

A UNIFIED FRAMEWORK OF THE ITERATIVE SCHEME GENERATING METHOD FOR EXTENDED MANN–ISHIKAWA-TYPE AND CESÀRO MEAN ITERATIONS

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ABSTRACT. This paper establishes a unification within the iterative scheme generating method (ISGM) framework by integrating iterations of extended Mann–Ishikawa type with full Cesàro mean iterations. Under the proposed unified ISGM schemes, we prove weak convergence of the sequences generated by quasi-nonexpansive mappings that are either demiclosed or mean-demiclosed. This work provides the first such unification within the ISGM framework and thereby expands the class of admissible iterative algorithms. As an application, the results are applied to a linear split feasibility problem.

1. INTRODUCTION

Throughout this paper, we denote by H a real Hilbert space with the norm $\|\cdot\|$ induced by an inner product $\langle \cdot, \cdot \rangle$. A mapping $T : C \rightarrow H$ is called nonexpansive if

$$(1.1) \quad \|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C,$$

where C is a nonempty subset of H . The set of fixed points of T is denoted by

$$F(T) = \{x \in C : Tx = x\}.$$

Numerous studies have investigated iterative methods to approximate fixed points of nonexpansive mappings.

The evolution of fixed point approximation theory can be viewed as a pursuit of two complementary goals: expanding the class of admissible mappings and developing robust, general-purpose algorithms. Historically, these streams progressed in tandem. The seminal works of Mann [32], Ishikawa [11], and Reich [38] established fundamental iterative schemes, while subsequent research, such as the mean iteration methods initiated by Bailon [2], Reich [37], and Shimizu and Takahashi [40], introduced alternative averaging-based paradigms.

In 2023, a shift in perspective occurred with the *iterative scheme generating method* (ISGM), which moved beyond proposing individual algorithms

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to providing a systematic framework for generating entire families of convergent iterations under verifiable conditions [19, 21]; see also Kondo [22, 23]:

Theorem 1.1 ([19]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let $S, T : C \rightarrow C$ be quasi-nonexpansive (3.7) and mean-demiclosed mappings (3.11) such that $F(S) \cap F(T) \neq \emptyset$. Let $P_{F(S) \cap F(T)}$ denote the metric projection from H onto $F(S) \cap F(T)$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\varliminf_{n \rightarrow \infty} a_n b_n > 0$, and $\varliminf_{n \rightarrow \infty} a_n c_n > 0$. Define a sequence $\{x_n\}$ in C as follows:*

$$(1.2) \quad \begin{aligned} x_1 &\in C : \text{ given,} \\ x_{n+1} &= a_n x_n + b_n \frac{1}{n} \sum_{k=0}^{n-1} S^k z_n + c_n \frac{1}{n} \sum_{k=0}^{n-1} T^k w_n \end{aligned}$$

for all $n \in \mathbb{N} = \{1, 2, \dots\}$, where $\{z_n\}$ and $\{w_n\}$ are sequences in C that satisfy

$$(1.3) \quad \|z_n - q\| \leq \|x_n - q\| \quad \text{and} \quad \|w_n - q\| \leq \|x_n - q\|$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges weakly to a point $\hat{x} \in F(S) \cap F(T)$, where $\hat{x} \equiv \lim_{n \rightarrow \infty} P_{F(S) \cap F(T)} x_n$.

In Theorem 1.1, the terms ‘quasi-nonexpansive mapping’ and ‘mean-demiclosed mapping’ will be formally defined in Section 3. (Intuitively, mean-demiclosedness ensures that weak cluster points of Cesàro-type averages of iterates remain fixed points.) A nonexpansive mapping that has a fixed point satisfies these conditions. The role of the sequences $\{z_n\}$ and $\{w_n\}$ in Theorem 1.1 is open and they are only required to satisfy the conditions in (1.3). Consequently, this method generates infinitely many iterative schemes, including the following iterative scheme:

$$(1.4) \quad \begin{aligned} w_n &= \mu_n x_n + (1 - \mu_n) S x_n, \\ z_n &= \lambda_n x_n + (1 - \lambda_n) T x_n, \\ x_{n+1} &= a_n x_n + b_n \frac{1}{n} \sum_{k=0}^{n-1} S^k z_n + c_n \frac{1}{n} \sum_{k=0}^{n-1} T^k w_n \end{aligned}$$

for all $n \in \mathbb{N}$ where $\lambda_n, \mu_n \in [0, 1]$. Theorem 1.1 was slightly generalized as Theorem 3.3 in Kondo [23].

In 2025, Kondo [25] proved the following:

Theorem 1.2 ([25]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H and let $S, T : C \rightarrow C$ be quasi-nonexpansive and demiclosed mappings (3.8) such that $F(S) \cap F(T) \neq \emptyset$. Denote by $P_{F(S) \cap F(T)}$ the metric projection from H onto $F(S) \cap F(T)$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + c_n = 1$*

for all $n \in \mathbb{N} \equiv \{1, 2, \dots\}$, $\underline{\lim}_{n \rightarrow \infty} a_n b_n > 0$, and $\underline{\lim}_{n \rightarrow \infty} a_n c_n > 0$. Define a sequence $\{x_n\}$ in C as follows:

$$x_1 \in C \text{ is given,}$$

$$x_{n+1} = a_n y_n + b_n S z_n + c_n T w_n$$

for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in C that satisfy the following conditions:

$$\|y_n - q\| \leq \|x_n - q\|, \quad \|z_n - q\| \leq \|x_n - q\|, \quad \|w_n - q\| \leq \|x_n - q\|$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$,

$$x_n - y_n \rightarrow 0, \quad x_n - z_n \rightarrow 0, \quad \text{and} \quad x_n - w_n \rightarrow 0.$$

Then, $\{x_n\}$ converges weakly to a point $\hat{x} \in F(S) \cap F(T)$, where $\hat{x} \equiv \lim_{n \rightarrow \infty} P_{F(S) \cap F(T)} x_n$.

This theorem has been extended in some directions: for strong convergence results using metric projections, see Kondo [24]; for a general class of mappings, see Kondo [26]; and for a Banach space setting, see Kondo [27]. In this paper, we present a new version of the ISGM that unifies two major generalizations in fixed-point approximation theory by Kondo [19, 25] and crystallizes them into a single iterative formalism. The proposed scheme encompasses extended Mann–Ishikawa-type terms and mean-valued terms with arbitrary weights. This unified approach is further motivated by Theorem 3.3 in Kondo [31]. Following these earlier contributions, this study provides the framework that fully unifies the ideas of the major iterative methods, which have been studied separately, in the ISGM framework.

A distinctive element of this formalism is the inclusion of a mean-valued sequence over powers of the mapping T . This mean-valued term theoretically incorporates the essence of the average convergence theorem into the iterative method, allowing us to exploit the structure of the mapping T on a more macroscopic scale. We show that including this term essentially relaxes the conditions for the iterative sequence to converge to a fixed point; see Remarks 4.1 and 5.2. Furthermore, this term may have a certain stabilizing effect on the iterative scheme, which will be an important topic for future empirical research; see Polyak and Juditsky [36].

This paper is organized as follows: Section 2 provides background information. In Section 3, we review various types of nonlinear mappings which significantly extend nonexpansive mappings. We present examples of non-continuous mappings. In Section 4, we establish the main theorems of this study. Section 5 applies the main theorems to various nonlinear mappings. Section 6 deals with applications to split feasibility problems, with a particular focus on the linear case. Section 7 concisely concludes the present paper.

2. PRELIMINARIES

This section recalls some preliminary facts. Let $\{x_n\}$ be a sequence in a real Hilbert space H . We denote strong and weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively. It is known that $x_n \rightharpoonup x$ if and only if for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_{ij}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{ij}} \rightarrow x$. If $x_n \rightharpoonup x$ and $y_n \rightarrow y$, then $\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$.

The nearest point in a convex set is characterized by the following lemma.

Lemma 2.1. *Let C be a nonempty and convex subset of H . Let $x \in H$ and $p \in C$. Then, the following two statements are equivalent:*

- (i) $\langle x - p, p - z \rangle \geq 0$ for all $z \in C$;
- (ii) $\|x - p\| \leq \|x - z\|$ for all $z \in C$.

A proof of Lemma 2.1 can be found in Takahashi [41], page 89. Moreover, if C is closed in H , then for any $x \in H$, there exists a unique nearest point $p \in C$ from $x \in H$. This mapping $x \mapsto p$ is called the *metric projection* from H onto C , denoted by P_C . A metric projection P_C is nonexpansive and, by Lemma 2.1, it satisfies the inequality

$$(2.1) \quad \langle x - P_C x, P_C x - z \rangle \geq 0 \quad \text{for all } x \in H \text{ and } z \in C.$$

The following result is due to Takahashi and Toyoda [44].

