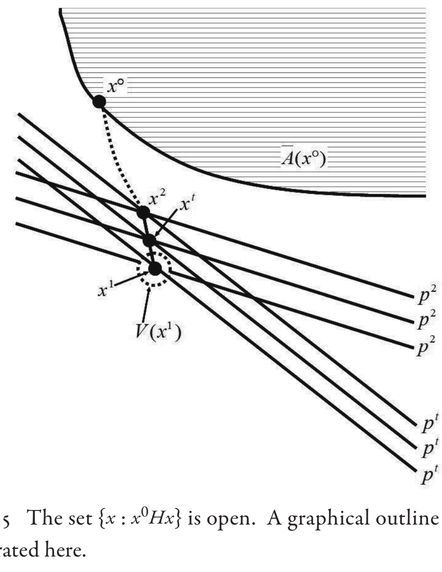

Fig. 5 The set $\{x : x^0 Hx\}$ is open. A graphical outline is illustrated here.

As is seen above, the proof of Lemma 4.3 is tedious but straightforward. Its graphical and intuitive outline, showing that the set $\{x : x^0 Hx\}$ is open, may be given in Fig. 5. This lemma will effectively be utilized in the equivalence theorem.

To reach the final goal of the equivalence theorem, we need to pass through the expenditure compensation functions to be stated below.

LEMMA 2 ($\tilde{a}$ and $\underline{a}$ are well-defined)

Let demand function h satisfy (D1, D2) and the strong axiom (S). Then, for any $(p, x^0) \in P \times X$, the expenditure compensation functions $\tilde{a}\,(p, x^0)$ and $\underline{a}\,(p, x^0)$ are well-defined.

Proof.[3] To see that the $\tilde{a}(p, x^0)$ is well-defined, it suffices to see that the following set is nonempty and bounded above (see Fig. 6 for an illustration) :

$$T(p, x^0) \equiv \{px : x \in \underline{A}(x^0) \cap E(p)\}. \qquad (17)$$

To this end, let us choose any $(p, x^0) \in P \times X$, and note $x^0 = h(p^0, m^0) \in B$. There are two possibilities for the value of m^0 : namely, $m^0 = 0$ or $m^0 > 0$. If $m^0 = 0$, then it is clear that $T(p, x^0) = \{0\}$. If $m^0 > 0$, then the set $\{x : 0 \leqq x \in Cls(X)\ \&\ p^0 x < m^0\}$ is nonempty. It is possible to find an $m^1 \in M$ such that, in $Cls(X)$, the hyperplane $px = m^1$ lies below the hyperplane $p^0 x = m^0$. Therefore, we have $p^0 h(p, m^1) < p^0 x^0$. This implies $x^0\ S\ h(p, m^1)$, so that $x^0 H\, h(p, m^1)$. It follows from (S) that we find $\sim h(p, m^1) H x^0$, implying that $p\, h(p, m^1) \in T\,(p, x^0)$.

This ensures that for any $(p, x^0) \in P \times X$, the set $T(p\ , x^0)$ is nonempty.

3) I am grateful to Uzawa (1960) for providing a guide to the proof of this lemma. Its completely revised version was contained in Chipman & Hurwicz & Richter & Sonnenshein (1971).

Next, for any $(p, x^0) \in P \times X$, let us choose an $m^2. \in M$ such that $px^0 < m^2$. It follows that $ph(p, m^2) = m^2 > px^0$, implying that $h(p, m^2)Hx^0$. This ensures that $ph(p, m^2)$ is a upper bound to the set $T(p, x\text{o})$.

The proof that $\underline{a}(p, x^0)$ is well-defined for any $(p, x^0) \in P \times X$ proceeds in a similar way.

Q.E.D.

