DISCUSSION PAPER SERIES E

SHIGA UNIVERSITY

Discussion Paper No. E-47

Loglinear and Linear Approximate Solutions for Finite-Time Consumption–Investment Problem

Kentaro Kikuchi Koji Kusuda

September 2025

The Institute for Economic and Business Research
Faculty of Economics
SHIGA UNIVERSITY

1-1-1 BANBA, HIKONE,
SHIGA 522-8522, JAPAN

Loglinear and Linear Approximate Solutions for Finite-Time Consumption–Investment Problem

Kentaro Kikuchi* Koji Kusuda†

3 September, 2025

Abstract

Assuming a quadratic security market model and homothetic robust utility in the finite-time consumption-investment problem, a second-order nonhomogeneous nonlinear partial differential equation is derived. This study introduces two types of time-dependent loglinear approximation methods related to the nonhomogeneous term and four types of time-dependent linear approximation methods for the nonlinear term. The study derives loglinear approximate solutions and showcases the results of linear approximate solutions. The study then compares the approximation accuracies of the approximate optimal portfolios based on these approximate solutions. The numerical analysis indicates that the accuracies of both loglinear approximate optimal portfolios are very low, whereas those of all linear approximate optimal portfolios are very high.

Keywords Linear approximation; Loglinear approximation; Time-dendent approximation; Consumption–investment problem; Homothetic robust utility; Quadratic model

1 Introduction

To analyze dynamic consumption–investment problems, we should establish a realistic security market model that captures actual asset price fluctuations. Prior empirical studies have shown that interest rates, market price of risk, asset volatilities, and inflation rates are stochastic and mean-reverting—such findings are now considered stylized facts. Batbold, Kikuchi, and Kusuda (2022) consider a finite-time consumption–investment problem for long-term investors with constant relative risk aversion (CRRA)

*Shiga University, 1-1-1 Banba, Hikone, Shiga 522-8522, Japan.

†Shiga University, 1-1-1 Banba, Hikone, Shiga 522-8522, Japan. kusuda@biwako.shiga-u.ac.jp (The corresponding author) ORCID: 0000-0002-5534-8149

utility under a quadratic security market (QSM) model[1] , wherein all the above-mentioned processes are stochastic and mean-reverting. Since the QSM model makes the investment opportunity set stochastic, intermediate utility generates a nonhomogeneous term in the second-order linear partial differential equation (PDE) for the indirect utility function. Batbold *et al.* (2022) derive a semi-analytical solution of the PDE and obtain an optimal portfolio which is decomposed into myopic, intertemporal hedging, and inflation-deflation hedging demands. They demonstrate that all the demands are nonlinear functions of the state vector. Their numerical analysis highlights the nonlinearity and significance of market timing effects. Nonlinearity stems from stochastic volatility, while significance is attributed primarily to inflation-deflation hedging demand, in addition to myopic demand. These results demonstrate that the QSM model is promising for consumption–investment problems.

We should also consider the need for robust dynamic investment control, which has been highlighted by the global financial crisis. Robust utility is proposed by Hansen and Sargent (2001); however robust utility does not have homotheticity[2] that CRRA utility has. Maenhout (2004) generalizes robust utility to endow it with homotheticity, which leads to homothetic robust utility. Homothetic robust utility has been used in various robust control studies including Skiadas (2003), Maenhout (2006), Liu (2010), Branger, Larsen, and Munk (2013), Munk and Rubtsov (2014), Yi, Viens, Law, and Li (2015), and Batbold, Kikuchi, and Kusuda (2019). The nonlinear term appears in the PDE along with the nonhomogeneous term under the stochastic investment opportunity set and homothetic robust utility.[3]

In infinite-time consumption–investment problems, the nonhomogeneous term in the PDE is equal to the stable optimal consumption–wealth ratio; thus, Campbell and Viceira (2002) use a loglinear approximation[4] of the nonhomogeneous term. Previous studies including Campbell and Viceira (2001) and Liu (2010) apply the loglinear approximation method of Campbell and Viceira (2002) to derive approximate solutions. Batbold

[1]QSM models are generalization of affine models independently developed by Ahn, Dittmar, and Gallant (2002) and Leippold and Wu (2002). The QSM models are employed in studies of empirical analysis (Leippold and Wu (2007), Kim and Singleton (2012), and Kikuchi (2024)), security pricing (Chen, Filipović, and Poor (2004), Boyarchenko and Levendorskii (2007), and Filipović, Gourier, and Mancini (2016)), and optimal consumption–investment (Batbold *et al.* (2022); Batbold, Kikuchi, and Kusuda (2025), and Kikuchi and Kusuda (2024)).

