Noname manuscript No.
(will be inserted by the editor)

Strategic Asset Allocation with Generalized Homothetic Robust Epstein-Zin Utility under a Quadratic Model

Kentaro Kikuchi · Koji Kusuda

4 December 2025

Abstract This study proposes "generalized homothetic robust Epstein-Zin (GHREZ) utility," in which the coefficient of relative ambiguity aversion in homothetic robust Epstein-Zin (HREZ) utility is generalized to a positive-definite matrix of relative ambiguity aversions. We analyze a robust consumption–investment problem with GHREZ utility under a quadratic security market model. We demonstrate that the volatility of the optimal robust portfolio can be expressed as a weighted average of the market price of risk and the "investor hedging value of intertemporal uncertainty." We derive an approximate optimal robust portfolio. We introduce the concept of "mean relative ambiguity aversion," and compare the relationship between the approximate optimal robust portfolios and relative ambiguity aversion matrices with an identical mean relative ambiguity aversion. Numerical analysis reveals that, as deviations from the case of HREZ utility increase, deviations from the approximate optimal asset allocation under HREZ utility also increase, and the sensitivity is quite large. This finding suggests that GHREZ utility is promising for solving the equity home bias puzzle.

Keywords Ambiguity aversion, Consumption-investment problem, Generarlized stochastic differential utility
JEL Classification C61 · G11

1 Introduction

Two key points should be considered when studying consumption–investment problems. The first point is the stylized fact that interest rates, the market

K. Kikuchi
Shiga University, Hikone, Japan
E-mail: kentaro-kikuchi@biwako.shiga-u.ac.jp

K. Kusuda
Shiga University, Hikone, Japan
E-mail: kusuda@biwako.shiga-u.ac.jp (The corresponding author)

price of risk, variances and covariances of asset returns, and inflation rates in the securities market are stochastic and mean-reverting. The second is the Knightian uncertainty, as confirmed by the global financial crisis. Regarding the first point, Batbold, Kikuchi, and Kusuda [2] examine the consumption–investment problem under a quadratic security market (QSM) model that satisfies the stylized fact. The QSM model is classified as a quadratic model[1], which is a generalization of the affine model (Duffie and Kan [11]).[2] Batbold *et al.* [2] derive an optimal portfolio under constant relative risk aversion (CRRA) utility, demonstrating that the optimal portfolio is decomposed into myopic, intertemporal hedging, and inflation–deflation hedging demands. Their numerical analysis highlights the nonlinearity and significance of market timing effects, attributing nonlinearity to the stochastic variances and covariances of asset returns, and significance to the inflation–deflation hedging demand, alongside myopic demand.

Investors with homothetic[3] robust utility, proposed by Maenhout [29] and theoretically justified by Skiadas [33] and Kusuda [22], regard the "base probability" as the most likely probability; however, they also consider other probabilities because the true probability is unknown. Homothetic robust utility is a homothetic robust version of CRRA utility, and is applied in various robust control studies such as Skiadas [33], Maenhout [30], Liu [28], Branger, Larsen, and Munk [5], Munk and Rubtsov [31], Yi, Viens, Law, and Li [35], and Kikuchi and Kusuda [20]. Homothetic robust Epstein-Zin (HREZ) utility, presented by Maenhout [29], is a homothetic robust version of Epstein-Zin utility (Epstein and Zin [13]) and a generalization of homothetic robust utility. The properties of HREZ utility are studied in Kusuda [22]. HREZ utility is applied to numerous studies, including asset pricing (Maenhout [29]), worst-case returns (Batbold *et al.* [4]), and consumption–investment (Kikuchi and Kusuda [21]). HREZ utility is characterized by subjective discount rate, elasticity of intertemporal substitution (EIS), relative risk aversion, and relative ambiguity aversion. Following Kikuchi and Kusuda [20], the sum of relative risk and ambiguity aversions is termed "relative uncertainty aversion," and the inverse of relative uncertainty aversion, "relative uncertainty tolerance." Kikuchi and Kusuda [21] examine a finite-time consumption–investment problem based on HREZ utility under the QSM model of Batbold *et al.* [2].

Investor ambiguity aversion toward a risk depends on the investor's level of knowledge regarding that risk. For instance, an investor with a lower level of knowledge about foreign risks than about domestic ones exhibits greater ambiguity aversion toward foreign risks than toward domestic ones. However,

[1] The quadratic model is independently developed by Ahn, Dittmar, and Gallant [1] and Leippold and Wu [26].

[2] Quadratic models are employed in studies of empirical analysis (Leippold and Wu [27], Kim and Singleton [24], and Kikuchi [18]), security pricing (Chen, Filipović, and Poor [7], Branger and Munk [6], Filipović, Gourier, and Mancini [14], and Batbold, Kikuchi, and Kusuda [4]), and optimal consumption–investment (Batbold *et al.* [2], Kikuchi and Kusuda [19], [20], and Batbold, Kikuchi, and Kusuda [3]).

[3] A utility function U is homothetic if, for any consumption plan c and $\tilde{c}$, and any scaler $\varepsilon_0 > 0$, $U(\varepsilon_0 \tilde{c}) \geq U(\varepsilon_0 c) \Leftrightarrow U(\tilde{c}) \geq U(c)$.

relative ambiguity aversions for all risks are equal in HREZ utility. We propose generalized HREZ (GHREZ) utility in which the coefficient of relative ambiguity aversion in HREZ utility is generalized to a positive-definite matrix of relative ambiguity aversions. This matrix is referred to as the "relative ambiguity aversion matrix." In this matrix, each element depends on the corresponding risk. The coefficient of relative uncertainty tolerance in HREZ utility is generalized to the "matrix of relative uncertainty tolerances" in GHREZ utility. We address the consumption–investment problem based on GHREZ utility under the QSM model of Batbold *et al.* [2]. The objective of this study is to derive the optimal robust portfolio under GHREZ utility and to examine the relationship between the optimal robust portfolio and relative ambiguity aversion matrix.

The main results of this study are summarized as follows. Firstly, we derive an optimal solution that includes an unknown function, which is a solution to a nonlinear partial differential equation (PDE). This solution is a generalization of the one derived by Kikuchi and Kusuda [21] for HREZ utility.

Secondly, the volatility of the optimal robust portfolio is expressed as a weighted average of the market price of risk and the investor hedging value of intertemporal uncertainty (IHVIU) introduced by Batbold *et al.* [4]. Here, the weight for the market price of risk is the matrix of relative uncertainty tolerances. In the QSM models, risks are generated by the N-dimensional standard Brownian motion. Let $n \in \{1, \cdots, N\}$. The n-th risk is generated by the n-th element of Brownian motion. The n-th element of the volatility of the optimal robust portfolio is interpreted as representing investment in the n-th risk. For HREZ utility, the volatility of the optimal robust portfolio for the n-th risk depends solely on the market price of n-th risk and the investor hedging value of n-th intertemporal uncertainty, with common weight across all risks. By contrast, for GHREZ utility, it depends on the entire vectors of market price of risk and IHVIU, with the weight different from that of the m-th element of the volatility of the optimal robust portfolio for all $m \neq n$.

Thirdly, applying the linear approximation method of Kikuchi and Kusuda [20] to the aforementioned nonlinear PDE results in the attainment of an approximate optimal solution, a generalization of Kikuchi and Kusuda [21]'s solution for HREZ utility.

Fourthly, we conduct a simple numerical analysis assuming that investors with GHREZ utility invest in the S&P 500, 10-year Treasury Inflation-Protected Securities (TIPS), and money market account under a simple two-factor QSM model. We introduce mean relative ambiguity aversion and compare the approximate optimal robust asset allocation of investors with different relative ambiguity aversion matrices that have identical mean relative ambiguity aversion. As deviations from the case of HREZ utility increase, deviations from the approximate optimal asset allocation under HREZ utility also increase, and the sensitivity is quite large. This finding implies that when specifying GHREZ utility, it is crucial to discern the overall structure of the relative ambiguity aversion matrix, rather than solely focusing on the mean relative ambiguity aversion.

The standard international capital asset pricing model (CAPM) demonstrates that the portfolio weight of each country's equity holdings should be equal to its relative market capitalization share in the world. However, in reality, most investors hold nearly all of their wealth in domestic equity. This under-diversification phenomenon is documented as equity home bias puzzle in the literature (French and Poterba [15], Coeurdacier and Rey [8], and Cooper, Sercu, and Vanpee [9]). The standard international CAPM assumes CRRA utility, with equal relative uncertainty aversions for both domestic and foreign risks. To address the equity home bias puzzle, we propose to assume GHREZ utility where the relative ambiguity aversion matrix is diagonal and the relative ambiguity aversions for foreign risks are greater than those for domestic risks. We expect the portfolio weight of each foreign country's equity holdings to be significantly less than its relative market capitalization share in the world. This is suggested by the finding that as deviations from the case of equal relative ambiguity aversions for all risks increase, deviations from the approximate optimal asset allocation under HREZ utility also increase, and the sensitivity is quite large.

The remainder of this paper is organized as follows. Section 2 introduces GHREZ utility and reviews the QSM model. Section 3 derives the optimal robust control and PDE for indirect utility and analyzes the volatility of the optimal robust portfolio. Section 4 derives a linear approximate optimal portfolio. Section 5 conducts a simple numerical analysis and Section 6 addresses future research. The appendix includes proofs of the lemma and proposition.

2 GHREZ Utility and the QSM Model

This section introduces GHREZ utility and reviews the QSM model. The investor's optimal robust control problem is subsequently presented.

2.1 Environment

We consider United States (US) markets over the period $[0, \infty)$.[4] Investors' common subjective probability and information structure are modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathrm{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$ is the natural filtration generated by an N-dimensional standard Brownian motion B_t. We denote the expectation operator under P by E and the conditional expectation operator given $\mathcal{F}_t$ by E_t.

Frictionless markets for a consumption commodity and securities exist at every date $t \in [0, \infty)$, and the consumer price index p_t is observed. The traded securities are K types of non-bond indices, the instantaneous nominal risk-free security (the money market account (MMA)), and a continuum of zero-coupon bonds and zero-coupon inflation-indexed bonds, with maturity dates $(t, t+\tau^*]$, where τ^* is the longest time to maturity of the bonds. Each zero-coupon bond

[4] Investors must consider infinite-time markets because stocks are priced as the discounted present value of the dividend stream over infinite time.

has a 1 US dollar (USD) payoff at maturity, and each zero-coupon inflation-indexed bond has a p_T USD payoff at maturity T. At each date t, P_t, P_t^T, Q_t^T, and S_t^k represent the USD prices of the MMA, zero-coupon bond with maturity date T, zero-coupon inflation-indexed bond maturing at T, and the k-th index, respectively. Let A' denote the transpose of A and I_n the $n \times n$ identity matrix. Let O_n and 0_n denote the $n \times n$ zero matrix and the n zero vector, respectively.