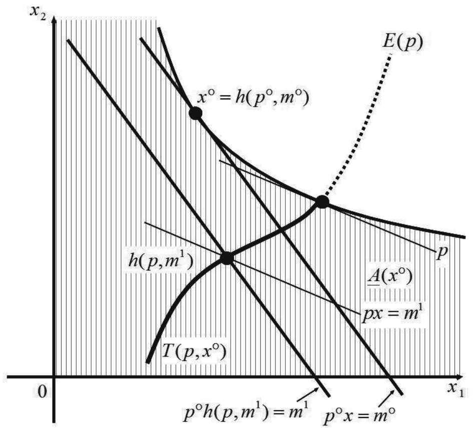

Fig. 6 The proof of LEMMA 2 : a graphical outline

With all the preparations aforementioned, we are now in a position to state and prove the equivalence theorem, which certainly represents the most important result of this chapter.

THEOREM 3 (EQUIVALENCE THEOREM)

Let the demand function h satisfy (D1, D2). Then, the *strong axiom* (S) holds if and only if the *weak axiom* (W) and the *regularity condition* (R) both hold.

Proof. (i) (S) $\Rightarrow$ (W) & (R) : [4)]

Let $h(p, m)$ satisfy the *strong axiom* (S). Then, by definition per se, the *weak axiom*(W) is obviously implied by (S). Note that, as was shown in Lemma 4.2, the expenditure compensation functions $\tilde{a}(p, x^0)$ and $\underline{a}(p, x^0)$ are well-defined for any $(p, x^0) \in P \times X$. In order to derive the *regularity condition* (R), we rely on proof by contradiction. To this end, let us dare to suppose to the contrary that $\tilde{a}(p, x^0) < \underline{a}(p, x^0)$ for some $(p, x^0) \in P \times X$. Then, choose an $m \in M$ so that $\tilde{a}(p, x^0) < m < \underline{a}(p, x^0)$, and let $x = h(p, m)$. Then, we immediately obtain the following:

$$\tilde{a}(p, x^0) < px < \underline{a}(p, x^0). \qquad (18)$$

By the definition of $\tilde{a}$ and $\underline{a}$, we obtain the following:

$$xHx^0 \text{ and } x^0Hx, \qquad (19)$$

which clearly contradicts the *strong axiom* (S). So, in order to get rid of the contradiction, we should have $\tilde{a}(p, x^0) \geqq \underline{a}(p, x^0)$, which is nothing but the *regularity condition* (R).

(ii) (W) & (R) $\Rightarrow$ (S) :

Next, we will prove the *weak axiom* (W) together with the *regularity condition* (R) implies the *strong axiom* (S) . Once again, we rely on the proof by contradiction. So, let us assume otherwise. Then, there exist some commodities $x, x^0 \in X$ such that the following relations hold:

$$xHx^0 \text{ and } x^0Hx. \qquad (20)$$

Let $x = h(p, m)$ for some $(p, m) \in B$. Then, we will show that Eq. (20) implies the following:

$$\tilde{a}(p, x^0) \leqq px \leqq \underline{a}(\mathrm{p}, x^0)\,. \qquad (21)$$

4) I am indebted to Lionel W. McKenzie for suggesting the present proof, which was better than my previous proof.

Indeed, if we assume on the contrary that $\bar{a}(p, x^0) > px$, we could take an $x^* \in \underline{A}(x^0) \cap E(p)$ for which $\bar{a}(p, x^0) \geqq px^* > px$. The last inequality yields $x^* \, Sx$, which together with xHx^0 entails $x^* \, Hx^0$. This contradicts $x^* \in \underline{A}(x^0)$. In order to get rid of the contradiction, we should originally have $\bar{a}(p, x^0) \leqq px$.

The proof that $x^0 \, Hx$, $x \in E(p)$, implies $px \leqq \underline{a}(p, x^0)$ proceeds in an analogous way.