[2]A utility function U is homothetic if, for any consumption plan c and $\tilde{c}$, and any scaler $\alpha > 0$, $U(\alpha\tilde{c}) \geq U(\alpha c) \Leftrightarrow U(\tilde{c}) \geq U(c)$.

[3]Note that in the case of the portfolio choice problem as studied in Maenhout (2006), Branger *et al.* (2013), Munk and Rubtsov (2014), and Yi *et al.* (2015), the PDE is homogeneous because there is no intermediate utility that generates a nonhomogeneous term, and an analytical solution can be derived.

[4]Their loglinear approximation method is a continuous-time version of the method proposed by Campbell (1993) in the discrete-time model.

et al. (2019) use another loglinear approximation method to derive another approximate solution. All the above-mentioned studies consider infinite-time consumption–investment problems. This study asseses the finite-time consumption–investment problems for log-term investors with homothetic robust utility under the QSM model assumed by Batbold *et al.* (2022). Under the finite-time setting, the nonhomogeneous term is time-dependent and unstable. Kikuchi and Kusuda (2025) propose a time-dependent linear approximation method for the nonlinear term to derive an approximate solution.[5] However, there exist alternative time-dependent approximation methods. One such approach involves the use of a loglinear approximation related to the nonhomogeneous term, while the other entails the use of an alternative linear approximation for the nonlinear term.

In this study, we present two types of loglinear approximation methods and four types of linear approximation methods including the method proposed by Kikuchi and Kusuda (2025). We then compare the accuracies of approximate optimal portfolios based on these solutions. The objective of this study is to explore superior approximation methods for second-order nonlinear nonhomogeneous PDEs derived from optimal conditions for consumption–investment problems. It is not the objective of this study to explore methods that are generally superior as approximations for second-order nonlinear nonhomogeneous PDEs of the same type. The main contributions of this study can be summarized as follows.

First, we introduce two types of time-dependent loglinear approximation methods related to the nonhomogeneous term. One of them is a time-dependent version of the loglinear approximation method of Batbold *et al.* (2019). We then derive approximate solutions.

Second, we introduce four types of time-dependent linear approximation methods, including the proposed method by Kikuchi and Kusuda (2025), and present approximate solutions demonstrated by Kikuchi and Kusuda (2025).

Third, we examine the case of a long-term investor who plans to invest in the S&P500 and 10-year U.S. Treasury Inflation-Protected Securities (TIPS) in addition to the money market account over a 35-year period under the two-factor QSM model estimated by Batbold *et al.* (2022). When relative ambiguity aversion is zero, *i.e.*, in the case of CRRA utility, the semi-analytical solution presented by Batbold *et al.* (2022) is obtained. Therefore, we confirm that the optimal portfolio based on a numerical solution of the PDE approximates the one based on the semi-analytical solution with sufficiently high accuracy and can be regarded as the true optimal portfolio. Then, we consider the case of homothetic robust utility and compare

[5]The proposed time-dependent linear approximation method is applied to finite-time consumption-investment problems in the case of "age-dependent robust utility" (Kikuchi and Kusuda (2024)).

the accuracies of the six types of approximate optimal portfolios based on the numerical solution of the PDE. The results show that the accuracies of both loglinear approximate optimal portfolios are very low, whereas those of all linear approximate optimal portfolios are very high. Among the high-precision linear approximation methods, three methods appear to demonstrate slightly higher accuracy and stability than the remaining method. One method among them is recommended for adoption due to its simplicity in implementation.