Lemma 2.2. *Let F be a nonempty, closed, and convex subset of H , let P_F be the metric projection from H onto F , and let $\{x_n\}$ be a sequence in H . If $\|x_{n+1} - q\| \leq \|x_n - q\|$ for all $q \in F$ and $n \in \mathbb{N}$, then $\{P_F x_n\}$ is convergent in F .*

The following lemma is widely known; see Maruyama *et al.* [33] and Zegeye and Shahzad [47].

Lemma 2.3. *Let $x_n \in H$ and let $a_n \in \mathbb{R}$ ($n = 1, 2, \dots, N$) such that $\sum_{n=1}^N a_n = 1$. Then, the following equation holds:*

$$\left\| \sum_{n=1}^N a_n x_n \right\|^2 = \sum_{n=1}^N a_n \|x_n\|^2 - \sum_{\substack{m,n=1, \\ m < n}}^N a_m a_n \|x_m - x_n\|^2.$$

In the case $N = 4$ in Lemma 2.3, the following holds: if $a + b + c + d = 1$, then

$$(2.2) \quad \begin{aligned} \|ax + by + cz + dw\|^2 &= a\|x\|^2 + b\|y\|^2 + c\|z\|^2 + d\|w\|^2 \\ &\quad - ab\|x - y\|^2 - ac\|x - z\|^2 - ad\|x - w\|^2 \\ &\quad - bc\|y - z\|^2 - bd\|y - w\|^2 - cd\|z - w\|^2, \end{aligned}$$

where $x, y, z, w \in H$. A proof of this case can be found in Takahashi and Kondo [30], page 282. In (2.2), no assumption that $a, b, c, d \in [0, 1]$ is needed.

If $a, b, c, d \in [0, 1]$ is additionally satisfied, then it holds that

$$(2.3) \quad \begin{aligned} & \|ax + by + cz + dw\|^2 \\ & \leq a\|x\|^2 + b\|y\|^2 + c\|z\|^2 + d\|w\|^2 \\ & \quad - ab\|x - y\|^2 - ac\|x - z\|^2 - ad\|x - w\|^2. \end{aligned}$$

Letting $d = 0$ results in

$$(2.4) \quad \|ax + by + cz\|^2 \leq a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x - y\|^2 - ac\|x - z\|^2.$$

The inequalities (2.3) and (2.4) are employed in the proofs of Theorems 4.2 and 4.1, respectively.

3. NONLINEAR MAPPINGS

In this section, we introduce various nonlinear mappings that generalize nonexpansive mappings. For further details, see the appendices in Kondo [21, 25].

Kocourek *et al.* [13] introduced a class of mappings that unifies nonexpansive mappings and some other important classes of mappings. A mapping $T : C \rightarrow H$ is called *generalized hybrid* [13] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$(3.1) \quad \alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. Setting $\alpha = 1$ and $\beta = 0$ in (3.1) yields the defining inequality of nonexpansive mappings (1.1), showing that every nonexpansive mapping is a generalized hybrid mapping. Also, this class of mappings includes nonspreading mappings [14], hybrid mappings [42], as well as λ -hybrid mappings [1].

A mapping $T : C \rightarrow H$ is called *normally generalized hybrid* [45] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$(3.2) \quad \alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0$$

for all $x, y \in C$, subject to the conditions (i) $\alpha + \beta + \gamma + \delta \geq 0$, and (ii) $\alpha + \beta > 0$ or $\alpha + \gamma > 0$. The class of normally generalized hybrid mappings contains generalized hybrid mappings as a case of $\alpha + \beta = 1$ and $\gamma + \delta = -1$. Moreover, if $\alpha + \beta + \gamma + \delta > 0$, a normally generalized hybrid mapping admits at most one fixed point. In the case where $\alpha = 1$, $\beta = \gamma = 0$, and $\delta = -\rho$ with $\rho \in [0, 1)$, the inequality (3.2) reduces to

$$\|Tx - Ty\| \leq \sqrt{\rho}\|x - y\|;$$

hence, the class includes $\sqrt{\rho}$ -contraction mappings.

A mapping $T : C \rightarrow C$ is called *normally 2-generalized hybrid* [28] if there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that

$$(3.3) \quad \begin{aligned} & \alpha_2\|T^2x - Ty\|^2 + \alpha_1\|Tx - Ty\|^2 + \alpha_0\|x - Ty\|^2 \\ & + \beta_2\|T^2x - y\|^2 + \beta_1\|Tx - y\|^2 + \beta_0\|x - y\|^2 \leq 0 \end{aligned}$$

for all $x, y \in C$, subject to the conditions (i) $\sum_{n=0}^2 (\alpha_n + \beta_n) \geq 0$ and (ii) $\alpha_2 + \alpha_1 + \alpha_0 > 0$. When $\alpha_2 + \alpha_1 + \alpha_0 = 1$ and $\beta_2 + \beta_1 + \beta_0 = -1$, such a mapping is referred to as a *2-generalized hybrid mapping* [33]. Moreover, if $\sum_{n=0}^2 (\alpha_n + \beta_n) > 0$, then it admits at most one fixed point.

The notion of a normally N -generalized hybrid mapping is defined analogously. For instance, the condition for a *normally 3-generalized hybrid mapping* is as follows:

$$(3.4) \quad \alpha_3 \|T^3x - Ty\|^2 + \alpha_2 \|T^2x - Ty\|^2 + \alpha_1 \|Tx - Ty\|^2 + \alpha_0 \|x - Ty\|^2 \\ + \beta_3 \|T^3x - y\|^2 + \beta_2 \|T^2x - y\|^2 + \beta_1 \|Tx - y\|^2 + \beta_0 \|x - y\|^2 \leq 0$$

for all $x, y \in C$, where the parameters satisfy (i) $\sum_{n=0}^3 (\alpha_n + \beta_n) \geq 0$ and (ii) $\alpha_3 + \alpha_2 + \alpha_1 + \alpha_0 > 0$.

We now present an example of a normally 2-generalized hybrid mapping; see also Kondo [10, 15, 20, 21]. In (3.3), set $\alpha_2 = \alpha$ and $\beta_2 = -\beta$ with $0 < \beta < \alpha$, and set all other coefficients to zero. Then, (3.3) reduces to

$$(3.5) \quad \alpha \|T^2x - Ty\|^2 \leq \beta \|T^2x - y\|^2.$$

The following function satisfies (3.5):

Example 3.1 . Let $H = C = \mathbb{R}$ and define $T : \mathbb{R} \rightarrow \mathbb{R}$ by

$$Tx = \begin{cases} 0 & \text{if } x \leq \sqrt{\alpha}; \\ \sqrt{\beta} & \text{if } \sqrt{\alpha} < x, \end{cases}$$

where $0 < \beta < \alpha$. Then, T satisfies (3.5); see Kondo [21], page 307. $\square$

Next, we present an example of a normally 3-generalized hybrid mapping. In (3.4), set $\alpha_3 = \alpha$ and $\beta_3 = -\beta$ with $0 < \alpha/4 < \beta < \alpha$, and set all other coefficients to zero. We have

$$(3.6) \quad \alpha \|T^3x - Ty\|^2 \leq \beta \|T^3x - y\|^2.$$

Example 3.2. Let $H = C = \mathbb{R}$ and define $T : \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$Tx = \begin{cases} 0 & \text{if } x \leq \frac{\sqrt{\alpha}}{2}; \\ \frac{\sqrt{\beta}}{2} & \text{if } \frac{\sqrt{\alpha}}{2} < x \leq \sqrt{\alpha}; \\ \sqrt{\beta} & \text{if } \sqrt{\alpha} < x, \end{cases}$$

where $0 < \alpha/4 < \beta < \alpha$. The mapping T satisfies (3.6). Indeed, under the assumption $0 < \alpha/4 < \beta < \alpha$, we have $T^3x = 0$ for all $x \in \mathbb{R}$. Consequently,

$$(3.6) \iff \alpha (Ty)^2 \leq \beta y^2 \iff |Ty| \leq \sqrt{\frac{\beta}{\alpha}} |y|.$$

We verify this inequality in three cases: (i) If $y \leq \sqrt{\alpha}/2$, then $Ty = 0$, so (3.6) holds. (ii) Suppose that $\sqrt{\alpha}/2 < y \leq \sqrt{\alpha}$. Then $Ty = \sqrt{\beta}/2$. Moreover,

$$\sqrt{\frac{\beta}{\alpha}}|y| > \sqrt{\frac{\beta}{\alpha}} \frac{\sqrt{\alpha}}{2} = \frac{\sqrt{\beta}}{2} = Ty.$$

Thus, (3.6) is met. (iii) If $\sqrt{\alpha} < y$, then $Ty = \sqrt{\beta}$. On the other hand,

$$\sqrt{\frac{\beta}{\alpha}}|y| > \sqrt{\frac{\beta}{\alpha}}\sqrt{\alpha} = \sqrt{\beta} = Ty.$$

From the above, the mapping satisfies (3.6) and therefore, it is a normally 3-generalized hybrid mapping. $\square$

Let C be a nonempty, closed, and convex subset of H . A mapping $T : C \rightarrow H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if

$$(3.7) \quad \|Tx - q\| \leq \|x - q\| \quad \text{for all } x \in C \text{ and } q \in F(T).$$

The fixed point set of a quasi-nonexpansive mapping is closed and convex; see Itoh and Takahashi [12] or Kocourek *et al.* [13].