Now, let us specifically assume $px = m = \underline{a}(p, x^0)$. Then, take a sequence $\{px^i\}$ converging to px such that $x^i \in \bar{A}(x^0) \cap E(p)$. This is possible by the definition of $\underline{a}$. Define the new set $Y(p^0, x^0)$ as follows (see Fig. 7 for an illustration) :

$$Y(p^0, x^0) = \bar{A}(x^0) \cap \{x \in Cls(X) \colon \underline{a}(p, x^0) \leqq px < \underline{a}(p, x^0) + 1\}. \qquad (22)$$

Let us focus on the sequence $\{x^i\}$ aforementioned. Since $x^i \in Y(p^0, x^0)$ for sufficiently large i, we may without loss of generality assume that the original sequence $\{x^i\}$ itself is a sequence in $Y(p^0, x^0)$. Since $\bar{A}(x^0)$ is a closed set in $Cls(X)$ by means of LEMMA 1, $Y(p^0, x^0)$ is indeed a compact set in $Cls(X)$. Therefore, we can take a convergent subsequence $\{x^k\} \subset \{x^i\}$ in $Y(p^0, x^0)$. Let $x^1 = lim \, x^k \in Y(p^0, x^0) \subset \bar{A}(x^0)$. Then, we obtain $px^1 = px$. Here, note that either $x^1 = x$ or $x^1 \neq x$ since h may, or may not, be continuous. If $x^1 = x$ then, from Eq. (4.20), we have x^0Hx^1, contradicting that $x^1 \in \bar{A}(x^0)$. If $x^1 \neq x$ then the equality $px = px^1$ yields xSx^1, which together with Eq. (20) implies x^0Hx^1. This also contradicts that $x^i \in \bar{A}(x^0)$.

To get out of the contradiction, we can not specifically assume $px = \underline{a}(p, x^0)$.

Hence, in the light of Eq. (21), we must have the following;

$$\bar{a}(p, x^0) \leqq px < \underline{a}(p, x^0). \qquad (23)$$

This clearly contradicts the regularity condition (R). And now, we can happily declare that the proof is complete. Q.E.D.

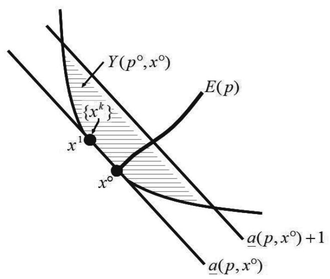

Fig. 7 The set $Y(p^0, x^0)$ in the proof of THEOREM 4.3. is illustrated.

Needless to say, THEOREM 3 (the equivalence theorem) represents the most important result in this chapter. As is easily seen in the detailed proof aforementioned, LEMMA 1, demonstrating that the set $\bar{A}(x^0)$ is closed under (W), plays a critical role in the proof part. Indeed, if the set $\bar{A}(x^0)$ is NOT closed, we may have the case in which $x \notin \bar{A}(x^0)$ and $x \notin$

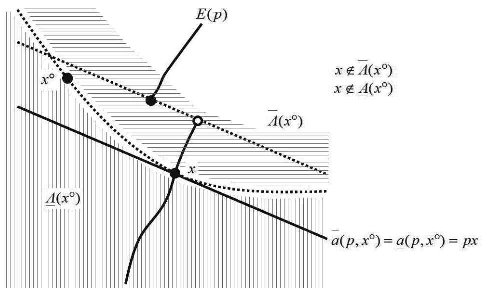

Fig. 8 The set $Y(p^0, x^0)$ is illustrated here for the proof of Theorem 3.

$\underline{A}(x^0)$, meaning that x^0Hx and xHx^0, and $\tilde{a}(p, x^0) = \underline{a}(p, x^0) = px, x \in E(p)$. For a graphical outline of this point, see Fig. 8. In such a case, h surely satisfies the regularity condition (R), but not the strong axiom (S).

It is worthwhile to discuss how our equivalence result distinguishes itself from the previous results in the related literature. First of all, we can say that the hypothesis of THEREM 4.3 is definitely weaker than that of van Moeseke's Theorem 2.5 [4, p. 99] since : (i) neither Lipschitz nor continuity conditions are imposed on the demand function h, (ii) X need not be the positive or nonnegative orthant of R^n ; it can be any convex subset.