For infinite-time consumption-investment problems, loglinear approximation methods have been exclusively used. However, the findings strongly indicate that linear approximation methods should be used instead of loglinear approximation methods for finite-time problems.

The remainder of this paper is organized as follows. Section 2 reviews the optimal robust control problem and the PDE. Section 3 introduce two types of loglinear approximation methods and derive the approximate optimal solutions. Section 4 introduces four types of linear approximation methods. Section 5 compares the accuracies of the approximation methods. Section 6 concludes this study and address future research issues. Appendix includes the proof of proposition.

2 Review of Optimal Robust Control and PDE

We review the QSM model and robust consumption–investment problem. We then show the optimal robust control and PDE derived by Kikuchi and Kusuda (2025).

2.1 QSM Model

We consider frictionless US markets over the period $[0, \infty)$. Investors' common subjective probability and information structure are modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathrm{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,\infty)}$ is the natural filtration generated by an N-dimensional standard Brownian motion B_t. We denote the expectation operator under P by E and the conditional expectation operator given $\mathcal{F}_t$ by E_t.

There are markets for a consumption good and securities at every date $t \in [0, \infty)$, and the consumer price index p_t is observed. The traded securities are K-types of stock price indices, the instantaneously nominal risk-free security called the money market account and a continuum of zero-coupon inflation-indexed bonds whose maturity dates are $(t, t + \tau^*]$, where τ^* is the longest time to maturity of the bonds. The payoff of bond is p_T US dollar at maturity T. On every date t, Q_t^T, and S_t^k denote the USD prices of the money market account, zero-coupon inflation-indexed bond with maturity date T, and k-th index, respectively. Let A' and I_n denote the transpose of A and $n \times n$ identity matrix, respectively.

We assume the following QSM model introduced by Batbold *et al.* (2022).

Assumption 1. *Let* $(\rho_0, \iota_0, \delta_{0k}, \sigma_{0k})$, $(\lambda, \rho, \iota, \sigma_p, \delta_k, \sigma_k)$, *and* $(\mathcal{R}, \mathit{\Delta}_k, \Sigma_k)$ *denote scalers, N-dimensional vectors, and $N \times N$ positive-definite symmetric matrices, respectively, where* $k \in \{1, \cdots, K\}$.

1. *State vector process X_t is N-dimensional and satisfies the following stochastic differential equation (SDE):*
$$dX_t = -\mathcal{K} X_t \, dt + I_N \, dB_t, \tag{2.1}$$
where $\mathcal{K}$ is an $N \times N$ lower triangular matrix, and all the diagonal elements are positive.

2. *Market price λ_t of risk and instantaneous nominal risk-free rate r_t are provided as*
$$\lambda_t = \lambda + \Lambda X_t, \qquad r_t = \rho_0 + \rho' X_t + \frac{1}{2} X_t' \mathcal{R} X_t, \tag{2.2}$$
where Λ is an $N \times N$ matrix.

3. *Consumer price index p_t satisfies*
$$\frac{dp_t}{p_t} = \mu^p(X_t) \, dt + \sigma^p(X_t)' dB_t, \qquad p_0 = 1, \tag{2.3}$$
where $\mu^p(X_t)$ and $\sigma^p(X_t)$ are given by
$$\mu^p(X_t) = \iota_0 + \iota' X_t + \frac{1}{2} X_t' \mathcal{I} X_t, \tag{2.4}$$
$$\sigma^p(X_t) = \sigma_p + \Sigma_p X_t, \tag{2.5}$$
where $\mathcal{I}$ is an $N \times N$positive-semidefinite symmetric matrix.

4. *The dividend of the k-th stock price index is given by*
$$D_t^k = \left(\delta_{0k} + \delta_k' X_t + \frac{1}{2} X_t' \mathit{\Delta}_k X_t \right) \exp \left(\sigma_{0k} t + \sigma_k' X_t + \frac{1}{2} X_t' \Sigma_k X_t \right). \tag{2.6}$$

5. *Markets are complete and arbitrage-free.*

2.2 Homothetic Robust Utility and Portfolio

Let $\mathbb{P}$ denote the set of all equivalent probability measures.[6] We use the follwoing notation: β is the subjective discount rate, $\gamma > 1$ is the relative risk aversion coefficient, $\theta > 0$ is the relative ambiguity aversion coefficient, and $\alpha \in [0, 1]$ represents the relative importance of the intermediate and terminal utility.