2.2 GHREZ Utility and Directional Ambiguity Aversion

2.2.1 GHREZ Utility

First, we begin with Epstein-Zin utility. Let V^{EZ} denote the Epstein-Zin utility process defined by

$$V_t^{\mathrm{EZ}} = \mathrm{E}_t\left[\int_t^{T^*} f(c_s, V_s^{\mathrm{EZ}})ds\right], \quad \forall t \in [0, T^*], \tag{2.1}$$

where the normalized aggregator from Duffie and Epstein [10] is given by

$$f(c,v) = \frac{\beta}{1-\psi^{-1}} c^{1-\psi^{-1}} \big((1-\gamma)v\big)^{1-\frac{1-\psi^{-1}}{1-\gamma}} - \frac{\beta(1-\gamma)}{1-\psi^{-1}} v, \tag{2.2}$$

where $\beta > 0$ is the subjective discount rate, $\gamma > 0$ is the relative risk aversion, and $\psi > 0$ is the EIS. Schroder and Skiadas [34] prove that if $(\gamma, \psi) \in (0,1) \times (1,\infty) \cup (1,\infty) \times (0,1)$, then for any consumption plan c such that $\mathrm{E}\big[\int_0^{T^*} c_t^l dt + c_{T^*}^l\big] < \infty$ for all $l \in \mathbb{R}$, there exists a unique V such that $\mathrm{E}\big[\mathrm{ess}\sup_{t\in[0,T^*]} |V_t^{\mathrm{EZ}}|^l\big] < \infty$ for every $l > 0$. Many empirical analyses indicate $\gamma > 1 > \psi > 0$. Skiadas [32] demonstrates that if $\gamma\psi > 1$, then the agent is information seeking.

Agents with robust utility regard probability P as the most likely probability, but also consider other probabilities because the true probability is unknown. Thus, the agents assume set $\mathbb{P}$ of all equivalent probability measures[5] as the alternative probabilities. According to Girsanov's theorem, any equivalent probability measure is characterized by a measurable process $(\xi_t)_{t\in[0,T]}$ with Novikov's integrability condition as the following Radon-Nikodym derivative:

$$\mathrm{E}_T\left[\frac{d\mathrm{P}^\xi}{d\mathrm{P}}\right] = \exp\left(\int_0^T \xi_t \, dB_t - \frac{1}{2}\int_0^T |\xi_t|^2 dt\right) \quad \forall T \in (0,\infty). \tag{2.3}$$

Therefore, the agents rationally determine the worst-case probability by considering deviations from P.

[5] A probability measure $\tilde{\mathrm{P}}$ is said to be an *equivalent probability measure of* P if and only if $\mathrm{P}(A) = 0 \Leftrightarrow \tilde{\mathrm{P}}(A) = 0$.

Definition 1 GHREZ utility is defined by

$$U(c) = \inf_{\mathrm{P}^\xi \in \mathbb{P}} \mathrm{E}^\xi \left[\int_0^{T^*} \left(f(c_t, V_t^\xi) + \frac{(1-\gamma)V_t^\xi}{2} \xi_t' \Theta^{-1} \xi_t \right) dt \right], \tag{2.4}$$

where Θ is a positive-definite matrix, and V_t^ξ is the utility process defined recursively as

$$V_t^\xi = \mathrm{E}_t^\xi \left[\int_t^{T^*} \left(f(c_s, V_s^\xi) + \frac{(1-\gamma)V_s^\xi}{2} \xi_s' \Theta^{-1} \xi_s \right) ds \right]. \tag{2.5}$$

We refer to Θ as the relative ambiguity aversion matrix. An interesting subclass of GHREZ utilities is one for which the relative ambiguity aversion matrix is diagonal, that is, $\Theta = \mathrm{diag}(\theta_1, \theta_2, \cdots, \theta_N)$ where $\theta_n > 0$ for all $n \in \{1, \cdots, N\}$. If $\theta_n = \theta$ for every $n \in \{1, \cdots, N\}$, then GHREZ utility reduces to HREZ utility.

Remark 1 Kusuda [23] derives the generalized stochastic differential utility (GSDU) (Lazrak and Quenez [25]) representation of $U(c)$ given by Eq. (2.4).

$$V_t = \mathrm{E}_t \left[\int_t^{T^*} \left(f(c_s, V_s) - \frac{1}{2(1-\gamma)V_s} \sigma_s' \Theta \sigma_s . \right) ds \right], \quad \forall t \in [0, T^*], \tag{2.6}$$

where σ is the volatility of V. El Karoui, Peng, and Quenez [12] demonstrate that GSDU is time consistent and increasing in consumption. Kusuda [23] implies the following: i) GHREZ utility is risk averse, and ii) GHREZ utility with $(\beta, \gamma^*, \psi, \Theta^*)$ is more risk averse than GHREZ utility with $(\beta, \gamma, \psi, \Theta)$ if $\gamma^* \geq \gamma$. Kusuda [23] also demonstrates the following properties: i) GHREZ utility is SDU if and only if it is reduced to HREZ utility, ii) GHREZ utility is homothetic and ambiguity averse, and iii) GHREZ utility with $(\beta, \gamma, \psi, \Theta^*)$ is more ambiguity averse than GHREZ utility with $(\beta, \gamma, \psi, \Theta)$ if $\Theta^* \geq \Theta$, *i.e.*, $(\Theta^* - \Theta)$ is positive semidefinite.

2.2.2 Directional Ambiguity Aversion

Following Kusuda [23], we introduce the notion of aversion to directional ambiguity. We assume, without loss of generality, that Brownian motion $B = (B_1, B_2)'$ is two-dimensional, that is, $N = 2$. The *unit direction vector* $d(\alpha)$ and *direction* B^α are defined by

$$d(\alpha) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \quad B^\alpha = d(\alpha)' B. \tag{2.7}$$

We assume that GHREZ utility U is expressed by Eq. (2.6). U^α is defined by the following utility process:

$$dV_t^\alpha = -\left(f(c_t, V_t^\alpha) - \frac{1}{2(1-\gamma)V_t^\alpha} d(\alpha)' \Theta d(\alpha) |\bar{\sigma}_t|^2 \right) dt + \bar{\sigma}_t dB_t^\alpha, \quad V_{T^*}^\alpha = 0, \tag{2.8}$$

where $\bar{\sigma}_t = |\sigma_t|$.

Definition 2 Let U be GHREZ utility with $(\beta, \gamma, \psi, \Theta)$. U is more ambiguity averse in the direction $B^{\hat{\alpha}}$ than in B^{α} if for every consumption plan c, $U^{\hat{\alpha}}(c) \leq U^{\alpha}(c)$.

Remark 2 Let U be GHREZ utility with parameters $(\beta, \gamma, \psi, \Theta)$. Kusuda [23] demonstrates a proposition that if $d(\hat{\alpha})'\Theta d(\hat{\alpha}) \geq d(\alpha)'\Theta d(\alpha)$, then U is more ambiguity averse in $B^{\hat{\alpha}}$ than in B^{α}. For example, assume $\alpha = 0$ and $\hat{\alpha} = \frac{\pi}{2}$. Let

$$\Theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{12} & \theta_{22} \end{pmatrix}. \tag{2.9}$$

Then, $B^{\alpha} = B_1, B^{\hat{\alpha}} = B_2$, and $d(\alpha)'\Theta d(\alpha) = \theta_{11}, d(\hat{\alpha})'\Theta d(\hat{\alpha}) = \theta_{22}$. Therefore, the above proposition demonstrates that if $\theta_{11} < \theta_{22}$, then U is more ambiguity averse in B_2 than in B_1.

2.3 QSM Model

We assume the following QSM model introduced by Batbold *et al.* [2] and Kikuchi and Kusuda [19].

Assumption 1 *Let $k \in \{1, \cdots, K\}$. Let $(\rho_0, \iota_0, \delta_{0k}, \sigma_{0k})$, $(\lambda, \rho, \iota, \sigma_p, \delta_k, \sigma_k)$, and $(\mathcal{R}, \Delta_k, \Sigma_k)$ denote scalers, N-dimensional vectors, and $N \times N$ positive-definite symmetric matrices, respectively.*

1. *The state vector process X_t is N-dimensional and satisfies the following stochastic differential equation (SDE):*
$$dX_t = -\mathcal{K} X_t \, dt + I_N \, dB_t, \tag{2.10}$$
where $\mathcal{K}$ is an $N \times N$ lower triangular matrix with positive diagonal elements.
2. *The market price λ_t of risk (MPR) and the instantaneous nominal risk-free rate r_t are given by*
$$\lambda_t = \lambda + \Lambda X_t, \tag{2.11}$$
$$r_t = \rho_0 + \rho' X_t + \frac{1}{2} X_t' \mathcal{R} X_t, \tag{2.12}$$
where Λ is an $N \times N$ lower triangular matrix, and $(\rho_0, \rho, \mathcal{R})$ satisfies $\rho_0 = \frac{1}{2}\rho' \mathcal{R}^{-1} \rho$.[6]
3. *The consumer price index p_t satisfies*
$$\frac{dp_t}{p_t} = \mu_t^p \, dt + (\sigma_t^p)' dB_t, \qquad p_0 = 1, \tag{2.13}$$
where μ_t^p and σ_t^p are given by
$$\mu_t^p = \iota_0 + \iota' X_t + \frac{1}{2} X_t' \mathcal{I} X_t, \quad \sigma_t^p = \sigma_p + \Sigma_p X_t, \tag{2.14}$$
where $\mathcal{I}$ is an $N \times N$positive-semidefinite symmetric matrix.

[6] This condition ensures that the instantaneous nominal risk-free rate is non-negative.

4. *The dividend rate of the k-th index is given by*

$$D_t^k = \left(\delta_{0k} + \delta_k' X_t + \frac{1}{2} X_t' \Delta_k X_t\right) \exp\left(\sigma_{0k} t + \sigma_k' X_t + \frac{1}{2} X_t' \Sigma_k X_t\right), \tag{2.15}$$

where $(\delta_{0k}, \delta_k, \Delta_k)$ *satisfies* $\delta_{0k} = \frac{1}{2}\delta_k' \Delta_j^{-1} k \delta_j$.[7]

5. *Markets are complete and arbitrage-free.*

Let $\tau = T - t$ denote the time to maturity of bond P_t^T or inflation-indexed bond Q_t^T. Batbold *et al.* [2] show that the dynamics of security price processes satisfy the following (for proof, see Appendix A.1 in Batbold *et al.* [2]):

1. The default-free bond with time τ to maturity:

$$\frac{dP_t^T}{P_t^T} = (r_t + (\sigma(\tau) + \Sigma(\tau)X_t)'\lambda_t)\, dt + (\sigma(\tau) + \Sigma(\tau)X_t)'\, dB_t, \quad P_T^T = 1, \tag{2.16}$$

where $(\Sigma(\tau), \sigma(\tau))$ is a solution to the system of ordinary differentail equations (ODEs) (A.1) and (A.2).