Let us recall that Sonnenschein's Example 2 (1971, p.274) shows that (D1, D2) and (S) do not imply the continuity of h. In view of THEOREM 4.3, it is easily seen that the regularity condition (R) [even together with (D1, D2) and (S)] does not suffice to guarantee the continuity of h. Besides, based upon the strong axiom of revealed preference (S), Hurwicz and Richter (1971) carefully studied revealed preference theory *without demand continuity assumptions*.[5)]

In the light of these works mentioned above, the significance of our equivalence theorem *without demand continuity assumptions* is quite evident. Generally speaking, continuity assumptions may be useful yet strong in any subject of research. They may unfortunately be too strong in some research areas including the present consumer demand theory.

5) When I established the equivalence theorem with no continuity assumptions, I received a very positive response from Edmond Malinvaud, one of the leading theoretical economists. In fact, in his famous advanced text, Malinvaud (1972, revised 1985) praised my early paper (1973) by saying that Sakai finished the equivalence theorem in consumer demand theory. It was certainly a nice surprise to me. Note that besides my regularity condition (R), there are several other regularity conditions conceivable (see Sakai (1973)).

V The Effectiveness and Limitations of Rational Decisions

In his first printing of *Foundations of Economic Analysis* (1947), Paul A. Samuelson remarked as follows;

> I was aware that each field involved interdependent unknowns determined by presumably efficacious, interdependent equilibrium conditions — a fact which has always been generally realized. But, and this leads me to the second fundamental purpose of this work, it had not been pointed out to my knowledge that there exist formally identical *meaningful* theorems in these fields, each derived by an essentially analogous method. This is not surprising since only the smallest fraction of economic writings, theoretical and applied, has been concerned with the derivation of *operationally meaningful* theorems.
> (Samuelson 1947, p. 3)

The main purpose of Samuelson's *Foundations* (1947) was to obtain a series of *operationally meaningful* theorems, which had unfortunately been very rare in the 1940s. In fact, when it published at long last after an unavoidable delay due to the Second World War, it received a sort of mixed blessing in the then dominant economics profession. For instance, Kenneth Boulding (1948), one of the finest contemporaries at that time, wrote the following rather critical and cynical review :

> The *Foundations* is an important book. It should be studied not only by the mathematically baptized but also by those who, like myself, hang on to n-dimensions by the skin of their teeth. No economist who studies it can fail to profit by it. Nevertheless, the present reviewer cannot help feeling a certain sense of rapidly diminishing marginal productivity. in the application of mathematics to economics. There is an elusive flavor of John Stuart Mill about the *Foundations* which makes it seem less like a foundation than a coping stone, finishing an edifice which does not have much further to go. It may well be that the slovenly literary borderland between economics and sociology will be the most fruitful building during the years to come and that mathematical economics will remain too flawless in its perfection to be very fruitful. (Boulding 1948, p. 199.)

As Boulding eloquently tells us, Samuelson's *Foundations* is an important book. Every economist would agree with this. Indeed, by making use of the weak axiom of revealed preference with help of his friend Houthakker, Samuelson succeeded in deriving many *meaningful theorems* in consumer choice theory. So far so good. There exist a certain number of economists, however, who might rather put themselves in Boulding's position. We should be very careful of Boulding's remark on Samuelson's *Foundations*, which never admires that *Foundations* is a *very important book*, only saying that it is just an important book. This descriptive difference looks small, but may sometimes be larger than we imagine.[6)]

As far as I am concerned, my own position on *Foundations* is rather delicate. When I was young and gay, I found much interest in mathematical economics, and read Samuelson's many books and papers repeatedly and energetically. To tell the truth, this paper is no more than a completely revised version of Chapter One of my Ph.D. thesis *Foundations of Consumption and Production Theories* submitted to the University of Rochester in September 1972. Professors Lionel W. McKenzie and James W. Friedman were my thesis advisers, thus giving me kind suggestions and constant encouragement. When I was later teaching economic theory at the University of Pittsburgh, my attitude on the application of mathematics to economics gradually changed into the direction of more economic reality rather than pure theory per se, with growing interest in building a bridge between Theory and History. Since John R. Hicks happens to be McKenzie's teacher at Oxford, I am a sort of Hicks' grand child, and then have followed the new Hicks who has found growing interest in *Theory of Economic History*.[7)]

In hindsight, although Boulding's prediction that "the slovenly literary borderland

6) Before Boulding's review appeared, the ordinal utility approach represented by Allen (1936) and Hicks (1939, revised 1946) dominated consumer demand theory for a long time. Although Hicks was influenced by the rise of revealed preference approach, he took an independent middle path between the two approaches. For such a subtle position, see Hicks (1956).