[6]A probability measure $\tilde{\mathrm{P}}$ is said to be an equivalent probability measure of P if and only if $\mathrm{P}(A) = 0 \Leftrightarrow \tilde{\mathrm{P}}(A) = 0$.

Assumption 2. *The investor's utility is the homothetic robust utility of the form:*

$$U(c) = \inf_{\mathrm{P}^{\xi}\in\mathbb{P}} \mathrm{E}^{\xi}\left[\int_0^{T^*} e^{-\beta t}\left(\alpha\frac{c_t^{1-\gamma}}{1-\gamma} + \frac{(1-\gamma)V_t^{\xi}}{2\theta}\,|\xi_t|^2\right) dt + (1-\alpha)e^{-\beta T^*}\frac{c_{T^*}^{1-\gamma}}{1-\gamma}\right], \tag{2.7}$$

where c is a consumption plan such that $c = (c_t)_{t\in[0,T^*)}$ is an adapted non-negative consumption-rate process, E^{ξ} is the expectation under P^{ξ}, and V_t is the utility process defined recursively as

$$V_t^{\xi} = \mathrm{E}_t^{\xi}\left[\int_t^{T^*} e^{-\beta(s-t)}\left(\alpha\frac{c_s^{1-\gamma}}{1-\gamma} + \frac{(1-\gamma)V_s^{\xi}}{2\theta}\,|\xi_s|^2\right) ds + (1-\alpha)e^{-\beta(T^*-t)}\frac{c_{T^*}^{1-\gamma}}{1-\gamma}\right]. \tag{2.8}$$

Let $Q_t(\tau) = Q_t^T$ where $\tilde{\tau} = T - t$.

Assumption 3. *The investor invests in $P_t, Q_t(\tilde{\tau}_1), \cdots, Q_t(\tilde{\tau}_J)$, and $S_t^1, \cdots, S_t^K$ at time t where $J + K = N$.*

Let $\Phi(\tilde{\tau})$ and Φ^k denote the portfolio weight of inflation-indexed bond with $\tilde{\tau}$-time to maturity and that of the k-th stock price index, respectively. Let $\hat{\sigma}_{\tilde{\tau}}(X_t)$ and $\hat{\sigma}^k(X_t)$ denote the volatility of inflation-indexed bond with $\tilde{\tau}$-time to maturity and that of the k-th stock price index, respectively. Let Φ and $\Sigma(X_t)$ denote the portfolio weight vector and portfolio volatility matirx. Then, Φ_t and $\Sigma(X_t)$ are expressed as

$$\Phi_t' = (\Phi_t(\tilde{\tau}_1), \cdots, \Phi_t(\tilde{\tau}_J), \Phi_t^1, \cdots, \Phi_t^K), \tag{2.9}$$

$$\Sigma(X_t)' = (\hat{\sigma}_{\tilde{\tau}_1}(X_t), \cdots, \hat{\sigma}_{\tilde{\tau}_J}(X_t), \hat{\sigma}^1(X_t), \cdots, \hat{\sigma}^k(X_t)). \tag{2.10}$$

2.3 Optimal Robust Portfolio and PDE

We define the real market price of risk $\bar{\lambda}(X_t)$ and real instantaneous interest rate $\bar{r}(X_t)$ as

$$\bar{\lambda}(X_t) = \lambda_t - \sigma^p(X_t), \tag{2.11}$$

$$\bar{r}(X_t) = r_t - \mu^p(X_t) + \lambda_t'\sigma^p(X_t). \tag{2.12}$$

Then, $\bar{\lambda}(X_t)$ and $\bar{r}(X_t)$ are expressed as

$$\bar{\lambda}(X_t) = \bar{\lambda} + \bar{\Lambda}X_t, \tag{2.13}$$

$$\bar{r}(X_t) = \bar{\rho}_0 + \bar{\rho}'X_t + \frac{1}{2}X_t'\bar{\mathcal{R}}X_t, \tag{2.14}$$