2. The default-free inflation-indexed bond with time τ to maturity:

$$\frac{dQ_t^T}{Q_t^T} = \left(r_t + \left(\sigma_q(\tau) + \Sigma_q(\tau)X_t\right)'\lambda_t\right) dt + \left(\sigma_q(\tau) + \Sigma_q(\tau)X_t\right)' dB_t, \tag{2.17}$$

where $(\Sigma_q(\tau), \sigma_q(\tau)) = (\bar{\Sigma}_q(\tau) + \Sigma_p, \bar{\sigma}_q(\tau) + \sigma_p)$ and $(\bar{\Sigma}_q(\tau), \bar{\sigma}_q(\tau))$ is a solution to the system of ODEs (A.3) and (A.4).

3. The k-th index:

$$\frac{dS_t^k + D_t^k dt}{S_t^k} = (r_t + (\sigma_k + \Sigma_k X_t)'\lambda_t)\, dt + (\sigma_k + \Sigma_k X_t)'\, dB_t, \tag{2.18}$$

where Σ_k is a solution to Eq. (A.5) and σ_k is given by Eq. (A.6).

2.4 Investor

Let $P_t(\tau) = P_t^T$ and $Q_t(\tau) = Q_t^T$ where $\tau = T - t$.

Assumption 2 *The investor has the GHREZ utility of the form (2.4), where* $\gamma > 1 > \psi > 0$ *and* $\gamma\psi > 1$. *The investor invests in* $P_t(\tau_1), \cdots, P_t(\tau_I), Q_t(\tau_1^q), \cdots, Q_t(\tau_J^q)$, *and* $S_t^1, \cdots, S_t^K$ *at time t, where* $I + J + K = N$. *Let* $\Phi^P(\tau), \Phi^Q(\tau^q)$, *and* Φ^k *denote the portfolio weights on the bond with* τ*-time to maturity, inflation-indexed bond with* τ^q*-time to maturity, and k-th index, respectively. Let* Φ_t

[7] This condition ensures that dividend rates are non-negative processes.

and $\Sigma(X_t)$ denote the portfolio weight and volatility matrix at time t. Φ_t and $\Sigma(X_t)$ are expressed as follows:

$$\Phi_t = \begin{pmatrix} \Phi_t^P(\tau_1) \\ \vdots \\ \Phi_t^P(\tau_I) \\ \Phi_t^Q(\tau_1^q) \\ \vdots \\ \Phi_t^Q(\tau_J^q) \\ \Phi_t^1 \\ \vdots \\ \Phi_t^K \end{pmatrix}, \quad \Sigma(X_t) = \begin{pmatrix} \big(\sigma(\tau_1) + \Sigma(\tau_1)X_t\big)' \\ \vdots \\ \big(\sigma(\tau_I) + \Sigma(\tau_I)X_t\big)' \\ \big(\sigma_q(\tau_1^q) + \Sigma_q(\tau_1^q)X_t\big)' \\ \vdots \\ \big(\sigma_q(\tau_J^q) + \Sigma_q(\tau_J^q)X_t\big)' \\ (\sigma_1 + \Sigma_1 X_t)' \\ \vdots \\ (\sigma_K + \Sigma_K X_t)' \end{pmatrix}. \tag{2.19}$$

2.5 Real Budget Constraint and Robust Control Problem

To derive the real budget constraint, we define the real MPR $\bar{\lambda}(X_t)$ and the instantaneous real risk-free rate $\bar{r}(X_t)$ by

$$\bar{\lambda}(X_t) = \lambda_t - \sigma_t^p, \tag{2.20}$$
$$\bar{r}(X_t) = r_t - \mu_t^p + \lambda_t'\sigma_t^p. \tag{2.21}$$

Then, $\bar{\lambda}(X_t)$ and $\bar{r}(X_t)$ are expressed as

$$\bar{\lambda}(X_t) = \bar{\lambda} + \bar{\Lambda}X_t, \tag{2.22}$$
$$\bar{r}(X_t) = \bar{\rho}_0 + \bar{\rho}'X_t + \frac{1}{2}X_t'\bar{\mathcal{R}}X_t, \tag{2.23}$$

where

$$\bar{\lambda} = \lambda - \sigma_p, \quad \bar{\Lambda} = \Lambda - \Sigma_p, \tag{2.24}$$
$$\bar{\rho}_0 = \rho_0 - \iota_0 + \lambda'\sigma^p, \quad \bar{\rho} = \rho - \iota + \Lambda'\sigma_p + \Sigma_p'\lambda, \tag{2.25}$$
$$\bar{\mathcal{R}} = \mathcal{R} - \mathcal{I} + \Sigma_p'\Lambda + \Lambda'\Sigma_p. \tag{2.26}$$

Let W and $\bar{W}$ denote the nominal wealth process and real wealth process, respectively. Let $\mathbf{X} = (\bar{W}, X')'$ and $\bar{W}_0 > 0$. Kikuchi and Kusuda [19] show that, given an initial state $\mathbf{X}_0$, a consumption plan c, and a self-financing portfolio weight Φ, the real budget constraint equation is given by

$$\frac{d\bar{W}_t}{\bar{W}_t} = \left(\bar{r}(X_t) + \bar{\varsigma}_t'\bar{\lambda}(X_t) - \frac{c_t}{\bar{W}_t}\right) dt + \bar{\varsigma}_t'\, dB_t, \quad \bar{W}_t > 0, \quad \forall t \in (0, T^*), \tag{2.27}$$

where

$$\bar{\varsigma}_t := \Sigma(X_t)'\Phi_t - \sigma_t^p. \tag{2.28}$$

Remark 3 Eq. (2.27) indicates that $\bar{\varsigma}_t$ represents the volatility of portfolio Φ_t. The n-th element $\bar{\varsigma}_{nt}$ of portfolio volatility represents the investment in the risk in the direction B_{nt}. Eq. (2.27) also exhibits that the n-th element $\bar{\lambda}_{nt}$ of the (real) MPR is the reward for unit investment in the risk in the direction B_{nt}.

We say that a control $(c, \bar{\varsigma})$ is admissible if it satisfies the real budget constraint (2.27) with initial state $\mathbf{X}_0$. Let $\mathcal{B}(\mathbf{X}_0)$ denote the set of admissible controls. The investor's robust consumption–investment problem is given by

$$\sup_{(c,\bar{\varsigma})\in\mathcal{B}(\mathbf{X}_0)} \inf_{\mathrm{P}^\xi\in\mathbb{P}} V_0^\xi. \tag{2.29}$$

3 Optimal Robust Control

This section derives the optimal robust control and PDE for indirect utility, followed by an analysis of the volatility of the optimal robust portfolio.

3.1 Optimal Control and the PDE for Indirect Utility

Let $\Gamma = \gamma I, \Psi = \psi I$ and $\tau^* = T^* - t$, hereafter. Let $\bar{W}^*$ and $(c^*, \bar{\varsigma}^*)$ denote the optimal real wealth and optimal control, respectively. The following lemma generalizes Lemma 1 by Kikuchi and Kusuda [21], in the case of HREZ utility.

Lemma 1 *Under Assumptions 1 and 2, the indirect utility function, optimal wealth, and optimal control for problem (2.29) satisfy Eqs. (3.1)–(3.4), and G is a solution to the PDE (3.5).*

$$J(t, \mathbf{X}_t^*) = \frac{\bar{W}_t^{*1-\gamma}}{1-\gamma}\big(G(\tau^*, X_t)\big)^{\frac{1-\gamma}{\psi-1}}, \tag{3.1}$$

$$\bar{W}_t^* = W_0 \exp\left(\int_0^t \left(\bar{r}_s + (\bar{\varsigma}_s^*)'\bar{\lambda}_s - \frac{\beta^\psi}{G(\tau, s)} - \frac{1}{2}|\bar{\varsigma}_s^*|^2\right) ds + \int_0^t (\bar{\varsigma}_s^*)'\, dB_s\right), \tag{3.2}$$

$$c_t^* = \beta^\psi \frac{\bar{W}_t^*}{G(\tau^*, X_t)}, \tag{3.3}$$

$$\bar{\varsigma}_t^* = (\Gamma + \Theta)^{-1}\bar{\lambda}_t + \big(I_N - (\Gamma+\Theta)^{-1}\big)\left(\frac{1}{1-\psi}\frac{G_x(\tau^*, X_t)}{G(\tau^*, X_t)}\right), \tag{3.4}$$

$$\begin{aligned} G_\tau &= \frac{1}{2}\,\mathrm{tr}\,[G_{xx}] + \frac{1}{2(1-\psi)}\frac{G_x'}{G}\big(\Psi - (\Gamma+\Theta)^{-1}\big)G_x \\ &-\Big(\mathcal{K}x + \big(I_N - (\Gamma+\Theta)^{-1}\big)\bar{\lambda}(x)\Big)' G_x - \left(\frac{1-\psi}{2}\bar{\lambda}(x)'(\Gamma+\Theta)^{-1}\bar{\lambda}(x) + (1-\psi)\,\bar{r}(x) + \beta\psi\right) G + \beta^\psi, \\ &G(0, x) = 0. \end{aligned} \tag{3.5}$$

Proof See Appendix B.1.

Remark 4 Strictly speaking, $(c^*, \bar{\varsigma}^*)$ is a candidate for optimal control because we do not provide a verification theorem; we tentatively call $(c^*, \bar{\varsigma}^*)$ optimal control in this study.

Note that $(\Gamma + \Theta)^{-1}$ in Eqs. (3.4) and (3.5) is well-defined because Γ is positive definite. Kikuchi and Kusuda [20] refer to the sum of relative risk aversion and relative ambiguity aversion as the relative uncertainty aversion. They also refer to the reciprocal of the relative uncertainty aversion as the relative uncertainty tolerance. We refer to $(\Gamma + \Theta)$ and $(\Gamma + \Theta)^{-1}$ as the matrix of relative uncertainty aversions and the matrix of relative uncertainty tolerances, respectively. These concepts generalize the relative uncertainty aversion and relative uncertainty tolerance.