A mapping $T : C \rightarrow H$ is called *demiclosed* if for every sequence $\{x_n\}$ in C ,

$$(3.8) \quad x_n - Tx_n \rightarrow 0 \text{ and } x_n \rightharpoonup v \implies v \in F(T).$$

It is often said that $I - T$ is demiclosed if (3.8) holds, where I denotes the identity mapping. Kondo [17] called a mapping $T : C \rightarrow C$ *2-demiclosed* if for every sequence $\{x_n\}$ in C ,

$$(3.9) \quad x_n - Tx_n \rightarrow 0, x_n - T^2x_n \rightarrow 0, \text{ and } x_n \rightharpoonup v \implies v \in F(T).$$

Similarly, an *M-demiclosed mapping* can be defined for any $M \in \mathbb{N}$. Clearly,

$$(3.10) \quad \text{demiclosed mappings} \subset 2\text{-demiclosed mappings} \subset \dots$$

Given a bounded sequence $\{w_n\}$ in C , define $W_n \equiv \frac{1}{n} \sum_{l=0}^{n-1} T^l w_n (\in C)$ for all $n \in \mathbb{N}$. Kondo [16] defined a mapping T to be *mean-demiclosed* if for every subsequence $\{W_{n_i}\}$ of $\{W_n\}$,

$$(3.11) \quad W_{n_i} \rightharpoonup v \implies v \in F(T).$$

Regarding these types of mappings, the following lemmas are known; see [13, 28, 29, 33, 45]:

Lemma 3.1. *Let $S : C \rightarrow C$ be a normally generalized hybrid mapping (3.2) with a fixed point, where C is a nonempty, closed, and convex subset of H . Then, S is quasi-nonexpansive, demiclosed, and mean-demiclosed.*

Lemma 3.2. *Let $T : C \rightarrow C$ be a normally N -generalized hybrid mapping (3.3) with a fixed point, where C is a nonempty, closed, and convex subset of H . Then, T is quasi-nonexpansive, N -demiclosed, and mean-demiclosed.*

Kondo [26] also proved the above lemmas for the case $N = 2$ in its appendix.

In what follows, we assume that the nonlinear mappings under consideration have a common fixed point. The following result, due to Hojo [8], guarantees the existence of such a point:

Theorem 3.1. *Let C be a nonempty, closed, convex, and bounded subset of H . Let $S, T : C \rightarrow C$ be a normally M -generalized hybrid mapping and a normally N -generalized hybrid mapping, respectively. Assume that $ST = TS$. Then, S and T have a common fixed point.*

See also Hojo *et al.* [9] and Kondo [18], and the references therein. According to Theorem 3.1, the existence of a common fixed point is guaranteed if it is assumed that C is bounded and $ST = TS$, in addition to the standard assumptions in the main theorems (Theorem 4.1–4.3).

4. MAIN RESULTS

In this section, we present an ISGM that combines Mann–Ishikawa-type and mean-valued iterations. This framework generates infinitely many iterative schemes that weakly converge to common fixed points. Although Theorem 4.1 treats the case of a quasi-nonexpansive and demiclosed mapping, the result can be readily extended to the quasi-nonexpansive and M -demiclosed case. We provide a proof for the quasi-nonexpansive and 2-demiclosed case in Theorem 4.2. Theorem 4.3, which addresses the general quasi-nonexpansive and M -demiclosed case, is stated without proof. The essential ingredients of the proof have been developed and refined in numerous prior works; see the references cited in Kondo [15].

Theorem 4.1. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let $S : C \rightarrow C$ be a quasi-nonexpansive and demiclosed mapping. Let $T : C \rightarrow C$ be a quasi-nonexpansive and mean-demiclosed mapping. Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $P_{F(S) \cap F(T)}$ denote the metric projection from H onto $F(S) \cap F(T)$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n \rightarrow \infty} a_n b_n > 0$, and $\underline{\lim}_{n \rightarrow \infty} a_n c_n > 0$. Define a sequence $\{x_n\}$ in C as follows:*

$$x_1 \in C : \text{ given,}$$

$$x_{n+1} = a_n y_n + b_n S z_n + c_n \frac{1}{n} \sum_{k=0}^{n-1} T^k w_n$$

for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in C that satisfy

$$(4.1) \quad \|y_n - q\| \leq \|x_n - q\|, \quad \|z_n - q\| \leq \|x_n - q\|, \quad \text{and} \quad \|w_n - q\| \leq \|x_n - q\|$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ and

$$(4.2) \quad x_n - y_n \rightarrow 0 \quad \text{and} \quad x_n - z_n \rightarrow 0.$$

Then, the sequence $\{x_n\}$ converges weakly to a point $\hat{x} \in F(S) \cap F(T)$, where $\hat{x} \equiv \lim_{n \rightarrow \infty} P_{F(S) \cap F(T)} x_n$.

Proof. Recall that S and T are quasi-nonexpansive (3.7). Consequently, $F(S)$ and $F(T)$ are closed and convex. Therefore, $F(S) \cap F(T)$ is also closed and convex. Since $F(S) \cap F(T) \neq \emptyset$ by assumption, the metric projection $P_{F(S) \cap F(T)}$ from H onto $F(S) \cap F(T)$ is properly defined. Define $W_n = \frac{1}{n} \sum_{k=0}^{n-1} T^k w_n \in C$. Then, we have $x_{n+1} = a_n y_n + b_n S z_n + c_n W_n$.

We prove that

$$(4.3) \quad \|W_n - q\| \leq \|x_n - q\|$$

for any $q \in F(T)$ and $n \in \mathbb{N}$. In fact, using the hypothesis that T is quasi-nonexpansive (3.7) and (4.1) yields

$$\begin{aligned} \|W_n - q\| &= \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k w_n - q \right\| = \frac{1}{n} \left\| \sum_{k=0}^{n-1} T^k w_n - nq \right\| \\ &= \frac{1}{n} \left\| \sum_{k=0}^{n-1} (T^k w_n - q) \right\| \leq \frac{1}{n} \sum_{k=0}^{n-1} \|T^k w_n - q\| \\ &\leq \frac{1}{n} \sum_{k=0}^{n-1} \|w_n - q\| = \|w_n - q\| \leq \|x_n - q\|, \end{aligned}$$

which establishes (4.3).

Next, we prove that

$$\|x_{n+1} - q\| \leq \|x_n - q\|$$

for any $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Choose $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ arbitrarily. Since S and T are quasi-nonexpansive, by (4.1) and (4.3), we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|a_n y_n + b_n S z_n + c_n W_n - q\| \\ &= \|a_n (y_n - q) + b_n (S z_n - q) + c_n (W_n - q)\| \\ &\leq a_n \|y_n - q\| + b_n \|S z_n - q\| + c_n \|W_n - q\| \\ &\leq a_n \|y_n - q\| + b_n \|z_n - q\| + c_n \|W_n - q\| \\ &\leq a_n \|x_n - q\| + b_n \|x_n - q\| + c_n \|x_n - q\| \\ &= \|x_n - q\|. \end{aligned}$$

From this inequality, we observe the following: (i) $\{\|x_n - q\|\}$ converges in $\mathbb{R}$ for all $q \in F(S) \cap F(T)$; (ii) $\{x_n\}$ is bounded; (iii) $\{P_{F(S) \cap F(T)} x_n\}$ converges in $F(S) \cap F(T)$. (iii) follows from Lemma 2.2. We denote the limit by $\hat{x} \equiv \lim_{n \rightarrow \infty} P_{F(S) \cap F(T)} x_n$.

Note that $\{z_n\}$, $\{w_n\}$, $\{S z_n\}$, and $\{T w_n\}$ are bounded. Indeed, since $\{x_n\}$ is bounded, from (4.1), $\{z_n\}$ and $\{w_n\}$ are also bounded. Let $q \in F(S)$.

Since S is quasi-nonexpansive, we have

$$\begin{aligned}\|Sz_n\| &\leq \|Sz_n - q\| + \|q\| \\ &\leq \|z_n - q\| + \|q\|.\end{aligned}$$

Since $\{z_n\}$ is bounded, so is $\{Sz_n\}$. Analogously, one can show that $\{S^2z_n\}$ and $\{Tw_n\}$ are also bounded.