where

$$\bar{\lambda} = \lambda - \sigma_p, \qquad \bar{\Lambda} = \Lambda - \Sigma_p, \tag{2.15}$$

$$\bar{\rho}_0 = \rho_0 - \iota_0 + \lambda'\sigma^p, \qquad \bar{\rho} = \rho - \iota + \Lambda'\sigma_p + \Sigma_p'\lambda, \tag{2.16}$$

$$\bar{\mathcal{R}} = \mathcal{R} - \mathcal{I} + \Sigma_p'\Lambda + \Lambda'\Sigma_p. \tag{2.17}$$

Let c_t and $\bar{W}_t$ denote the consumption rate process and the real wealth process, respectively. Let $\mathbf{X}_t = (\bar{W}_t, X_t')'$ and let $\bar{W}_0 > 0$. Batbold *et al.* (2022) show the following:

1. $\hat{\sigma}_\tau(X_t)$ and $\hat{\sigma}^k(X_t)$ are given by

$$\hat{\sigma}_{\tilde{\tau}}(X_t) = \bar{\sigma}(\tilde{\tau}) + \sigma_p + \big(\bar{\Sigma}(\tilde{\tau}) + \Sigma_p\big)X_t, \tag{2.18}$$

$$\hat{\sigma}^k(X_t) = \sigma_k + \Sigma_k X_t, \tag{2.19}$$

   where $(\bar{\Sigma}(\tilde{\tau}), \bar{\sigma}(\tilde{\tau}))$ is a solution to ordinary differential equations (ODEs) (A.1) and (A.2), and (Σ_k, σ_k) is a solution to Eqs. (A.3) and (A.4).

2. The investor's consumption–investment problem and the value function are defined by

$$V(\mathbf{X}_0) = \sup_{(c,\bar{\sigma})\in\mathcal{B}(\mathbf{X}_0)} \inf_{\mathrm{P}^\xi\in\mathbb{P}} U_0. \tag{2.20}$$

   where $\mathcal{B}(\mathbf{X}_0)$ is the set of admissible controls defined by the following real budget constraint:

$$\frac{d\bar{W}_t}{\bar{W}_t} = \left(\bar{r}_t + \bar{\sigma}_t'\bar{\lambda}_t - \frac{c_t}{\bar{W}_t}\right) dt + \bar{\sigma}_t'\, dB_t, \quad t \in [0, T^*], \tag{2.21}$$

   with initial state $\mathbf{X}_0 = (\bar{W}_0, X_0')'$, and $\bar{\sigma}$ is the investment control given by

$$\bar{\sigma}_t = \Sigma(X_t)'\varPhi_t - \sigma_t^p. \tag{2.22}$$

Let $\tau = T^* - t$, hereafter. Kikuchi and Kusuda (2025) present that the indirect utility function, optimal consumption, and optimal investment for problem (2.20) satisfy Eqs. (2.23), (2.24), and (2.25), respectively, and G is a solution of the PDE (2.26).

$$J(t, \mathbf{X}_t) = e^{-\beta t}\frac{\bar{W}_t^{1-\gamma}}{1-\gamma}\big(G(\tau, X_t)\big)^\gamma, \tag{2.23}$$

$$c_t^* = \alpha^{\frac{1}{\gamma}}\frac{\bar{W}_t^*}{G(\tau, X_t)}, \tag{2.24}$$

$$\bar{\sigma}_t^* = \frac{1}{\gamma+\theta}\bar{\lambda}_t + \left(1 - \frac{1}{\gamma+\theta}\right)\frac{\gamma}{\gamma-1}\frac{G_X(\tau, X_t)}{G(\tau, X_t)}, \tag{2.25}$$