3.2 Volatility of the Optimal Robust Portfolio

Let $\bar{\eta}_t^* := \dfrac{1}{1-\psi}\dfrac{G_x(\tau^*, X_t)}{G(\tau^*, X_t)}$. Following Batbold *et al.* [4], we refer to $\bar{\eta}_t^*$ as the investor hedging value of intertemporal uncertainty (IHVIU). From Eq. (2.28), the volatility of optimal robust portfolio satisfies

$$\bar{\varsigma}_t^* = \Sigma(X_t)'\Phi_t^* = (\Gamma + \Theta)^{-1}\lambda_t + \big(I_N - (\Gamma + \Theta)^{-1}\big)\bar{\eta}_t^*. \tag{3.6}$$

Eq. (3.6) demonstrates that the volatility of the optimal robust portfolio is a weighted average of the MPR and the IHVIU, where the weight for the nominal market price of risk is the matrix of relative uncertainty tolerances.

Let v_{nt} denote the n-th element of vector process v_t. In the case of HREZ utility, GHREZ utility with $\Theta = \mathrm{diag}(\theta_1, \cdots, \theta_N)$, and general GHREZ utility, Eq. (3.6) is rewritten as (3.7), (3.8), and (3.9), respectively.

$$\bar{\varsigma}_{nt}^* = (\gamma + \theta)^{-1}\lambda_{nt} + \big(1 - (\gamma + \theta)^{-1}\big)\bar{\eta}_{nt}^*, \tag{3.7}$$

$$\bar{\varsigma}_{nt}^* = (\gamma + \theta_n)^{-1}\lambda_{nt} + \big(1 - (\gamma + \theta_n)^{-1}\big)\bar{\eta}_{nt}^*, \tag{3.8}$$

$$\bar{\varsigma}_{nt}^* = \big((\Gamma + \Theta)^{-1}\big)_n\lambda_t + \big(I_N - (\Gamma + \Theta)^{-1}\big)_n\bar{\eta}_t^*. \tag{3.9}$$

In Eq. (3.9), M_n is the n-th row of matrix M.

Remark 5 In all of the Eqs. (3.7)–(3.9), the volatility of the optimal robust portfolio is a weighted average of the MPR and the IHVIU. In the case of HREZ utility and GHREZ utility with $\Theta = \mathrm{diag}(\theta_1, \cdots, \theta_N)$, $\bar{\varsigma}_{nt}^*$ depends only on the corresponding market price λ_{nt} of risk and investor hedging value η_{nt} of intertemporal uncertainty. However, in the case of HREZ utility, its weight is the common relative uncertainty tolerance $(\gamma + \theta)^{-1}$, whereas in the case of GHREZ utility with $\Theta = \mathrm{diag}(\theta_1, \cdots, \theta_N)$, its weight is the corresponding relative uncertainty tolerance $(\gamma + \theta_n)^{-1}$. Furthermore, in the case of general GHREZ utility, $\bar{\varsigma}_{nt}^*$ depends not only on $(\lambda_{nt}, \bar{\eta}_{nt}^*)$ but also on $(\lambda_{mt}, \bar{\eta}_{mt}^*)$ for all $m \neq n$ and its weight depends on the n-th row elements of the matrix of the relative uncertainty tolerances.

4 Approximate Optimal Robust Portfolio

This section applies the time-dependent linear approximation method of Kikuchi and Kusuda [20] to the nonlinear PDE (3.5) and derives an approximate optimal robust portfolio.

4.1 Linear Approximate Solution

The PDE (3.5) is rewritten as

$$\begin{aligned} G_\tau &= \frac{1}{2}\operatorname{tr}\left[G_{xx}\right] + \frac{1}{2(1-\psi)}\frac{G_x'}{G}\big(\Psi - (\Gamma+\Theta)^{-1}\big)G_x \\ &\quad - \Big(\mathcal{K}x + \big(I_N - (\Gamma+\Theta)^{-1}\big)(\bar\lambda + \bar\Lambda x)\Big)' G_x \\ &- \left\{\frac{1-\psi}{2}(\bar\lambda+\bar\Lambda x)'(\Gamma+\Theta)^{-1}(\bar\lambda+\bar\Lambda x) + (1-\psi)\left(\bar\rho_0 + \bar\rho' x + \frac{1}{2}x'\bar{\mathcal{R}}x\right) + \beta\psi\right\} G + \beta^\psi. \end{aligned} \tag{4.1}$$

Let $\tilde G$ denote a time-dependent linear approximate solution of the PDE (4.1). The linear approximation method of Kikuchi and Kusuda [20] approximates $\dfrac{G_x}{G}$ in the nonlinear term of the PDE (4.1) by a linear function of x.

$$\frac{\tilde G_x}{\tilde G} = a(\tau^*) + A(\tau^*)x, \tag{4.2}$$

where $(a(\tau), A(\tau))$ is specified at the end of this subsection. The following approximate nonhomogeneous linear PDE is obtained.

$$\tilde G_\tau = \mathcal{L}\tilde G + \beta^\psi, \quad \tilde G(0,x) = 0, \tag{4.3}$$

where $\mathcal{L}$ is the linear differential operator defined by

$$\begin{aligned} \mathcal{L}\tilde G &= \frac{1}{2}\operatorname{tr}\left[\tilde G_{xx}\right] \\ &+ \left(\frac{1}{2(1-\psi)}\big(a(\tau)+A(\tau)x\big)\big(\Psi - (\Gamma+\Theta)^{-1}\big) + \mathcal{K}X_t + \big(I_N - (\Gamma+\Theta)^{-1}\big)\left(\bar\lambda + \bar\Lambda x\right)\right)' \tilde G_x \\ &- \left\{\frac{1-\psi}{2}(\bar\lambda+\bar\Lambda x)'(\Gamma+\Theta)^{-1}(\bar\lambda+\bar\Lambda x) + (1-\psi)\left(\bar\rho_0 + \bar\rho' x + \frac{1}{2}x'\bar{\mathcal{R}}x\right) + \beta\psi\right\} \tilde G. \end{aligned} \tag{4.4}$$

To solve the PDE (4.3), we first consider the following homogeneous linear PDE:

$$\frac{\partial}{\partial\tau}\tilde g(\tau^*, x) = \mathcal{L}\tilde g(\tau^*, x), \quad \tilde g(0,x) = 1. \tag{4.5}$$

An analytical solution to the PDE (4.5) is expressed as

$$\tilde g(\tau, x) = \exp\left(b_0(\tau^*) + b(\tau^*)'x + \frac{1}{2}x'B(\tau^*)x\right), \quad (a_0, a, A)(0) = 0, \tag{4.6}$$

where $B(\tau)$ is a symmetric matrix. Then, a semi-analytical solution to the PDE (4.3) is expressed as

$$\tilde{G}(\tau^*, x) = \int_0^{\tau^*} \tilde{g}(s, x)\, ds. \tag{4.7}$$

Define $\tilde{b}(\tau,^* x)$ and $\tilde{B}(\tau^*, x)$ by

$$\begin{aligned} \tilde{b}(\tau, x) &= \frac{1}{\tilde{G}(\tau^*, x)} \int_0^{\tau^*} \tilde{g}(s, x) b(s)\, ds, \\ \tilde{B}(\tau, x) &= \frac{1}{\tilde{G}(\tau^*, x)} \int_0^{\tau^*} \tilde{g}(s, x) B(s)\, ds. \end{aligned} \tag{4.8}$$

In Eq. (4.2), we set $(a(\tau^*), A(\tau^*)) = (\tilde{b}(\tau^*, 0), \tilde{B}(\tau^*, 0))$, that is,

$$\frac{\tilde{G}_x}{\tilde{G}} = \tilde{b}(\tau^*, 0) + \tilde{B}(\tau^*, 0)x. \tag{4.9}$$

4.2 Approximate Optimal Portfolio

Let $H = \mathcal{K} + \big(I_N - (\Gamma + \Theta)^{-1}\big)\bar{\Lambda}$. Define functions m_2, m_1, and m_0 by

$$m_2(B) = B^2 - H'B - BH - (1-\psi)\Big(\bar{\Lambda}'(\Gamma+\Theta)^{-1}\bar{\Lambda} + \bar{\mathcal{R}}\Big),$$

$$m_1(B, b) = \Big(B - H'\Big)b - \big(I_N - (\Gamma+\Theta)^{-1}\big)B\bar{\lambda} - (1-\psi)\Big(\bar{\Lambda}'(\Gamma+\Theta)^{-1}\bar{\lambda} + \bar{\rho}\Big),$$

$$m_0(B, b) = \frac{1}{2}\big(\mathrm{tr}[B] + |b|^2\big) - \bar{\lambda}'\big(I_N - (\Gamma+\Theta)^{-1}\big)b - (1-\psi)\left(\frac{1}{2}\bar{\lambda}'(\Gamma+\Theta)^{-1}\bar{\lambda} + \bar{\rho}_0\right) + \beta\psi.$$

The approximate optimal wealth and optimal control based on the solution of the linear approximate PDE (4.3) are denoted by $\tilde{W}^*$ and $(\tilde{c}^*, \tilde{\varsigma}^*)$, respectively. The following proposition generalizes Proposition 1 by Kikuchi and Kusuda [21] in the HREZ utility case.

Proposition 1 *Under Assumptions 1 and 2, the approximate optimal wealth and optimal control for problem (2.29) satisfy Eqs. (4.10)–(4.12).*

$$\tilde{W}_t^* = W_0 \exp\left(\int_0^t \left(\bar{r}(X_s) + (\tilde{\varsigma}_s^*)'\bar{\lambda}(X_s) - \frac{\beta^\psi}{\tilde{G}(\tau^*, X_t)} - \frac{1}{2}|\tilde{\varsigma}_s^*|^2\right) ds + \int_0^t (\tilde{\varsigma}_s^*)'\, dB_s\right), \tag{4.10}$$

$$\tilde{c}_t^* = \beta^\psi \frac{\tilde{W}_t^*}{\tilde{G}(\tau^*, X_t)}, \tag{4.11}$$

$$\tilde{\varsigma}_t^* = (\Gamma+\Theta)^{-1}\big(\bar{\lambda} + \bar{\Lambda}X_t\big) + \big(I_N - (\Gamma+\Theta)^{-1}\big)\left(\frac{1}{1-\psi}\big(\tilde{b}(\tau, X_t) + \tilde{B}(\tau, X_t)X_t\big)\right), \tag{4.12}$$

where $(\tilde{B}, \tilde{b})$ *is given by Eq. (4.8), and* (B, b, b_0) *is a solution of the system of ODEs:*

$$
\begin{aligned}
\frac{dB}{d\tau} &= m_2(B) + \frac{1}{1-\psi}\tilde{B}(\tau,0)'\Big(\Psi - (\Gamma+\Theta)^{-1}\Big)B(\tau),\\
\frac{db}{d\tau} &= m_1(B,b)\\
&+\frac{1}{2(1-\psi)}\left\{\tilde{B}(\tau,0)'\Big(\Psi-(\Gamma+\Theta)^{-1}\Big)b(\tau) + B(\tau)'\Big(\Psi-(\Gamma+\Theta)^{-1}\Big)\tilde{b}(\tau,0)\right\},\\
\frac{db_0}{d\tau} &= m_0(B,b) + \frac{1}{2(1-\psi)}\tilde{b}(\tau,0)'\Big(\Psi-(\Gamma+\Theta)^{-1}\Big)b(\tau),
\end{aligned}
\tag{4.13}
$$

with $(B, b, b_0)(0) = (O_N, 0_N, 0)$.