We now prove that

$$(4.4) \quad y_n - Sz_n \rightarrow 0 \text{ and } y_n - W_n \rightarrow 0.$$

Using any element $q \in F(S) \cap F(T)$, (2.4), (4.1), and (4.3), we obtain

$$\begin{aligned}&\|x_{n+1} - q\|^2 \\ &= \|a_n(y_n - q) + b_n(Sz_n - q) + c_n(W_n - q)\|^2 \\ &\leq a_n\|y_n - q\|^2 + b_n\|Sz_n - q\|^2 + c_n\|W_n - q\|^2 \\ &\quad - a_nb_n\|y_n - Sz_n\|^2 - a_nc_n\|y_n - W_n\|^2 \\ &\leq a_n\|y_n - q\|^2 + b_n\|z_n - q\|^2 + c_n\|W_n - q\|^2 \\ &\quad - a_nb_n\|y_n - Sz_n\|^2 - a_nc_n\|y_n - W_n\|^2 \\ &\leq \|x_n - q\|^2 - a_nb_n\|y_n - Sz_n\|^2 - a_nc_n\|y_n - W_n\|^2,\end{aligned}$$

which results in

$$a_nb_n\|y_n - Sz_n\|^2 + a_nc_n\|y_n - W_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2.$$

By (i), the right-hand side tends to 0. From the assumptions $\lim_{n \rightarrow \infty} a_nb_n > 0$ and $\lim_{n \rightarrow \infty} a_nc_n > 0$, we obtain (4.4), as claimed. Consequently, it holds that

$$(4.5) \quad z_n - Sz_n \rightarrow 0 \text{ and } x_n - W_n \rightarrow 0.$$

This can be verified using (4.2) and (4.4) as follows:

$$\begin{aligned}\|z_n - Sz_n\| &\leq \|z_n - x_n\| + \|x_n - y_n\| + \|y_n - Sz_n\| \rightarrow 0; \\ \|x_n - W_n\| &\leq \|x_n - y_n\| + \|y_n - W_n\| \rightarrow 0.\end{aligned}$$

Our aim is to prove that $x_n \rightharpoonup \hat{x}$ ($\equiv \lim_{k \rightarrow \infty} P_{F(S) \cap F(T)} x_k$). Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. From (ii), $\{x_{n_i}\}$ is bounded and thus, there exists a subsequence $\{x_{n_{ij}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{ij}} \rightharpoonup v$ for some $v \in H$. From (4.2) and (4.5), we obtain $z_{n_{ij}} \rightharpoonup v$ and $W_{n_{ij}} \rightharpoonup v$. Recall that S is demiclosed. Consequently, from (4.5), $v \in F(S)$. Since T is mean-demiclosed, from $W_{n_{ij}} \rightharpoonup v$, it follows that $v \in F(T)$. Since $v \in F(S) \cap F(T)$, from a property (2.1) of the metric projection,

$$\langle x_{n_{ij}} - P_{F(S) \cap F(T)} x_{n_{ij}}, P_{F(S) \cap F(T)} x_{n_{ij}} - v \rangle \geq 0$$

for all $j \in \mathbb{N}$. Since $x_{n_{ij}} \rightharpoonup v$ and $P_{F(S) \cap F(T)} x_n \rightarrow \hat{x}$, passing to the limit yields $\langle v - \hat{x}, \hat{x} - v \rangle \geq 0$, which implies that $v = \hat{x}$. Therefore, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_{ij}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{ij}} \rightharpoonup v = \hat{x}$. Therefore, $x_n \rightharpoonup \hat{x}$. This completes the proof. $\square$

Remark 4.1. Using a mean-valued sequence, the condition $x_n - w_n \rightarrow 0$ becomes dispensable for Theorem 4.1. See also Theorems 1.1 and 1.2 in Introduction.

The next theorem addresses the quasi-nonexpansive and 2-demiclosed case:

Theorem 4.2. Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let $S : C \rightarrow C$ be a quasi-nonexpansive and 2-demiclosed mapping. Let $T : C \rightarrow C$ be a quasi-nonexpansive and mean-demiclosed mapping. Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $P_{F(S) \cap F(T)}$ denote the metric projection from H onto $F(S) \cap F(T)$. Let $\{a_n\}$, $\{b_n\}$, $\{b'_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + b'_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n \rightarrow \infty} a_n b_n > 0$, $\underline{\lim}_{n \rightarrow \infty} a_n b'_n > 0$, and $\underline{\lim}_{n \rightarrow \infty} a_n c_n > 0$. Define a sequence $\{x_n\}$ in C as follows:

$$x_1 \in C : \text{ given,}$$

$$x_{n+1} = a_n y_n + b_n S z_n + b'_n S^2 z_n + c_n \frac{1}{n} \sum_{k=0}^{n-1} T^k w_n$$

for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in C that satisfy (4.6)

$$\|y_n - q\| \leq \|x_n - q\|, \|z_n - q\| \leq \|x_n - q\|, \text{ and } \|w_n - q\| \leq \|x_n - q\|$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ and

$$(4.7) \quad x_n - y_n \rightarrow 0 \text{ and } x_n - z_n \rightarrow 0.$$

Then, the sequence $\{x_n\}$ converges weakly to a point $\hat{x} \in F(S) \cap F(T)$, where $\hat{x} \equiv \lim_{n \rightarrow \infty} P_{F(S) \cap F(T)} x_n$.

Proof. Since S and T are quasi-nonexpansive (3.7), $F(S) \cap F(T)$ is closed and convex. Since $F(S) \cap F(T) \neq \emptyset$ is assumed, the metric projection $P_{F(S) \cap F(T)}$ from H onto $F(S) \cap F(T)$ is properly defined. Letting $W_n \equiv \frac{1}{n} \sum_{k=0}^{n-1} T^k w_n \in C$, we have $x_{n+1} = a_n y_n + b_n S z_n + b'_n S^2 z_n + c_n W_n$. Observe that

$$(4.8) \quad \|W_n - q\| \leq \|x_n - q\|$$

for any $q \in F(T)$ and $n \in \mathbb{N}$. This can be shown in the same manner as (4.3) in the proof of the previous theorem.

We verify that

$$\|x_{n+1} - q\| \leq \|x_n - q\|$$

for any $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Since S and T are quasi-nonexpansive, using (4.6) and (4.8), we have

$$\begin{aligned}
& \|x_{n+1} - q\| \\
&= \|a_n(y_n - q) + b_n(Sz_n - q) + b'_n(S^2z_n - q) + c_n(W_n - q)\| \\
&\leq a_n\|y_n - q\| + b_n\|Sz_n - q\| + b'_n\|S^2z_n - q\| + c_n\|W_n - q\| \\
&\leq a_n\|y_n - q\| + b_n\|z_n - q\| + b'_n\|z_n - q\| + c_n\|W_n - q\| \\
&\leq a_n\|x_n - q\| + b_n\|x_n - q\| + b'_n\|x_n - q\| + c_n\|x_n - q\| \\
&= \|x_n - q\|,
\end{aligned}$$

where $q \in F(S) \cap F(T)$ is arbitrarily selected. It follows that (i) $\{\|x_n - q\|\}$ converges in $\mathbb{R}$ for all $q \in F(S) \cap F(T)$; (ii) $\{x_n\}$ is bounded; (iii) $\{P_{F(S) \cap F(T)}x_n\}$ converges in $F(S) \cap F(T)$ (from Lemma 2.2). We denote $\hat{x} \equiv \lim_{n \rightarrow \infty} P_{F(S) \cap F(T)}x_n$. Since $\{x_n\}$ is bounded, $\{z_n\}$, $\{w_n\}$, $\{Sz_n\}$, $\{S^2z_n\}$, and $\{Tw_n\}$ are bounded.