$$\frac{G_\tau}{G} = \frac{1}{2}\operatorname{tr}\left[\frac{G_{XX}}{G}\right] + \frac{\theta}{2(\gamma-1)(\gamma+\theta)}\left|\frac{G_X}{G}\right|^2 - \left(\mathcal{K}X_t + \frac{\gamma+\theta-1}{\gamma+\theta}(\bar{\lambda} + \bar{\Lambda}X_t)\right)'\frac{G_X}{G}$$
$$+ \frac{\alpha^{\frac{1}{\gamma}}}{G} - \frac{\gamma-1}{2\gamma(\gamma+\theta)}|\bar{\lambda} + \bar{\Lambda}X_t|^2 - \frac{\gamma-1}{\gamma}\left(\bar{\rho}_0 + \bar{\rho}'X_t + \frac{1}{2}X_t'\bar{\mathcal{R}}X_t\right) - \frac{\beta}{\gamma},$$
$$G(0, X_{T^*}) = (1-\alpha)^{\frac{1}{\gamma}}. \tag{2.26}$$

3 Loglinear Approximations

We introduce two types of loglinear approximation methods, wherein the nonhomogeneous term in the PDE (2.26) is loglinearly approximated.

3.1 Loglinear Approximation Method I

We apply a time-dependent loglinear approximation to the nonhomogeneous term in the PDE (2.26) as follows:

$$\frac{1}{G(\tau, X_t)} \approx k(\tau)\big(1 - \log k(\tau)\big) - k(\tau) \log G(\tau, X_t), \tag{3.1}$$

where

$$k(\tau) = \frac{1}{G(\tau, 0)}. \tag{3.2}$$

We refer to this as the loglinear approximation method I (LLM I). Approximating the nonhomogeneous term in PDE (2.26) by Eq. (3.1) leads to the following approximate PDE:

$$\begin{aligned}\frac{G_\tau}{G} = \frac{1}{2} \operatorname{tr}\left[\frac{G_{XX}}{G}\right] + \frac{\theta}{2(\gamma-1)(\gamma+\theta)} \left|\frac{G_X}{G}\right|^2 \\ - \left(\mathcal{K} X_t + \frac{\gamma+\theta-1}{\gamma+\theta}\left(\bar{\lambda} + \bar{\Lambda} X_t\right)\right)' \frac{G_X}{G} - \alpha^{\frac{1}{\gamma}} k \log G \\ + \alpha^{\frac{1}{\gamma}} k(1 - \log k) - \frac{\gamma-1}{2\gamma(\gamma+\theta)} |\bar{\lambda} + \bar{\Lambda} X_t|^2 - \frac{\gamma-1}{\gamma}\left(\bar{\rho}_0 + \bar{\rho}' X_t + \frac{1}{2} X_t' \bar{\mathcal{R}} X_t\right) - \frac{\beta}{\gamma}.\end{aligned} \tag{3.3}$$

An analytical solution of the PDE (3.3) is expressed as

$$G(\tau, X_t) = (1-\alpha)^{\frac{1}{\gamma}} \exp\left(a_0(\tau) + a(\tau)' X_t + \frac{1}{2} X_t' A(\tau) X_t\right), \tag{3.4}$$

where $A(\tau)$ is a symmetric matrix. Thus, k is computed as

$$k(\tau) = (1-\alpha)^{-\frac{1}{\gamma}} \exp\left(-a_0(\tau)\right). \tag{3.5}$$

3.2 Loglinear Approximation Method II

In the loglinear approximation method II (LLM II), we generalize the method following Batbold *et al.* (2019) for infinite-time problems to a time-dependent approximation and redefine k in (3.2) as follows.

$$k(\tau) = \exp\left(-\mathrm{E}\left[\lim_{t\to\infty} \log G(\tau, X_t)\right]\right). \tag{3.6}$$

Let $X_\infty = \lim_{t\to\infty} X_t$. Then,

$$k(\tau) = \exp\left(-\frac{1}{\gamma}\log(1-\alpha) - a_0(\tau) - a(\tau)'[X_\infty] - \frac{1}{2}\mathrm{E}[X_\infty' A(\tau) X_\infty]\right), \tag{3.7}$$

is calculated by the following lemma.