7) It was in 1969 that noted theoretician John Hicks published a very interesting book *A Theory of Economic History*, in which two apparently different subjects – Theory and History – were nicely combined into one. In Japan also, well-known economist Takashi Negishi (1989) wrote an outstanding book *History of Economic Theory* a la John Hicks. In postscript of the book, Negishi eloquently remarked. " The currently dominating mainstream theory is not the only possible theory. The study of the history of economic theory is important, not only from the heretical point of view, to show how and why the current mainstream theory is wrong, but also from the mainstream point of view, to spur on the development of its own thinking." (Negishi 1989, p. 385.)

between economics and sociology will be the most fruitful building during the years to come" may not be totally right, I do think that it has been fairly correct. In fact, from the 1980s to the present 2020s, there have gradually yet surely emerged new waves of social sciences toward "more psychology, more sociology and still more history". Amartia Sen (1977, 1987) is one of those economists, which has constantly issued a strong warning against *excessively rational and unduly consistent human behavior*, for it would possibly lead us to the non-realistic world where all men and women as "rational fools" make neither mistakes nor contradictory behaviors. As the saying goes, extremely spicy things are not good for the stomach. Very noted Japanese essayist Shichihei Yamamoto (1983) wrote a very popular yet sarcastic essay *A study of "koo-ki" or social pressure*, arguing that the heart of ordinary people with cool head sometimes tend to *get hot and excited under impersonal and social pressure*. For instance, at the end of the Second World War, many young and talented Japanese pilots dared to take on zero fighter planes and carry out *"kamikaze" or suicidal attack missions with no hope of victory*. And now, in parallel with the emergence of the Economics of Uncertainty, we can see the coming and growing wave of Behavioral Economics, with Akerlof & Schiller (2009) , Kahneman (2003) and Thaler (2015) being recent leaders, in which many biases and irregular human behavior cannot be neglected and should be taken care of in standard social sciences.

In conclusion, we can argue that revealed preference theory a la Samuelson and Houthaker is no longer perfect, having many weaknesses and limitations. Not only that, all theories in economics and related fields should not be free of shortcomings and defects. I do declare, however, that Samuelson's reveal preference theory is still alive today and can be further developed.[8)]

We can still learn new lessons from old teachings.

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8) For the reevaluation of revealed preference in the twenty-first century, see Niholson (2005), Szenberg & Ramrattan & Gottsman (2006), Varian (2006), and Chambers & Cristopher (2016) .

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Consumer Decisions and Revealed Preference

Reevaluating the Foundations of Samuelson's Economic Analysis

Yasuhiro Sakai

This paper discusses consumer decision and revealed preference from many new angles, thus carefully reexamining the foundations of economic analysis. It is Paul A. Samuelson's contribution who boldly introduced to microeconomic theory the brand-new concept of *revealed preference* against the then standard doctrine of ordinal utility. The main result of this chapter is that Houthakker's *strong axiom of revealed preference* holds if and only if Samuelson's *weak axiom of revealed preference* and *a certain "regularity condition" newly introduced here* both hold. This equivalence result distinguishes itself from all the previous works in the sense that no continuity assumptions are imposed on the demand function. In the light of uncertainty and behavioral economics in the present times, the significance and limitations of Samuelson-type economics are also discussed. Throughout this research, many graphical illustrations are attempted for giving a helping hand to understand mathematical lemmas and theorems.

Keywords: revealed preference, Samuelson, the weak axiom, Houthakker, the strong axiom, regularity conditions, behavioral economics