We prove that

$$(4.9) \quad y_n - Sz_n \rightarrow 0, \quad y_n - S^2z_n \rightarrow 0, \quad \text{and} \quad y_n - W_n \rightarrow 0.$$

Employing (2.3), we have

$$\begin{aligned}
& \|x_{n+1} - q\|^2 \\
&= \|a_n(y_n - q) + b_n(Sz_n - q) + b'_n(S^2z_n - q) + c_n(W_n - q)\|^2 \\
&\leq a_n\|y_n - q\|^2 + b_n\|Sz_n - q\|^2 + b'_n\|S^2z_n - q\|^2 + c_n\|W_n - q\|^2 \\
&\quad - a_nb_n\|y_n - Sz_n\|^2 - a_nb'_n\|y_n - S^2z_n\|^2 - a_nc_n\|y_n - W_n\|^2 \\
&\leq a_n\|y_n - q\|^2 + b_n\|z_n - q\|^2 + b'_n\|z_n - q\|^2 + c_n\|W_n - q\|^2 \\
&\quad - a_nb_n\|y_n - Sz_n\|^2 - a_nb'_n\|y_n - S^2z_n\|^2 - a_nc_n\|y_n - W_n\|^2 \\
&\leq \|x_n - q\|^2 \\
&\quad - a_nb_n\|y_n - Sz_n\|^2 - a_nb'_n\|y_n - S^2z_n\|^2 - a_nc_n\|y_n - W_n\|^2,
\end{aligned}$$

where $q \in F(S) \cap F(T)$. This results in

$$\begin{aligned}
& a_nb_n\|y_n - Sz_n\|^2 + a_nb'_n\|y_n - S^2z_n\|^2 + a_nc_n\|y_n - W_n\|^2 \\
&\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2.
\end{aligned}$$

Since $\{\|x_n - q\|\}$ is convergent, the right-hand side converges to 0. Using the assumptions on $\{a_n\}$, $\{b_n\}$, $\{b'_n\}$, and $\{c_n\}$, we obtain (4.9), as claimed.

We now prove that

$$(4.10) \quad z_n - Sz_n \rightarrow 0, \quad z_n - S^2z_n \rightarrow 0, \quad \text{and} \quad x_n - W_n \rightarrow 0.$$

Using (4.7) and (4.9) yields

$$\|z_n - Sz_n\| \leq \|z_n - x_n\| + \|x_n - y_n\| + \|y_n - Sz_n\| \rightarrow 0.$$

Similarly, the same argument shows that $z_n - S^2z_n \rightarrow 0$ and $x_n - W_n \rightarrow 0$.

We aim to prove that $x_n \rightharpoonup \hat{x}$ ($\equiv \lim_{k \rightarrow \infty} P_{F(S) \cap F(T)} x_k$). Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{ij}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{ij}} \rightharpoonup v$ for some $v \in H$. From (4.7), we have $y_{n_{ij}} \rightharpoonup v$, $z_{n_{ij}} \rightharpoonup v$, and $W_{n_{ij}} \rightharpoonup v$. Since S is 2-demiclosed and T is mean-demiclosed, using (4.10), we obtain $v \in F(S) \cap F(T)$. From (2.1), it holds that

$$\langle x_{n_{ij}} - P_{F(S) \cap F(T)} x_{n_{ij}}, P_{F(S) \cap F(T)} x_{n_{ij}} - v \rangle \geq 0$$

for all $j \in \mathbb{N}$. Since $x_{n_{ij}} \rightharpoonup v$ and $P_{F(S) \cap F(T)} x_n \rightarrow \hat{x}$, we have $\langle v - \hat{x}, \hat{x} - v \rangle \geq 0$ and therefore, $v = \hat{x}$. We have shown that, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_{ij}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{ij}} \rightharpoonup v = \hat{x}$. This implies that $x_n \rightharpoonup \hat{x}$. This completes the proof. $\square$

A weak approximation method for quasi-nonexpansive and M -demiclosed mappings can be established:

Theorem 4.3. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let $S : C \rightarrow C$ be a quasi-nonexpansive and M -demiclosed mapping. Let $T : C \rightarrow C$ be a mean-demiclosed mapping. Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $P_{F(S) \cap F(T)}$ denote the metric projection from H onto $F(S) \cap F(T)$. Let $\{a_n\}$, $\{b_n^{(m)}\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + \sum_{m=1}^M b_n^{(m)} + c_n = 1$ for all $n \in \mathbb{N}$, $\lim_{n \rightarrow \infty} a_n b_n^{(m)} > 0$ ($\forall m = 1, \dots, M$), and $\lim_{n \rightarrow \infty} a_n c_n > 0$. Define a sequence $\{x_n\}$ in C as follows:*

$$x_1 \in C : \text{ given,}$$

$$x_{n+1} = a_n y_n + \sum_{m=1}^M b_n^{(m)} S^m z_n + c_n \frac{1}{n} \sum_{k=0}^{n-1} T^k w_n$$

for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in C that satisfy

$$\|y_n - q\| \leq \|x_n - q\|, \quad \|z_n - q\| \leq \|x_n - q\|, \quad \text{and} \quad \|w_n - q\| \leq \|x_n - q\|$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ and

$$x_n - y_n \rightarrow 0 \quad \text{and} \quad x_n - z_n \rightarrow 0.$$

Then, the sequence $\{x_n\}$ converges weakly to a point $\hat{x} \in F(S) \cap F(T)$, where $\hat{x} \equiv \lim_{n \rightarrow \infty} P_{F(S) \cap F(T)} x_n$.

Proof. This result can be verified by applying Lemma 2.3 in place of (2.3) or (2.4). Since the proof is similar to that of Theorem 4.2, we omit it here. $\square$

Remark 4.2. *This sequence of theorems highlights a key strength of the mean-valued term. Although the Mann–Ishikawa component must be adapted according to M , the structure of the mean-valued term remains unchanged.*

5. APPLICATION TO NONLINEAR MAPPINGS

In this section, we apply the main theorems to various types of nonlinear mappings introduced in Section 3. Recall that, according to Lemmas 3.1 and 3.2,

- a normally generalized hybrid mapping (3.2) is quasi-nonexpansive, demiclosed, and mean-demiclosed, if it has a fixed point;
- a normally N -generalized hybrid mapping is quasi-nonexpansive, N -demiclosed, and mean-demiclosed, if it has a fixed point;

For the definition of normally N -generalized hybrid mappings, see (3.3) and (3.4). Therefore, the following corollary is obtained from Theorem 4.1:

Corollary 5.1. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let $S : C \rightarrow C$ be a normally generalized hybrid mapping. Let $T : C \rightarrow C$ be a normally N -generalized hybrid mapping for some $N \in \mathbb{N}$. Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $P_{F(S) \cap F(T)}$ denote the metric projection from H onto $F(S) \cap F(T)$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n \rightarrow \infty} a_n b_n > 0$, and $\underline{\lim}_{n \rightarrow \infty} a_n c_n > 0$. Define a sequence $\{x_n\}$ in C as follows:*

$$x_1 \in C : \text{ given,}$$

$$x_{n+1} = a_n y_n + b_n S z_n + c_n \frac{1}{n} \sum_{k=0}^{n-1} T^k w_n$$

for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in C that satisfy

$$\|y_n - q\| \leq \|x_n - q\|, \quad \|z_n - q\| \leq \|x_n - q\|, \quad \text{and} \quad \|w_n - q\| \leq \|x_n - q\|$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ and

$$x_n - y_n \rightarrow 0 \quad \text{and} \quad x_n - z_n \rightarrow 0.$$

Then, the sequence $\{x_n\}$ converges weakly to a point $\hat{x} \in F(S) \cap F(T)$, where $\hat{x} \equiv \lim_{n \rightarrow \infty} P_{F(S) \cap F(T)} x_n$.

Remark 5.1. (i) Nonexpansive mappings and generalized hybrid mappings are particular cases of both normally generalized hybrid mappings and normally N -generalized hybrid mappings. Therefore, the mappings S and T in Corollary 5.1 may be taken as such mappings.

(ii) If we additionally assume that C is bounded and $ST = TS$, then $F(S) \cap F(T) \neq \emptyset$ is automatically satisfied; see Theorem 3.1.

By specifying the sequences $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$, we obtain the following two corollaries:

Corollary 5.2. *Let C be a nonempty, closed, and convex subset of H . Let $S : C \rightarrow C$ be a nonexpansive mapping. Let $T : C \rightarrow C$ be a normally N -generalized hybrid mapping for some $N \in \mathbb{N}$. Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be sequences of real numbers in the interval $[0, 1]$ such*

that $\lambda_n \rightarrow 1$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\lim_{n \rightarrow \infty} a_n b_n > 0$, and $\lim_{n \rightarrow \infty} a_n c_n > 0$. Define a sequence $\{x_n\}$ in C as follows:

$$(5.1) \quad \begin{aligned} x_1 &\in C : \text{ given,} \\ w_n &= \mu_n x_n + (1 - \mu_n) S x_n, \\ z_n &= \lambda_n x_n + (1 - \lambda_n) \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n. \\ x_{n+1} &= a_n x_n + b_n S z_n + c_n \frac{1}{n} \sum_{k=0}^{n-1} T^k w_n \end{aligned}$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges weakly to a point $\hat{x} \in F(S) \cap F(T)$, where $\hat{x} \equiv \lim_{n \rightarrow \infty} P_{F(S) \cap F(T)} x_n$.