Lemma 1. *Under Assumptions 1–3, k is represented by the following equation with (a_0, a, A):*

$$k(\tau) = \exp\left(-\frac{1}{\gamma}\log(1-\alpha) - a_0(\tau) - \frac{1}{2}\mathrm{tr}\left[(\mathcal{Q}^{-1})'\mathcal{M}(\tau)\mathcal{Q}^{-1}\right]\right), \tag{3.8}$$

where $\mathcal{Q}$ and $\mathcal{M}(\tau)$ are matrices such that

$$\mathcal{Q}^{-1}\mathcal{K}\mathcal{Q} = L = \mathrm{diag}(l_1, l_2, \cdots, l_N), \tag{3.9}$$

$$\mathcal{M}_{ij}(\tau) = \frac{1}{l_i + l_j}(\mathcal{Q}' A(\tau)\mathcal{Q})_{ij}. \tag{3.10}$$

where $\mathcal{M}_{ij}(\tau)$ and $(\mathcal{Q}'A(\tau)\mathcal{Q})_{ij}$ are the (i,j)-th element of $\mathcal{M}(\tau)$ and $\mathcal{Q}'A(\tau)\mathcal{Q}$, respectively.

Proof. See Appendix B.1. □

3.3 Loglinear Approximate Solution

Note that in Eqs. (3.5) and (3.8), (A, a, a_0) can be transformed to (A, a, k), since k is a monotonic function of a_0. We use the following notation.

$$H = \mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}. \tag{3.11}$$

Define functions h_2, h_1, and h_0 by

$$\begin{aligned}
h_2(A) &= \frac{\gamma(\gamma+\theta-1)}{(\gamma-1)(\gamma+\theta)}A^2 - H'A - AH - \frac{\gamma-1}{\gamma(\gamma+\theta)}\bar{\Lambda}'\bar{\Lambda} - \frac{\gamma-1}{\gamma}\bar{\mathcal{R}},\\
h_1(A,a) &= \left(\frac{\gamma(\gamma+\theta-1)}{(\gamma-1)(\gamma+\theta)}A - H'\right)a - \frac{\gamma+\theta-1}{\gamma+\theta}A\bar{\lambda} - \frac{\gamma-1}{\gamma(\gamma+\theta)}\bar{\Lambda}'\bar{\lambda} - \frac{\gamma-1}{\gamma}\bar{\rho},\\
h_0(A,a) &= \frac{1}{2}\mathrm{tr}[A] + \frac{\gamma(\gamma+\theta-1)}{2(\gamma-1)(\gamma+\theta)}|a|^2 - \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}'a - \frac{\gamma-1}{2\gamma(\gamma+\theta)}|\bar{\lambda}|^2 - \frac{\gamma-1}{\gamma}\bar{\rho}_0 - \frac{\beta}{\gamma}.
\end{aligned} \tag{3.12}$$

The solution of the approximate PDE (3.3) is referred to as the loglinear approximate optimal control and is denoted by $(\tilde{c}^*, \tilde{\sigma}^*)$. We have the following proposition.

Proposition 1. *Under Assumptions 1–3, the loglinear approximate optimal consumption and investment for problem (2.20) satisfy Eqs. (3.13) and (3.14), respectively.*

$$\tilde{c}_t^* = \alpha^{\frac{1}{\gamma}} \bar{W}_t \exp\left[-\left(a_0(\tau) + a(\tau)' X_t + \frac{1}{2} X_t' A(\tau) X_t \right) \right], \tag{3.13}$$

$$\tilde{\sigma}_t^* = \frac{1}{\gamma+\theta} \left(\bar{\lambda} + \bar{\Lambda} X_t \right) + \left(1 - \frac{1}{\gamma+\theta} \right) \frac{\gamma}{\gamma-1} \big(a(\tau) + A(\tau) X_t \big), \tag{3.14}$$

where (A, a, k) *is a solution of the system of ODEs:*

$$\frac{dA}{d\tau} = h_2(A) - \alpha^{\frac{1}{\gamma}} k A, \tag{3.15}$$

$$\frac{da}{d\tau} = h_1(A, a) - \alpha^{\frac{1}{\gamma}} k a, \tag{3.16}$$

$$\frac{dk}{d\tau} = -k \Big(h_0(A, a) + \alpha^{\frac{1}{\gamma}} k \Big), \tag{3.17}$$

with $(A(0), a(0), k(0)) = (0, 0, (1-\alpha)^{-\frac{1}{\gamma}})$.