Proof See Appendix B.2.

It follows from Eqs. (2.28) and (4.12) that the approximate optimal robust portfolio $\tilde{\Phi}^*_t$ is given by

$$
\begin{aligned}
\tilde{\Phi}^*_t &= \Sigma(X_t)'^{-1}(\Gamma+\Theta)^{-1}\Big(\lambda + \Lambda X_t\Big)\\
&\quad + \Sigma(X_t)'^{-1}\big(I_N - (\Gamma+\Theta)^{-1}\big)\left(\frac{1}{1-\psi}\Big(\tilde{a}(\tau^*, X_t) + \tilde{A}(\tau^*, X_t)X_t\Big)\right)\\
&\qquad + \Sigma(X_t)'^{-1}\big(I_N - (\Gamma+\Theta)^{-1}\big)\Big(\sigma_p + \Sigma_p X_t\Big).
\end{aligned}
\tag{4.14}
$$

5 Simple Numerical Analysis

This section quantitatively analyzes the relationship between relative ambiguity aversion matrices and optimal robust asset allocations.

5.1 Mean Relative Ambiguity Aversion Matrix

We introduce the notion of mean relative ambiguity aversion. Let Θ denote a relative ambiguity aversion matrix. As Θ is assumed to be positive definite, there exists an orthogonal matrix U and a diagonal matrix D such that $\Theta = UDU'$, where $D = \mathrm{diag}(\theta_1, \cdots, \theta_N)$. We define the *mean relative ambiguity aversion of* Θ by $\bar{\theta} = \frac{1}{N}\sum_{n=1}^{N}\theta_n$.

5.2 Basic Setup

We assume the two-factor QSM model as a simple QSM model. We consider an investor who plans to invest in the S&P 500 and 10-year TIPS in addition to the money market account over 35 years. We assume GHREZ utility with $(\beta, \gamma, \psi) = (0.01, 2.5, 0.5)$ following Kikuchi and Kusuda [21]. We compare the optimal asset allocations of GHREZ utility investors with the same mean relative risk aversion $\bar{\theta} = 1.5$.

5.3 Interpretation of State Processes and Brownian Motions

For the parameters in the two-factor QSM model, we employ the results estimated by Kikuchi and Kusuda [21] (Appendix C has the details). From Eq. (C.1), the estimated state vector process is subject to the following SDE:

$$dX_{1t} = -0.01318 X_{1t}\, dt + dB_{1t}, \tag{5.1}$$

$$dX_{2t} = -0.01117\big(X_{2t} - 1.315 X_{1t}\big) dt + dB_{2t}. \tag{5.2}$$

Eqs. (5.1) and (5.2) indicate that X_{1t} is a relatively short-period mean-reverting state process with a mean of 0 and X_{2t} is a relatively long-period mean-reverting state process with a short-term mean of $1.315X_{1t}$ and a long-term mean of 0. As X_1 and X_2 are driven by the uncertainty sources B_1 and B_2, respectively, B_1 and B_2 represent a relatively short- and long-period uncertainty, respectively. In general, it is unclear whether investors are more ambiguity averse toward short- or long-period uncertainty.

5.4 Two Types of Relative Ambiguity Aversion Matries

We consider two types of relative ambiguity aversion matrices.

5.4.1 GHREZ Utilities with Diagonal Relative Ambiguity Aversion Matrices

First, we compare the approximate optimal asset allocations of GHREZ utilities with the following diagonal relative ambiguity aversion matrices.

$$\Theta^{(k)} = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix} = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} + k \begin{pmatrix} 0.125 & 0 \\ 0 & -0.125 \end{pmatrix}, \tag{5.3}$$

where $k = -11, -10, \cdots, 11$. Note that in the case of $k = 0$, GHREZ utility reduces to HREZ utility. Assume $\theta_1 < \theta_2$ or equivalently $k < 0$. Then, as shown in Remark 2, investors are more ambiguity averse in the direction B_2 (long-period uncertainty) than in B_1 (short-period uncertainty).

5.4.2 GHREZ Utilities with Non-diagonal Relative Ambiguity Aversion Matrices

Second, we compare the approximate optimal asset allocations of GHREZ utilities with the following non-diagonal relative ambiguity aversion matrices.

$$\Theta^{(\tilde{\rho})} = 1.5 \begin{pmatrix} 1 & \tilde{\rho} \\ \tilde{\rho} & 1 \end{pmatrix}, \tag{5.4}$$

where $\tilde{\rho} = -0.9, -0.8, \cdots, 0.9$. We easily confirm that the mean relative ambiguity aversion of $\Theta^{(\tilde{\rho})}$ is 1.5 for any $\tilde{\rho}$. Note that when $\tilde{\rho} = 0.0$, GHREZ utility reduces to HREZ utility.

5.5 Relative Ambiguity Aversion Matrix and Optimal Asset Allocation

We analyze the relationship between relative ambiguity aversion matrices and optimal asset allocations. Given that $\mathrm{E}[\lim_{t\to\infty} X_t] = 0_N$, we set $X_t = 0_N$. Then, the relationship between the relative ambiguity aversion matrices and approximate optimal allocations to the S&P 500 and 10-year TIPS is illustrated in Fig. 1.

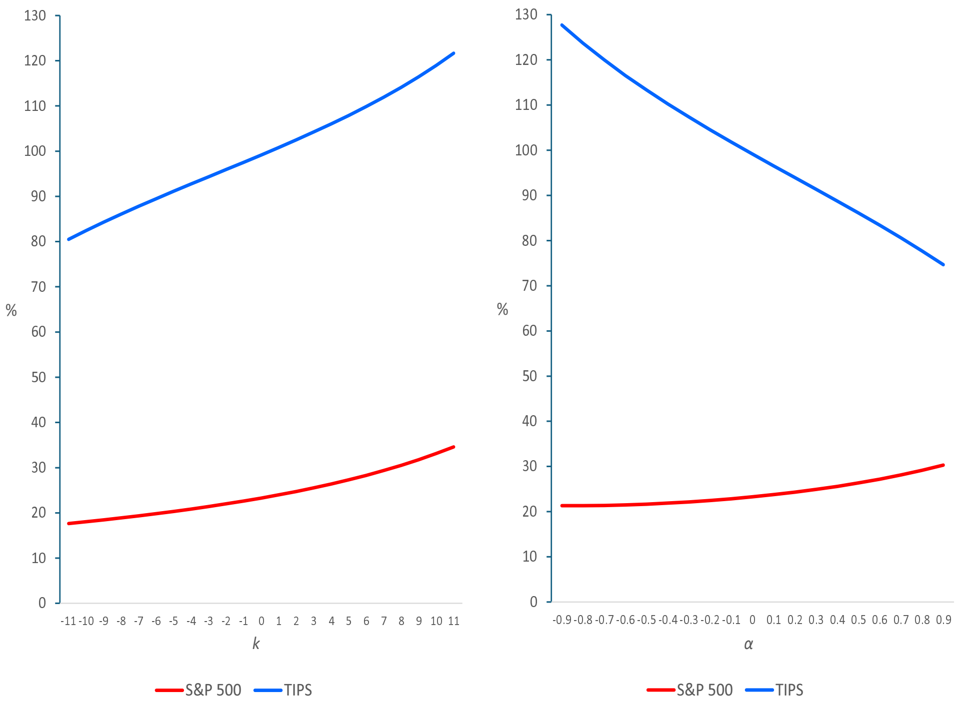

Fig. 1 Approximate optimal allocations (%) plotted against the diagonal (left) and nondiagonal (right) relative ambiguity aversion matrices.

As deviations from the case of equal relative ambiguity aversions for all risks (*i.e.*, HREZ utility) increase, deviations from the approximate optimal asset allocation under HREZ utility also increase, and the sensitivity is quite large. The findings suggest that when specifying the GHREZ utility, focusing solely on mean relative ambiguity aversion is insufficient; it is necessary to discern the overall structure of the relative ambiguity aversion matrix.

5.6 Factor Decompositions

The approximate optimal portfolio is decomposed into the sum of myopic, intertemporal hedging, and inflation–deflatiion hedging demands, as shown in

Eq. (4.14). First, the factor decomposition of the relationship between relative ambiguity aversion matrices and approximate optimal asset allocations to the 10-year TIPS yields the results in Fig. 2.

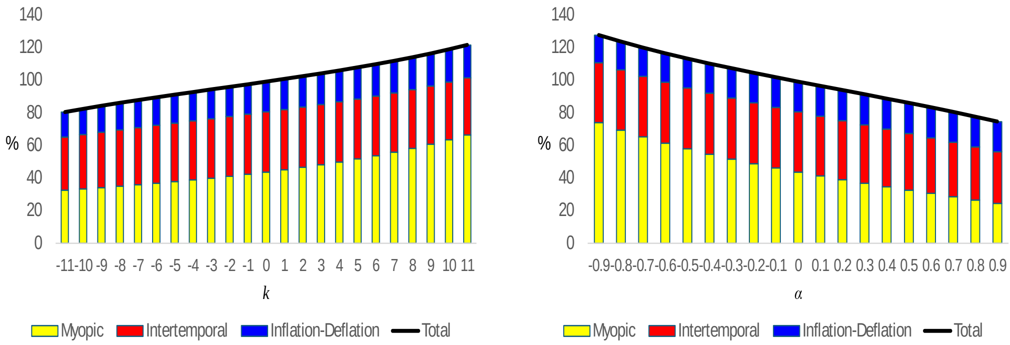

Fig. 2 Factor decomposition of approximate optimal allocations (%) to the 10-year TIPS plotted against the diagonal (left) and nondiagonal (right) relative ambiguity aversion matrices.

Changes in the optimal asset allocation to the 10-year TIPS are driven solely by shifts in myopic demand under both diagonal and nondiagonal relative ambiguity aversion matrices. Next, the factor decomposition of the relationship between relative ambiguity aversion matrices and approximate optimal asset allocations to the S&P 500 is illustrated in Fig. 3.