Proof. By Corollary 5.1, it suffices to verify that $\|z_n - q\| \leq \|x_n - q\|$, $\|w_n - q\| \leq \|x_n - q\|$, and $x_n - z_n \rightarrow 0$. Since S is nonexpansive and has a fixed point, it is quasi-nonexpansive. Thus, $\|w_n - q\| \leq \|x_n - q\|$ for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Similarly, $\|z_n - q\| \leq \|x_n - q\|$ holds. Using these facts, we have

$$(5.2) \quad \begin{aligned} \|x_{n+1} - q\| &= \left\| a_n x_n + b_n S z_n + c_n \frac{1}{n} \sum_{k=0}^{n-1} T^k w_n - q \right\| \\ &\leq a_n \|x_n - q\| + b_n \|S z_n - q\| + c_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k w_n - q \right\| \\ &\leq a_n \|x_n - q\| + b_n \|z_n - q\| + c_n \|w_n - q\| \\ &\leq a_n \|x_n - q\| + b_n \|x_n - q\| + c_n \|x_n - q\| \\ &= \|x_n - q\| \end{aligned}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded. Consequently, the sequence $\left\{ \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n \right\}$ is also bounded.

Next, we prove that $x_n - z_n \rightarrow 0$. It holds that

$$\begin{aligned} \|x_n - z_n\| &= \left\| x_n - \left(\lambda_n x_n + (1 - \lambda_n) \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n \right) \right\| \\ &= (1 - \lambda_n) \left\| x_n - \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n \right\|. \end{aligned}$$

Since $\{x_n\}$ and $\left\{ \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n \right\}$ are bounded, from the assumption $\lambda_n \rightarrow 1$, we obtain $x_n - z_n \rightarrow 0$. This completes the proof. $\square$

Remark 5.2. *The condition $\mu_n \rightarrow 1$ is not required because $x_n - w_n \rightarrow 0$ is not necessary to apply Corollary 5.1. This advantage stems from the use of the mean-valued sequence $\left\{ \frac{1}{n} \sum_{k=0}^{n-1} T^k w_n \right\}$. See also Remark 4.1.*

Two-step iterative methods such as (5.1) or (1.4) have been studied in Ishikawa [11], Berinde [3, 4], and Tan and Xu [46]. Our framework also generates three-step and more general iterative schemes. For three-step iterative methods, see Noor [34] and Phuengrattana and Suantai [35].

We also obtain the following result:

Corollary 5.3. *Let C be a nonempty, closed, and convex subset of H . Let $S : C \rightarrow C$ be a nonexpansive mapping. Let $T : C \rightarrow C$ be a normally N -generalized hybrid mapping for some $N \in \mathbb{N}$. Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $\{\lambda_n\}$, $\{\mu_n\}$, and $\{\nu_n\}$ be sequences of real numbers in $[0, 1]$ such that $\lambda_n + \mu_n + \nu_n = 1$ and $\lambda_n \rightarrow 1$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\liminf_{n \rightarrow \infty} a_n b_n > 0$, and $\liminf_{n \rightarrow \infty} a_n c_n > 0$. Define a sequence $\{x_n\}$ in C as follows:*

$$(5.3) \quad \begin{aligned} x_1 &\in C : \text{ given,} \\ z_n &= \lambda_n x_n + \mu_n T x_n + \nu_n T^2 x_n, \\ x_{n+1} &= a_n x_n + b_n S z_n + c_n \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n \end{aligned}$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges weakly to a point $\hat{x} \in F(S) \cap F(T)$, where $\hat{x} \equiv \lim_{n \rightarrow \infty} P_{F(S) \cap F(T)} x_n$.

Proof. By Corollary 5.1, it suffices to show that $\|z_n - q\| \leq \|x_n - q\|$ and $x_n - z_n \rightarrow 0$. Let $q \in F(S) \cap F(T)$. Since T is a normally N -generalized hybrid mapping with a fixed point, it is quasi-nonexpansive. Thus, it follows that

$$\begin{aligned} \|z_n - q\| &= \|\lambda_n x_n + \mu_n T x_n + \nu_n T^2 x_n - q\| \\ &\leq \lambda_n \|x_n - q\| + \mu_n \|T x_n - q\| + \nu_n \|T^2 x_n - q\| \\ &\leq \lambda_n \|x_n - q\| + \mu_n \|x_n - q\| + \nu_n \|T x_n - q\| \\ &\leq \lambda_n \|x_n - q\| + \mu_n \|x_n - q\| + \nu_n \|x_n - q\| \\ &= \|x_n - q\| \end{aligned}$$

for all $n \in \mathbb{N}$. Using this, we have $\|x_{n+1} - q\| \leq \|x_n - q\|$. This part can be shown in a similar manner as (5.2). Thus, $\{x_n\}$ is bounded. Furthermore, $\{T x_n\}$ and $\{T^2 x_n\}$ are also bounded. Indeed,

$$\begin{aligned} \|T^2 x_n\| &\leq \|T^2 x_n - q\| + \|q\| \\ &\leq \|T x_n - q\| + \|q\| \\ &\leq \|x_n - q\| + \|q\|. \end{aligned}$$

Since $\{x_n\}$ is bounded, $\{T x_n\}$ and $\{T^2 x_n\}$ are also bounded, as claimed.

Note that since $\lambda_n \rightarrow 1$, it holds that $\mu_n, \nu_n \rightarrow 0$. Therefore,

$$\begin{aligned} \|x_n - z_n\| &= \|x_n - [\lambda_n x_n + \mu_n T x_n + \nu_n T^2 x_n]\| \\ &\leq (1 - \lambda_n) \|x_n\| + \mu_n \|T x_n\| + \nu_n \|T^2 x_n\| \rightarrow 0. \end{aligned}$$

This completes the proof. $\square$

Setting $\nu_n = 0$ for all $n \in \mathbb{N}$ in (5.3), we obtain the following iterative rule:

$$(5.4) \quad \begin{aligned} z_n &= \lambda_n x_n + (1 - \lambda_n) T x_n, \\ x_{n+1} &= a_n x_n + b_n S z_n + c_n \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n. \end{aligned}$$

Similarly, setting $\mu_n = 0$, we obtain

$$\begin{aligned} z_n &= \lambda_n x_n + (1 - \lambda_n) T^2 x_n, \\ x_{n+1} &= a_n x_n + b_n S z_n + c_n \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n. \end{aligned}$$

Setting $\lambda_n = 1$ ($\Leftrightarrow \mu_n = \nu_n = 0$), we obtain the following basic iterative scheme:

$$x_{n+1} = a_n x_n + b_n S x_n + c_n \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n.$$

For other types of variation, see, e.g., Kondo [19, 21, 25, 27].

By Lemma 3.2, the case where S is a normally M -generalized hybrid mapping falls within the scope of Theorem 4.3. Hence, we obtain the following:

Corollary 5.4. *Let C be a nonempty, closed, and convex subset of H . Let $S, T : C \rightarrow C$ be normally M -generalized hybrid and normally N -generalized hybrid mappings for some $M, N \in \mathbb{N}$, respectively. Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n^{(m)}\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + \sum_{m=1}^M b_n^{(m)} + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n \rightarrow \infty} a_n b_n^{(m)} > 0$, and $\underline{\lim}_{n \rightarrow \infty} a_n c_n > 0$. Define a sequence $\{x_n\}$ in C as follows:*

$x_1 \in C$: given,

$$x_{n+1} = a_n y_n + \sum_{m=1}^M b_n^{(m)} S^m z_n + c_n \frac{1}{n} \sum_{k=0}^{n-1} T^k w_n$$

for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in C that satisfy

$$\|y_n - q\| \leq \|x_n - q\|, \quad \|z_n - q\| \leq \|x_n - q\|, \quad \text{and} \quad \|w_n - q\| \leq \|x_n - q\|$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ and

$$x_n - y_n \rightarrow 0 \quad \text{and} \quad x_n - z_n \rightarrow 0.$$

Then, the sequence $\{x_n\}$ converges weakly to a point $\hat{x} \in F(S) \cap F(T)$, where $\hat{x} \equiv \lim_{n \rightarrow \infty} P_{F(S) \cap F(T)} x_n$.

A normally generalized hybrid mapping (3.2) with a fixed point is quasi-nonexpansive and demiclosed (Lemma 3.1) and every demiclosed mapping is 2-demiclosed (3.9); recall the inclusion (3.10). Consequently, Theorem 4.2 yields the following:

Corollary 5.5. *Let C be a nonempty, closed, and convex subset of H . Let $S : C \rightarrow C$ be a normally generalized hybrid mapping. Let $T : C \rightarrow C$ be a normally N -generalized hybrid mapping for some $N \in \mathbb{N}$. Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, $\{b'_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + b'_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n \rightarrow \infty} a_n b_n > 0$, $\underline{\lim}_{n \rightarrow \infty} a_n b'_n > 0$, and $\underline{\lim}_{n \rightarrow \infty} a_n c_n > 0$. Define a sequence $\{x_n\}$ in C as follows:*

$$x_1 \in C : \text{ given,}$$

$$x_{n+1} = a_n y_n + b_n S z_n + b'_n S^2 z_n + c_n \frac{1}{n} \sum_{k=0}^{n-1} T^k w_n$$

for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in C that satisfy

$$\|y_n - q\| \leq \|x_n - q\|, \quad \|z_n - q\| \leq \|x_n - q\|, \quad \text{and} \quad \|w_n - q\| \leq \|x_n - q\|$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ and

$$x_n - y_n \rightarrow 0 \quad \text{and} \quad x_n - z_n \rightarrow 0.$$

Then, the sequence $\{x_n\}$ converges weakly to a point $\hat{x} \in F(S) \cap F(T)$, where $\hat{x} \equiv \lim_{n \rightarrow \infty} P_{F(S) \cap F(T)} x_n$.