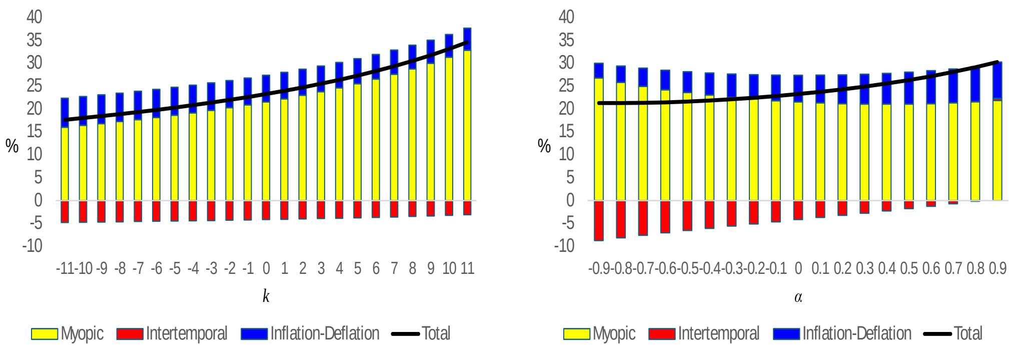

Fig. 3 Factor decomposition of approximate optimal allocations (%) to the S&P 500 plotted against the diagonal (left) and nondiagonal (right) relative ambiguity aversion matrices.

Changes in the optimal asset allocation to the S&P 500 are driven solely by shifts in myopic demand under diagonal relative ambiguity aversion matrices. By contrast, under nondiagonal relative ambiguity aversion matrices, all types of demand contribute to these changes. Kikuchi and Kusuda [20], [21] have emphasized the importance of accurately measuring intertemporal hedging and inflation–deflation hedging demands in strategic asset allocation. These results suggest that these factors are also crucial in specifying the relative ambiguity aversion matrix of GHREZ utility.

6 Future Research

Batbold *et al.* [3] assume a quadratic international security market (QISM) model and derive a semi-analytical solution to the finite-time consumption–investment problem under CRRA utility. Their QISM model is a generalization of the quadratic international bond market model proposed by Leippold and Wu [27]. Batbold *et al.* [3] assume CRRA utility and derive a semi-analytical solution. Their QISM model has two issues. Firstly, the state vector is composed of two components: a global state vector and a currency-specific state vector. The concept of a currency-specific state is unduly restrictive in comparison to that of a global state. Secondly, exchange rates cannot be expressed as functions of the state vector. Consequently, this results in estimation inefficiencies, such as the necessity of employing unstable changes from the previous period instead of stable levels of the exchange rate as the observation variable in Kalman filter-based estimation. It should be noted that such structural issues in their QISM model are also present in the quadratic international bond market model of Leippold and Wu [27].

The authors are currently engaged in the development of a novel QISM model. The following research project is planned for the future: The objectives of this project are twofold: first, to derive a linear approximate optimal solution for the consumption–investment problem of investors with GHREZ utility under the novel QISM model; and second, to verify whether the approximate optimal asset allocation contributes to solving the equity home bias puzzle.

A The Parameters of Return Rates of Securities

$(\Sigma(\tau), \sigma(\tau))$ in Eq. (2.16) is a solution to the system of the following ODEs:

$$\frac{d\Sigma(\tau)}{d\tau} = \Sigma(\tau)^2 - (\mathcal{K} + \Lambda)'\Sigma(\tau) - \Sigma(\tau)(\mathcal{K} + \Lambda) - \mathcal{R} \tag{A.1}$$

$$\frac{d\sigma(\tau)}{d\tau} = -(\mathcal{K} + \Lambda - \Sigma(\tau))'\sigma(\tau) - (\Sigma(\tau)\lambda + \rho), \tag{A.2}$$

with $(\Sigma, \sigma)(0) = (O_N, 0_N)$.

$(\Sigma(\tau), \sigma(\tau))$ in Eq. (2.17) is a solution to the system of the following ODEs:

$$\frac{d\bar{\Sigma}_q(\tau)}{d\tau} = \bar{\Sigma}_q(\tau)^2 - (\mathcal{K} + \bar{\Lambda})'\bar{\Sigma}_q(\tau) - \bar{\Sigma}_q(\tau)(\mathcal{K} + \bar{\Lambda}) - \bar{\mathcal{R}}, \tag{A.3}$$

$$\frac{d\bar{\sigma}_q(\tau)}{d\tau} = -(\mathcal{K} + \bar{\Lambda} - \bar{\Sigma}_q(\tau))'\bar{\sigma}_q(\tau) - (\bar{\Sigma}_q(\tau)\bar{\lambda} + \bar{\rho}), \tag{A.4}$$

with $(\bar{\Sigma}_q, \bar{\sigma}_q)(0) = O_N, 0_N)$.

In Eq. (2.18), Σ_k is a solution to Eq. (A.5) and σ_k is given by Eq. (A.6), respectively.

$$0 = \Sigma_k^2 - (\mathcal{K} + \Lambda)'\Sigma_k - \Sigma_j(\mathcal{K} + \Lambda) + \Delta_k - \mathcal{R}, \tag{A.5}$$

$$\sigma_k = (\mathcal{K} + \Lambda - \Sigma_k)'^{-1}(\delta_k - \rho - \Sigma_k\lambda), \tag{A.6}$$

B Proofs

B.1 Proof of Lemma 1

As the standard Brownian motion under P^ξ is given by $B_t^\xi = B_t - \int_0^t \xi_s\, ds$, the SDE for $\mathbf{X}_t$ under P^ξ is rewritten as

$$d\mathbf{X}_t = \left(\begin{pmatrix} \bar{W}_t\left(\bar{r}(X_t) + \bar{\varsigma}'\bar{\lambda}(X_t)\right) - c_t \\ -\mathcal{K}X_t \end{pmatrix} + \begin{pmatrix} \bar{W}_t\bar{\varsigma}_t' \\ I_N \end{pmatrix} \xi_t \right) dt + \begin{pmatrix} \bar{W}_t\bar{\varsigma}_t' \\ I_N \end{pmatrix} dB_t^\xi. \tag{B.1}$$

Let $J = J(t, \mathbf{x})$ denote the indirect utility function. Given that $\mathbf{X}$ is a Markov process, our optimal investment problem is formulated as a Markov decision problem. Thus, there are measurable functions $\hat{c}(\mathbf{X}_t), \hat{\varsigma}(\mathbf{X}_t)$, and $\hat{\xi}(\mathbf{X}_t)$ such that $c_t = \hat{c}(\mathbf{X}_t), \bar{\varsigma}_t = \hat{\varsigma}(\mathbf{X}_t)$, and $\xi_t = \hat{\xi}(\mathbf{X}_t)$ for every $t \in [0, T^*)$. The Hamilton-Jacobi-Bellman (HJB) equation for problem (2.29) is expressed as

$$\begin{aligned} \sup_{(\hat{c}(\mathbf{x}),\hat{\varsigma}(\mathbf{x}))\in\mathbb{R}_+\times\mathbb{R}^N} \inf_{\hat{\xi}(\mathbf{x})\in\mathbb{R}^N} \Bigg\{ & J_t + \begin{pmatrix} w\left(\bar{r}(x) + \hat{\varsigma}(\mathbf{x})'\bar{\lambda}(x)\right) - \hat{c}(\mathbf{x}) \\ -\mathcal{K}x \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix} \\ & + \frac{1}{2}\operatorname{tr}\left[\begin{pmatrix} w\hat{\varsigma}(\mathbf{x})' \\ I_N \end{pmatrix} \begin{pmatrix} w\hat{\varsigma}(\mathbf{x})' \\ I_N \end{pmatrix}' \begin{pmatrix} J_{ww} & J_{wx} \\ J_{xw} & J_{xx} \end{pmatrix} \right] \\ & + f(\hat{c}(\mathbf{x}), J) + \frac{(1-\gamma)J}{2}\hat{\xi}(\mathbf{x})'\Theta^{-1}\hat{\xi}(\mathbf{x}) + \hat{\xi}(\mathbf{x})' \begin{pmatrix} w\hat{\varsigma}(\mathbf{x})' \\ I_N \end{pmatrix}' \begin{pmatrix} J_w \\ J_w \end{pmatrix} \Bigg\} = 0, \end{aligned} \tag{B.2}$$

with $J(T^*, \mathbf{x}) = 0$. Then, the worst-case probability $\hat{\xi}^*$ is given by

$$\hat{\xi}^*(\mathbf{x}) = -\frac{1}{(1-\gamma)J}\Theta \begin{pmatrix} w\hat{\varsigma}(\mathbf{x})' \\ I_N \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix}. \tag{B.3}$$

Variables on controls are omitted, hereafter. Substituting $\hat{\xi}^*$ into the HJB equation (B.2) yields

$$\begin{aligned} \sup_{(\hat{c},\hat{\varsigma})\in\mathbb{R}_+\times\mathbb{R}^N} \Bigg[& J_t + \begin{pmatrix} w\left(\bar{r}(x) + \hat{\varsigma}'\bar{\lambda}(x)\right) - \hat{c} \\ -\mathcal{K}x \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix} + \frac{1}{2}\operatorname{tr}\left[\begin{pmatrix} w\hat{\varsigma}' \\ I_N \end{pmatrix} \begin{pmatrix} w\hat{\varsigma}' \\ I_N \end{pmatrix}' \begin{pmatrix} J_{ww} & J_{wx} \\ J_{xw} & J_{xx} \end{pmatrix} \right] \\ & + f(\hat{c}, J) - \frac{1}{2(1-\gamma)J} \begin{pmatrix} J_w \\ J_x \end{pmatrix}' \begin{pmatrix} w\hat{\varsigma}' \\ I_N \end{pmatrix} \Theta \begin{pmatrix} w\hat{\varsigma}' \\ I_N \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix} \Bigg] = 0. \end{aligned} \tag{B.4}$$

It is straightforward to see that optimal control $(\hat{c}^*, \hat{\varsigma}^*)$ in the HJB equation (B.4) satisfies

$$\hat{c}^* = \beta^\psi J_w^{-\psi}\left((1-\gamma)J\right)^{\frac{\gamma\psi-1}{\gamma-1}}, \tag{B.5}$$

$$\hat{\varsigma}^* = \mathcal{T}_t\left(\bar{\lambda}(x) + \frac{J_{xw}}{J_w} + \frac{\Theta}{\gamma-1}\frac{J_x}{J}\right), \tag{B.6}$$

where $\mathcal{T}_t$ is given by

$$\mathcal{T}_t = \left(-\frac{wJ_{ww}}{J_w}I_N + \frac{wJ_w}{(1-\gamma)J}\Theta\right)^{-1}. \tag{B.7}$$

The consumption-related terms in the HJB equation (B.4) are computed as

$$-\hat{c}^*J_w + f(\hat{c}^*, J) = \hat{c}^*\left(-J_w + \frac{1}{1-\psi^{-1}}J_w\right) - \frac{\beta(1-\gamma)}{1-\psi^{-1}}J = \frac{1}{\psi-1}\hat{c}^*J_w - \frac{\beta(1-\gamma)}{1-\psi^{-1}}J. \tag{B.8}$$

The investment-related terms in the HJB equation (B.4) are computed as

$$
\begin{aligned}
wJ_w\bar{\lambda}_t'\hat{\varsigma}^* &+ \frac{1}{2}\,\mathrm{tr}\left[\begin{pmatrix} w(\hat{\varsigma}^*)' \\ I_N \end{pmatrix}\begin{pmatrix} w(\hat{\varsigma}^*)' \\ I_N \end{pmatrix}'\begin{pmatrix} J_{ww} & J_{wx} \\ J_{xw} & J_{xx} \end{pmatrix}\right] \\
&- \frac{1}{2(1-\gamma)J}\begin{pmatrix} J_w \\ J_x \end{pmatrix}'\begin{pmatrix} \bar{W}_t(\hat{\varsigma}^*)' \\ I_N \end{pmatrix}\Theta\begin{pmatrix} w(\hat{\varsigma}^*)' \\ I_N \end{pmatrix}'\begin{pmatrix} J_w \\ J_x \end{pmatrix} \\
&= \frac{1}{2}\,\mathrm{tr}\left[J_{xx}\right] - \frac{1}{2(1-\gamma)J}J_x'\Theta J_x - \zeta_t'\left(w^2 J_{ww} I_N - \frac{(wJ_w)^2}{(1-\gamma)J}\Theta\right)^{-1}\zeta_t, \quad \text{(B.9)}
\end{aligned}
$$

where

$$
\zeta_t = -wJ_w\left(\bar{\lambda}(x) + \frac{J_{xw}}{J_w} + \frac{\Theta}{\gamma-1}\frac{J_x}{J}\right). \quad \text{(B.10)}
$$

By substituting optimal control (B.5) and (B.6) into the HJB equation (B.4) and using Eqs. (B.8) and (B.9), the following PDE for J is obtained:

$$
\begin{aligned}
J_t + \frac{1}{2}\mathrm{tr}\left[J_{xx}\right] - \frac{1}{2(1-\gamma)J}J_x'\Theta J_x - \frac{1}{2}\zeta_t'\left(w^2 J_{ww} I_N - \frac{(wJ_w)^2}{(1-\gamma)J}\Theta\right)^{-1}\zeta_t \\
+ \bar{r}(x)wJ_w - (\mathcal{K}x)'J_x + \frac{1}{\psi-1}\hat{c}^* J_w - \frac{\beta(1-\gamma)}{1-\psi^{-1}}J = 0. \quad \text{(B.11)}
\end{aligned}
$$

From the above PDE, we conjecture that the indirect utility function takes the form in (3.1).

$$
J(t,x) = \frac{w^{1-\gamma}}{1-\gamma}\left(G(\tau^*, x)\right)^{\frac{1-\gamma}{\psi-1}}.
$$

Derivatives of J are given by

$$
\begin{aligned}
&J_t = -\frac{1-\gamma}{\psi-1}J\frac{G_\tau}{G}, \quad wJ_w = (1-\gamma)J, \quad J_x = \frac{1-\gamma}{\psi-1}J\frac{G_x}{G}, \quad w^2 J_{ww} = -\gamma(1-\gamma)J, \\
&wJ_{xw} = \frac{(1-\gamma)^2}{\psi-1}J\frac{G_x}{G}, \quad J_{xx} = \frac{1-\gamma}{\psi-1}J\left(\frac{2-\gamma-\psi}{\psi-1}\frac{G_x}{G}\frac{G_x'}{G} + \frac{G_{xx}}{G}\right).
\end{aligned}
$$

The optimal consumption control (3.3) follows from (B.5):

$$
\hat{c}^* = \beta^\psi\left(\frac{(1-\gamma)J}{w}\right)^{-\psi}\left((1-\gamma)J\right)^{\frac{\gamma\psi-1}{\gamma-1}} = \beta^\psi w^\psi\left(w^{1-\gamma}G^{\frac{1-\gamma}{\psi-1}}\right)^{\frac{\psi-1}{\gamma-1}} = \beta^\psi\frac{w}{G}. \quad \text{(B.12)}
$$

$\mathcal{T}_t$ in Eq. (B.7) and ζ_t in Eq. (B.10) are expressed as

$$
\mathcal{T}_t = (\Gamma + \Theta)^{-1}, \quad \text{(B.13)}
$$

$$
\zeta_t = (\gamma-1)J\left(\bar{\lambda}(x) + \frac{1}{1-\psi}(\Gamma + \Theta - I_N)\frac{G_x}{G}\right). \quad \text{(B.14)}
$$

Therefore, by inserting Eq. (B.14) and the derivatives of J into Eq. (B.6), we obtain Eq. (3.4). The second to fourth terms in the PDE (B.11) are calculated from Eqs. (B.14) as follows:

$$\begin{aligned}
&\frac{1}{2}\operatorname{tr}[J_{xx}] - \frac{1}{2(1-\gamma)J}J_x'\Theta J_x - \frac{1}{2}\zeta_t'\left(w^2 J_{ww} I_N - \frac{(wJ_w)^2}{(1-\gamma)J}\Theta\right)^{-1}\zeta_t \\
=&J\left\{\frac{1-\gamma}{2(\psi-1)}\operatorname{tr}\left[\frac{2-\gamma-\psi}{\psi-1}\frac{G_x}{G}\frac{G_x'}{G} + \frac{G_{xx}}{G}\right] - \frac{1-\gamma}{2(\psi-1)^2}\frac{G_x'}{G}\Theta\frac{G_x}{G}\right. \\
&\left. + \frac{1-\gamma}{2(\psi-1)^2}\left((\psi-1)\bar{\lambda}(x) - (\Gamma+\Theta-I_N)\frac{G_x}{G}\right)'(\Gamma+\Theta)^{-1}\left((\psi-1)\bar{\lambda}(x) - (\Gamma+\Theta-I_N)\frac{G_x}{G}\right)\right\} \\
=&\frac{1-\gamma}{\psi-1}J\left\{\frac{1}{2}\operatorname{tr}\left[\frac{2-\gamma-\psi}{\psi-1}\frac{G_x}{G}\frac{G_x'}{G} + \frac{G_{xx}}{G}\right] - \frac{1}{2(\psi-1)}\frac{G_x'}{G}\Theta\frac{G_x}{G}\right. \\
&\left. + \frac{1}{2(\psi-1)}\left((\psi-1)\bar{\lambda}(x) - (\Gamma+\Theta-I_N)\frac{G_x}{G}\right)'(\Gamma+\Theta)^{-1}\left((\psi-1)\bar{\lambda}(x) - (\Gamma+\Theta-I_N)\frac{G_x}{G}\right)\right\} \\
=&\frac{1-\gamma}{\psi-1}J\left\{\frac{1}{2}\operatorname{tr}\left[\frac{G_{xx}}{G}\right] + \frac{\psi-1}{2}\bar{\lambda}_t'(\Gamma+\Theta)^{-1}\bar{\lambda}(x) - \bar{\lambda}(x)'\big(I_N - (\Gamma+\Theta)^{-1}\big)\frac{G_x}{G}\right. \\
&\left. - \frac{1}{2(\psi-1)}\frac{G_x'}{G}\Big((\gamma+\psi-2)I_N + \Theta - \big(I_N - (\Gamma+\Theta)^{-1}\big)(\Gamma+\Theta-I_N)\Big)\frac{G_X}{G}\right\} \\
=&\frac{1-\gamma}{\psi-1}J\left\{\frac{1}{2}\operatorname{tr}\left[\frac{G_{xx}}{G}\right] + \frac{\psi-1}{2}\bar{\lambda}(x)'(\Gamma+\Theta)^{-1}\bar{\lambda}(x) - \bar{\lambda}_t'\big(I_N - (\Gamma+\Theta)^{-1}\big)\frac{G_x}{G}\right. \\
&\left. - \frac{1}{2(\psi-1)}\frac{G_x'}{G}\big(\Psi - (\Gamma+\Theta)^{-1}\big)\frac{G_X}{G}\right\}
\end{aligned} \tag{B.15}$$

The first, fifth, and sixth terms in the PDE (B.11) are computed as follows:

$$J_t + \bar{r}(x)wJ_w - (\mathcal{K}x)'J_x = \frac{1-\gamma}{\psi-1}J\left(-\frac{G_\tau}{G} + (\psi-1)\bar{r}(x) - (\mathcal{K}x)'\frac{G_x}{G}\right). \tag{B.16}$$

The seventh and eight terms in the PDE (B.11) are calculated from Eq. (B.12) as follows:

$$\frac{1}{\psi-1}\hat{c}^* J_w - \frac{\beta(1-\gamma)}{1-\psi^{-1}}J = \frac{1}{\psi-1}\Big(\beta^\psi\frac{w}{G}\frac{(1-\gamma)J}{w} + \beta(\gamma-1)\psi J\Big) = \frac{1-\gamma}{\psi-1}J\left(\frac{\beta^\psi}{G} - \beta\psi\right). \tag{B.17}$$

Substituting Eqs. (B.15)-(B.17) into Eq. (B.11) and dividing by $\frac{1-\gamma}{\psi-1}J$ yields the PDE (3.5).