A comparison between Corollaries 5.1 and 5.5 is instructive.

6. APPLICATION TO A LINEAR SPLIT FEASIBILITY PROBLEM

Fixed point theory is known to be effective for analyzing split feasibility problems (SFPs). SFPs were introduced by Censor and Elfving [6]. See also Takahashi [43], Hao and Zhao [7], and Kondo [24]. This type of problem has been extended to split common null point problems and other types of formulation; see [5, 39]. In this section, we investigate how to find a common solution of a SFP and a fixed point problem by applying Corollary 5.3 with (5.4). In particular, we focus on the linear case.

We first present a linear analogue of Lemma 2.1, which is presented in Takahashi [41] as Problem 5.2.3. For completeness, we reproduce a proof:

Lemma 6.1. *Let H be a real Hilbert space and L a subspace of H . Let $x \in H$ and $p \in L$. Then, the following two statements are equivalent:*

- (i) $\langle x - p, z \rangle = 0$ for all $z \in L$;
- (ii) $\|x - p\| \leq \|x - z\|$ for all $z \in L$.

Proof. We first prove the part (i) $\implies$ (ii). Let $z \in L$. From Lemma 2.1, it is sufficient to show that $\langle x - p, p - z \rangle \geq 0$. Since $p, z \in L$ and L is a linear subspace of H , we have $p - z \in L$. Thus, (i) implies $\langle x - p, p - z \rangle = 0 \geq 0$.

We now prove the converse implication (ii) $\implies$ (i). Let $z \in L$. Our aim is to prove that $\langle x - p, z \rangle = 0$. Since (ii) is assumed, Lemma 2.1 implies that the following holds:

$$(6.1) \quad \langle x - p, p - u \rangle \geq 0 \text{ for all } u \in L.$$

Setting $u = p - z \in L$ in (6.1) results in $\langle x - p, z \rangle \geq 0$, and setting $u = p + z \in L$ results in $\langle x - p, z \rangle \leq 0$. Consequently, $\langle x - p, z \rangle = 0$ holds. This completes the proof. $\square$

If L is a closed subspace of H , then the metric projection $P_L : H \rightarrow L$ exists and it is linear. Moreover, we have

$$(6.2) \quad \langle x - P_L x, z \rangle = 0 \text{ for all } x \in H \text{ and } z \in L$$

by (i) in Lemma 6.1.

Let H and H' be real Hilbert spaces and let $L (\subset H)$ and $Q (\subset H')$ be closed subspaces of H and H' , respectively. Let $A : H \rightarrow H'$ be a bounded linear operator such that $A \neq O$ and let $A^* : H' \rightarrow H$ denote its adjoint, where O is the zero mapping. The SFP is formulated as follows:

$$(\text{SFP}) \text{ Find } \hat{x} \in L \cap A^{-1}Q.$$

To apply fixed point theory to linear SFPs, we establish the following lemma:

Lemma 6.2. *Let H and H' be real Hilbert spaces and let $L (\subset H)$ and $Q (\subset H')$ be closed subspaces of H and H' , respectively. Let $P_L : H \rightarrow L$ and $P_Q : H' \rightarrow Q$ be the metric projections. Let $A : H \rightarrow H'$ be a bounded linear operator such that $A \neq O$ and let $A^* : H' \rightarrow H$ denote its adjoint. Then,*

$$L \cap A^{-1}Q = F(P_L(I - \eta A^*(I - P_Q)A))$$

holds for all $\eta \neq 0$.

Proof. Note that $L \cap A^{-1}Q \neq \emptyset$, since both L and $A^{-1}Q$ are subspaces of H . We verify the inclusion $(\subset)$. Let $x \in L \cap A^{-1}Q$, in other words, $x \in L$ and $Ax \in Q$. From $x \in L$, it holds that $P_L x = x$. Since $Ax \in Q$, we have that $P_Q Ax = Ax$. Consequently,

$$\begin{aligned} P_L(I - \eta A^*(I - P_Q)A)x &= P_L(x - \eta A^*(Ax - P_Q Ax)) \\ &= P_L(x - \eta A^*0) = P_L x = x, \end{aligned}$$

which implies that $x \in F(P_L(I - \eta A^*(I - P_Q)A))$.

Next, we prove the converse inclusion $(\supset)$. Let $x \in F(P_L(I - \eta A^*(I - P_Q)A))$, where $\eta \neq 0$ is arbitrary. Then,

$$x = P_L(x - \eta A^*(Ax - P_Q Ax)).$$

From this, $x \in L$ follows. Our aim is to prove that $Ax \in Q$. From (6.2), it follows that

$$\langle (x - \eta A^*(Ax - P_Q Ax)) - x, x - h \rangle = 0 \quad \text{for all } h \in L,$$

because $x - h \in L$. Since $\eta \neq 0$,

$$\langle A^*(Ax - P_Q Ax), h - x \rangle = 0 \quad \text{for all } h \in L.$$

Since A^* is the adjoint operator of A ,

$$(6.3) \quad \langle Ax - P_Q Ax, Ah - Ax \rangle = 0 \quad \text{for all } h \in L.$$

Let $x_0 \in L \cap A^{-1}Q$, that is, $x_0 \in L$ and $Ax_0 \in Q$. Setting $h = x_0 \in L$ in (6.3) yields

$$(6.4) \quad \langle Ax - P_Q Ax, Ax_0 - Ax \rangle = 0.$$

On the other hand, from $Ax_0 \in Q$ and (6.2),

$$(6.5) \quad \langle Ax - P_Q Ax, P_Q Ax - Ax_0 \rangle = 0,$$

since $P_Q Ax - Ax_0 \in Q$. Summing (6.4) and (6.5), we obtain

$$\langle Ax - P_Q Ax, P_Q Ax - Ax \rangle = 0.$$

Hence, $Ax = P_Q Ax$, and therefore $Ax \in Q$, which completes the proof. $\square$

Remark 6.1. *We compare this lemma with Lemma 7.2 in Kondo [24], which addresses the general case of nonempty closed convex sets. The following differences are noteworthy:*

(i) *Whereas Lemma 7.2 in Kondo [24] requires the assumption $L \cap A^{-1}Q \neq \emptyset$, this condition is unnecessary in Lemma 6.2, since it holds automatically when L and Q are linear subspaces.*

(ii) *In Lemma 6.2, the parameter η need not be positive, whereas $\eta > 0$ is assumed in Lemma 7.2 of Kondo [24].*

We present an example illustrating Lemma 6.2. For simplicity, let $Q = \{0\} \subset H'$. Then, by Lemma 6.2,

$$(6.6) \quad L \cap A^{-1}0 = F(P_L(I - \eta A^*A)) \quad \text{for all } \eta \neq 0.$$

Example 6.1. Set $H = \mathbb{R}^3$, $H' = \mathbb{R}$, $A = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}$, and $L = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : z = 0 \right\}$, where A is a 1×3 matrix. We verify identity (6.6). First, note that

$$A^{-1}Q = A^{-1}0 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x = z \right\}.$$

In this case,

$$L \cap A^{-1}Q = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x = z = 0 \right\} = y\text{-axis}.$$

Next, we will compute $F(P_L(I - \eta A^*A))$ for $\eta \neq 0$. Note that

$$P_L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $A = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}$, we have

$$\begin{aligned} A^*A &= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \\ I - \eta A^*A &= \begin{pmatrix} 1-\eta & 0 & \eta \\ 0 & 1 & 0 \\ \eta & 0 & 1-\eta \end{pmatrix}, \text{ and} \\ P_L(I - \eta A^*A) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1-\eta & 0 & \eta \\ 0 & 1 & 0 \\ \eta & 0 & 1-\eta \end{pmatrix} \\ &= \begin{pmatrix} 1-\eta & 0 & \eta \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

where I denotes the 3×3 identity matrix. To compute $F(P_L(I - \eta A^*A))$, we solve

$$\begin{pmatrix} 1-\eta & 0 & \eta \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

which yields $x = z = 0$. Therefore,

$$F(P_L(I - \eta A^*A)) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x = z = 0 \right\} = y\text{-axis}.$$