B.2 Proof of Proposition 1

Substituting Eqs. (4.7) and $G_x = \big(b^*(\tau,0) + B^*(\tau,0)X_t\big)G$ into Eqs. (3.3) and (3.4) yield the approximate optimal consumption (4.11) and investment (4.12). Substituting $\tilde{g}$ and its derivatives into the PDE (4.5), we obtain

$$\begin{aligned}
&\frac{db_0}{d\tau} + X'\frac{db}{d\tau} + \frac{1}{2}X'\frac{dB}{d\tau}X \\
=&m(X_t;(B,b)) - \frac{1}{2(\psi-1)}\Big(\tilde{b}(\tau,0) + \tilde{B}(\tau,0)X_t\Big)'\big(\Psi - (\Gamma+\Theta)^{-1}\big)\big(b(\tau) + B(\tau)X_t\big)\Big) \\
=&m(X_t;(B,b)) - \frac{1}{2(\psi-1)}\Big\{\tilde{b}(\tau,0)'\big(\Psi - (\Gamma+\Theta)^{-1}\big)b(\tau) \\
&+ X_t'\Big(\tilde{B}(\tau,0)'b(\tau) + B(\tau)'\big(\Psi - (\Gamma+\Theta)^{-1}\big)\tilde{b}(\tau,0)\Big) + X_t'\tilde{B}(\tau,0)'\big(\Psi - (\Gamma+\Theta)^{-1}\big)B(\tau)X_t\Big\},
\end{aligned} \tag{B.18}$$

where $m(X_t;(B,b))$ is given by

$$
\begin{aligned}
m(X_t;(B,b)) &= \frac{1}{2}\mathrm{tr}[B] + \frac{1}{2}\left(|b|^2 + 2X_t'Bb + X_t'B^2X_t\right) \\
&- \left\{\left(I_N - (\Gamma+\Theta)^{-1}\right)\bar{\lambda} + \left(\mathcal{K} + \left(I_N - (\Gamma+\Theta)^{-1}\right)\bar{\Lambda}\right)X_t\right\}' b - \left(I_N - (\Gamma+\Theta)^{-1}\right)\bar{\lambda}'BX_t \\
&- \frac{1}{2}X_t'\left(\mathcal{K} + \left(I_N - (\Gamma+\Theta)^{-1}\right)\bar{\Lambda}\right)' BX_t - \frac{1}{2}X_t'B\left(\mathcal{K} + \left(I_N - (\Gamma+\Theta)^{-1}\right)\bar{\Lambda}\right)X_t \\
&+ \frac{\psi-1}{2}\left(\bar{\lambda}'(\Gamma+\Theta)^{-1}\bar{\lambda} + 2\bar{\lambda}'(\Gamma+\Theta)^{-1}\bar{\Lambda}X_t + X_t'\bar{\Lambda}'(\Gamma+\Theta)^{-1}\bar{\Lambda}X_t\right) \\
&+ (\psi-1)\left(\bar{\rho}_0 + \bar{\rho}'X_t + \frac{1}{2}X_t'\bar{\mathcal{R}}X_t\right) - \beta\psi. \qquad \text{(B.19)}
\end{aligned}
$$

As Eq. (B.18) is identical on X, we get the system of ODEs (4.13).

C Esimated Parameters in the QSM Model

Kikuchi and Kusuda [21] estimate the two-factor QSM model based on the extended Kalman filter on 318 month-end data observed in the US security markets from January 1999 to June 2025. The time-series data used for estimation are 6-month and 2-year treasury spot rates[8], 10-year TIPS real spot rates[9], and S&P 500[10]. To reduce the estimation burden, they assume the second-order term of the instantaneous expected inflation rate to be zero, that is, $\mathcal{I} = 0$. Their estimation results are as follows:

$$
\begin{aligned}
&dX_t = -\begin{pmatrix} 0.01318 & 0 \\ -0.01469 & 0.01117 \end{pmatrix} X_t dt + I_2 dB_t, \qquad \lambda_t = \begin{pmatrix} 0.001275 \\ 0.7160 \end{pmatrix} + \begin{pmatrix} 0.01159 & 0 \\ 0.008200 & 0.00001890 \end{pmatrix} X_t, \\
&r_t = 0.2969 + \begin{pmatrix} -0.03305 \\ 0.01727 \end{pmatrix}' X_t + \frac{1}{2}X_t'\begin{pmatrix} 0.001889 & -0.0008790 \\ -0.0008790 & 0.0006379 \end{pmatrix} X_t, \\
&\mu_t^p = 0.0006826 + \begin{pmatrix} 0.007234 \\ -0.00007430 \end{pmatrix}' X_t, \qquad \sigma_t^p = \begin{pmatrix} 0.03077 \\ 0.05553 \end{pmatrix} + \begin{pmatrix} 0.02063 & 0 \\ 0 & 0.002814 \end{pmatrix} X_t, \\
&\frac{D_t}{S_t} = 0.03151 + \begin{pmatrix} -0.001017 \\ 0.001444 \end{pmatrix}' X_t + \frac{1}{2}X_t'\begin{pmatrix} 0.0001076 & 0.00001323 \\ 0.00001323 & 0.00004773 \end{pmatrix} X_t. \qquad \text{(C.1)}
\end{aligned}
$$

Acknowledgments

We would like to thank Editage [https://editage.com/] for English language editing and manuscript review.

Statements and Declarations

Competing Interests

The authors have no competing interests to declare that are relevant to the content of this article.

[8] The spot rate data are available on the FRB website. They are computed based on the estimation method by Gürkaynak, Sack, and Wright [16].

[9] The TIPS real spot rate data are available on the FRB website. They are computed based on the estimation method by Gürkaynak, Sack, and Wright [17].

[10] The data are available on the website of Robert Shiller.

Role of the Funding Source

A part of this study was supported by Japan Society for the Promotion of Science Grant-in-Aid for Scientific Research(C) Grant Number JP20K01768 and by Ishii Memorial Securities Research Promotion Foundation.

Data Availability Statements

The data sets used to estimate the quadratic secutiry market model are publicly available. The data sets generated and analyzed in this study are not publicly available, but are available from the authors only if the authors perceive a reasonable request based on special circumstances.

References

1. Ahn, D.H., Dittmar, R.F., Gallant, A.R.: Quadratic term structure models: theory and evidence. Rev. Financ. Stud., 15(1), 243–288 (2002)
2. Batbold, B., Kikuchi, K., Kusuda, K.: Semi-analytical solution for consumption and investment problem under quadratic security market model with inflation risk. Math. Financ. Econ., 16(3), 509–537 (2022)
3. Batbold, B., Kikuchi, K., Kusuda, K.: Strategic international asset allocation under a quadratic model with exchange rate and inflation–deflatiion risks. Decis. Econ. Financ. (2025a) `https://doi.org/10.1007/s10203-025-00527-8`
4. Batbold, B., Kikuchi, K., Kusuda, K.: Worst-case premiums and identification of homothetic robust Epstein-Zin Utility under a quadratic model. Dec. Econ. Financ. (2025b) `https://doi.org/10.1007/s10203-025-00545-6`
5. Branger, N., Larsen, L.S., Munk, C.: Robust portfolio choice with ambiguity and learning about return predictability. J. Bank. Financ., 37(5), 1397–1411 (2013)
6. Branger, N. Muck, M.: Keep on smiling? The pricing of quanto options when all covariances are stochastic. J. Bank. Financ. 36(6), 1577–1591 (2012)
7. Chen, L., Filipović, D., Poor, H.V.: Quadratic term structure models for risk-free and defaultable rates. Math. Financ. 14(4), 515–536 (2004)
8. Coeurdacier, N., Rey, H.: Home Bias in Open Economy Financial Macroeconomics. J. Econ. Literature 51, 63–115 (2013)
9. Cooper, I. A., Sercu, P., Vanpee, R.: The Equity Home Bias Puzzle: A Survey. Foundations Trends Financ. 7, 289–416 (2013)
10. Duffie, D., Epstein, L.: Asset pricing with stochastic differential utility. Rev. Financ. Stud. 5, 411–436 (1992)
11. Duffie, D., Kan, R.: A yield-factor model of interest rates. Math. Financ. 6(4), 379–406 (1996)
12. El Karoui, N. Peng, S., Quenez, M.C.: Backward stochastic differential equations in finance. Math. Financ. 7, 1–71 (1997)
13. Epstein, L., Zin, S.: Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. Econometrica 57, 937-969 (1989)
14. Filipović, D., Gourier, E., Mancini, L.: Quadratic variance swap models. J. Financ. Econ., 119(1), 44-68 (2016)
15. French, K.R., Poterba, J.M.: Investor diversification and international equity markets. American Economic Review, Papers and Proceedings of the Hundred and Third Annual Meeting of the American Economic Association 81, 222–226 (1991)
16. Gürkaynak, R.S., Sack, B., Wright, J.H.: The U.S. treasury yield curve: 1961 to the present. J Monet. Econ. 54, 2291–2304 (2007)
17. Gürkaynak, R. S., Sack, B., Wright, J.H.: The TIPS yield curve and inflation compensation. Amer Econ. J.: Macroecon. 2, 70-92 (2010)
18. Kikuchi, K.: A term structure interest rate model with the Brownian bridge lower bound. Ann. Financ. 20(3), 301–328 (2024)
19. Kikuchi, K., Kusuda, K.: Age-dependent robust strategic asset allocation with inflation–deflatiion hedging demand. Math. Financ. Econ. 18(4), 641–670 (2024)

20. Kikuchi, K., Kusuda, K.: A Linear approximate robust strategic asset allocation with inflation–deflatiion hedging demand. preprint (2025a) http://ssrn.com/abstract=5500839
21. Kikuchi, K., Kusuda, K.: Regularized robust strategic asset allocation under stochastic variance-covariance matrix of asset returns. preprint (2025b) http://ssrn.com/abstract=5670211
22. Kusuda, K.: Properties of generalized homothetic robust Epstein-Zin utility. preprint (2025a) http://ssrn.com/abstract=5237344
23. Kusuda, K.: Some properties of generalized homothetic robust Epstein-Zin utility. preprint (2025b) https://ssrn.com/abstract=5388086
24. Kim, D.H., Singleton, K.J.: Term structure models and the zero bound: An empirical investigation of Japanese yields. J. Econometrics 170(1), 32–49 (2012)
25. Lazrak, A. Quenez, M. C.: A generalized stochastic differential utility. Math. Oper. Res. 28(1), 154–180 (2003)
26. Leippold, M., Wu, L.: Asset pricing under the quadratic class. J. Financ. Quant. Anal. 37(2), 271–295 (2002)
27. Leippold, M. Wu, L.: Design and estimation of multi-currency quadratic models. Rev. Financ. 11(2), 167–207 (2007)
28. Liu, H.: Robust consumption and portfolio choice for time varying investment opportunities. Ann. Financ. 6(4), 435–454 (2010)
29. Maenhout, P.J.: Robust portfolio rules and asset pricing. Rev. Financ. Stud. 17(4), 951–983 (2004)
30. Maenhout, P.J.: Robust portfolio rules and detection-error probabilities for a mean-reverting risk premium. J. Econ. Theor. 128(1), 136–163 (2006)
31. Munk, C., Rubtsov, A.V.: Portfolio management with stochastic interest rates and inflation ambiguity. Ann. Financ. 10(3), 419–455 (2014)
32. Skiadas, C.: Recursive utility and preferences for information. Econ. Theory 12, 293–312 (1998)
33. Skiadas, C.: Robust control and recursive utility. Financ. Stoch. 7(4), 475–489 (1999)
34. Schroder, M., Skiadas, C.: Optimal consumption and portfolio selection with stochastic differential utility. J. Econ. Theor. 89, 68–126 (1999)
35. Yi, B., Viens, F., Law, B., Li, Z.: Dynamic portfolio selection with mispricing and model ambiguity. Ann. Financ. 11(1), 37–75 (2015)