This confirms that (6.6) holds true. $\square$

Assume in (6.6) that $L = H$. Then, we have

$$(6.7) \quad A^{-1}0 = F(I - \eta A^*A) \quad \text{for all } \eta \neq 0.$$

Example 6.2. Set $H = H' = \mathbb{R}^2$ and $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$. We verify identity (6.7). The kernel of A is

$$A^{-1}0 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x + y = 0 \right\}.$$

Furthermore,

$$A^*A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \text{ and}$$

$$I - \eta A^*A = \begin{pmatrix} 1 - 5\eta & -5\eta \\ -5\eta & 1 - 5\eta \end{pmatrix}.$$

We will compute $F(I - \eta A^*A)$. Solving

$$\begin{pmatrix} 1 - 5\eta & -5\eta \\ -5\eta & 1 - 5\eta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

for $\eta \neq 0$ yields $x + y = 0$. Thus,

$$F(I - \eta A^*A) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x + y = 0 \right\},$$

which confirms that (6.7) holds true. $\square$

Let C be a nonempty subset of a real Hilbert space H . A mapping $T : C \rightarrow H$ is said to be *K-Lipschitz continuous* if there exists $K > 0$ such that

$$\|Tx - Ty\| \leq K \|x - y\| \text{ for all } x, y \in C.$$

A mapping $T : C \rightarrow H$ is called *firmly nonexpansive* if

$$(6.8) \quad \|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle \text{ for all } x, y \in C.$$

A mapping is 1-Lipschitz continuous if and only if it is nonexpansive. Any firmly nonexpansive mapping is nonexpansive. The metric projection is firmly nonexpansive and hence nonexpansive.

We aim to present an iterative method to solve a SFP and fixed point problem simultaneously. The following lemma provides a key Lipschitz estimate:

Lemma 6.3. *Let H and H' be real Hilbert spaces. Let $A : H \rightarrow H'$ be a bounded linear operator such that $A \neq O$ and let $A^* : H' \rightarrow H$ denote its adjoint. Let L be a nonempty subset of H and $P : H \rightarrow L$ be a K -Lipschitz continuous mapping. Let $V : H' \rightarrow H'$ be a firmly nonexpansive mapping. Then, for every $\eta \in \left[0, \frac{2}{\|AA^*\|}\right]$, the mapping $P(I - \eta A^*(I - V)A) : H \rightarrow L$ is K -Lipschitz continuous. In particular, if $P : H \rightarrow L$ is nonexpansive, the mapping $P(I - \eta A^*(I - V)A) : H \rightarrow L$ is also nonexpansive for all $\eta \in \left[0, \frac{2}{\|AA^*\|}\right]$.*

For a proof of this lemma, see Lemma 7.6 in Kondo [24]. Using Lemmas 6.2 and 6.3 and Corollary 5.3 with (5.4), we obtain the following theorem, which demonstrates how to find a common solution of a linear SFP and a fixed point problem:

Theorem 6.1. *Let H and H' be real Hilbert spaces. Let $L (\subset H)$ and $Q (\subset H')$ be closed subspaces, let $P_L : H \rightarrow L$ and $P_Q : H' \rightarrow Q$ be the metric projections, and let $T : L \rightarrow L$ be a normally N -generalized hybrid*

mapping. Let $A : H \rightarrow H'$ be a bounded linear operator such that $A \neq O$. Assume that

$$\Omega \equiv L \cap A^{-1}Q \cap F(T) \neq \emptyset.$$

Denote by P_Ω the metric projection from H onto Ω . Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0, 1]$ such that $\lambda_n \rightarrow 1$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\lim_{n \rightarrow \infty} a_n b_n > 0$, and $\lim_{n \rightarrow \infty} a_n c_n > 0$. Let $\eta \in \left(0, \frac{2}{\|AA^*\|}\right]$, where A^* is the adjoint operator of A . Define a sequence $\{x_n\}$ in L as follows:

$$x_1 \in L : \text{ given,}$$

$$z_n = \lambda_n x_n + (1 - \lambda_n) T x_n,$$

$$x_{n+1} = a_n x_n + b_n P_L (I - \eta A^* (I - P_Q) A) z_n + c_n \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges weakly to an element $\hat{x} \in \Omega$, where $\hat{x} \equiv \lim_{n \rightarrow \infty} P_\Omega x_n$.

Proof. Since A is linear, $A^{-1}Q$ is a subspace of H . Since A is continuous, $A^{-1}Q$ is closed in H . Consequently, $L \cap A^{-1}Q$ is a closed subspace of H . Since $\Omega \equiv L \cap A^{-1}Q \cap F(T) \neq \emptyset$ is assumed and $\Omega \subset F(T)$, $F(T)$ is also nonempty. Since $T : L \rightarrow L$ is a normally N -generalized hybrid mapping with $F(T) \neq \emptyset$, from Lemma 3.2, it is quasi-nonexpansive. Therefore, $F(T)$ is closed and convex; for this point, see Itoh and Takahashi [12] or Kocourek *et al.* [13]. Since Ω is nonempty, closed, and convex subset of H , the metric projection P_Ω is properly defined.

Since the metric projections are firmly nonexpansive, they are nonexpansive. Since $\eta \in \left(0, \frac{2}{\|AA^*\|}\right] \subset \left[0, \frac{2}{\|AA^*\|}\right]$, from Lemma 6.3, the mapping $P_L (I - \eta A^* (I - P_Q) A)$ from H into L is nonexpansive. Since $T : L \rightarrow L$, the sequences $\{z_n\}$ and $\{x_n\}$ are properly defined as sequences in L .

Since $\eta \neq 0$, from Lemma 6.2, it holds that

$$F(P_L (I - \eta A^* (I - P_Q) A)) = L \cap A^{-1}Q,$$

which implies that

$$F(P_L (I - \eta A^* (I - P_Q) A)) \cap F(T) = L \cap A^{-1}Q \cap F(T) = \Omega.$$

From the above, we can apply Corollary 5.3 with (5.4) by taking

$$S = P_L (I - \eta A^* (I - P_Q) A),$$

which yields the desired result. $\square$

Recall that a normally N -generalized hybrid mapping is not necessarily continuous; see the examples presented in Section 3.

Let $T : L \rightarrow L$ be a linear and orthogonal transformation, i.e., $\|Tu\| = \|u\|$ for all $u \in H$. Then, it is nonexpansive. Since T is linear, $F(T)$ is a subspace of L . Since T is nonexpansive, it is normally N -generalized hybrid. Therefore, we obtain the following result:

Theorem 6.2. *Let H and H' be real Hilbert spaces. Let $L (\subset H)$ and $Q (\subset H')$ be closed subspaces, let $P_L : H \rightarrow L$ and $P_Q : H' \rightarrow Q$ be the metric projections, and let $T : L \rightarrow L$ be an orthogonal transformation. Let $A : H \rightarrow H'$ be a bounded linear operator such that $A \neq O$. Define*

$$\Omega \equiv L \cap A^{-1}Q \cap F(T)$$

and denote by P_Ω the metric projection from H onto Ω . Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0, 1]$ such that $\lambda_n \rightarrow 1$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\lim_{n \rightarrow \infty} a_n b_n > 0$, and $\lim_{n \rightarrow \infty} a_n c_n > 0$. Let $\eta \in \left(0, \frac{2}{\|AA^\|}\right]$, where A^* is the adjoint operator of A . Define a sequence $\{x_n\}$ in L as follows:*

$$x_1 \in L : \text{ given,}$$

$$z_n = \lambda_n x_n + (1 - \lambda_n) T x_n,$$

$$x_{n+1} = a_n x_n + b_n P_L (I - \eta A^* (I - P_Q) A) z_n + c_n \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges weakly to an element $\hat{x} \in \Omega$, where $\hat{x} \equiv \lim_{n \rightarrow \infty} P_\Omega x_n$.

The assumption $\Omega \neq \emptyset$ is unnecessary, since Ω is a subspace of H (and thus contains the origin). See also Remarks 5.1-(ii) and 6.1-(i).

7. CONCLUDING REMARKS

In this paper, the iterative scheme generating method (ISGM) was extended to incorporate a mixed structure of Mann–Ishikawa type iterations and averaging sequences, allowing the unified treatment of iterations based on finite powers of mappings and Cesàro-type means within a single framework. One advantage of using the mean-valued sequence is shown in Remarks 4.1 and 5.2.

The class of admissible mappings was expanded to cover wide classes of nonlinear mappings. In particular, continuity assumptions were not required, and several examples of discontinuous mappings satisfying the proposed conditions were presented. As an application, the proposed framework was applied to linear split feasibility problems, and concrete examples were provided to illustrate the targeted fixed point set.

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