Proof. See Appendix B.2. □

4 Linear Approximations

We introduce four types of linear approximation methods, including the proposed method by Kikuchi and Kusuda (2025), for the nonlinear term in the PDE (2.26), and present approximate solutions demonstrated by Kikuchi and Kusuda (2025).

The PDE (2.26) is rewritten as

$$G_\tau = \frac{1}{2} \operatorname{tr}[G_{XX}] + \frac{\theta}{2(\gamma-1)(\gamma+\theta)} \frac{G_X'}{G} G_X - \left(\mathcal{K} X_t + \frac{\gamma+\theta-1}{\gamma+\theta} \left(\bar{\lambda} + \bar{\Lambda} X_t \right) \right)' G_X \\ - \left\{ \frac{\gamma-1}{2\gamma(\gamma+\theta)} |\bar{\lambda} + \bar{\Lambda} X_t|^2 + \frac{\gamma-1}{\gamma} \left(\bar{\rho}_0 + \bar{\rho}' X_t + \frac{1}{2} X_t' \bar{\mathcal{R}} X_t \right) + \frac{\beta}{\gamma} \right\} G + \alpha^{\frac{1}{\gamma}}. \tag{4.1}$$

4.1 Four Types of Linear Approximation Methods

Linear approximation methods approximate $\frac{G_X}{G}$ in the nonlinear term of the PDE (4.1) by a time-dependent linear function of X_t.

$$\frac{G_X}{G} \approx d(\tau) + D(\tau) X_t. \tag{4.2}$$

Using Eq. (4.2), we obtain the following approximate nonhomogeneous linear PDE:

$$G_\tau = \mathcal{L}G + \frac{\theta}{2(\gamma-1)(\gamma+\theta)}\big(d(\tau)+D(\tau)X_t\big)'G_X + \alpha^{\frac{1}{\gamma}}, \quad G(0,X) = (1-\alpha)^{\frac{1}{\gamma}}, \tag{4.3}$$

where $\mathcal{L}$ is the linear differential operator defined by

$$\begin{aligned}\mathcal{L}G &= \frac{1}{2}\operatorname{tr}[G_{XX}] - \left(\mathcal{K}X_t + \frac{\gamma+\theta-1}{\gamma+\theta}\left(\bar{\lambda}+\bar{\Lambda}X_t\right)\right)'G_X \\ &- \left\{\frac{\gamma-1}{2\gamma(\gamma+\theta)}|\bar{\lambda}+\bar{\Lambda}X_t|^2 + \frac{\gamma-1}{\gamma}\left(\bar{\rho}_0 + \bar{\rho}'X_t + \frac{1}{2}X_t'\bar{\mathcal{R}}X_t\right) + \frac{\beta}{\gamma}\right\}G. \end{aligned}\tag{4.4}$$

Consider the following homogeneous linear PDE:

$$\frac{\partial}{\partial\tau}g(\tau,X) = \mathcal{L}g(\tau,X) + \frac{\theta}{2(\gamma-1)(\gamma+\theta)}\big(d(\tau)+D(\tau)X_t\big)'g_X, \quad g(0,X) = 1. \tag{4.5}$$

An analytical solution of the homogenous PDE (4.5) is expressed as

$$g(\tau,X) = \exp\left(b_0(\tau) + b(\tau)'X + \frac{1}{2}X'B(\tau)X\right), \tag{4.6}$$

where $B(\tau)$ is a symmetric matrix. Then, a semi-analytical solution of the approximate PDE (4.3) is expressed as

$$G(\tau,X_t) = \alpha^{\frac{1}{\gamma}}\int_0^\tau g(s,X_t)\,ds + (1-\alpha)^{\frac{1}{\gamma}}g(\tau,X_t). \tag{4.7}$$

Define $b^*(\tau,X_t)$ and $B^*(\tau,X_t)$ by

$$\begin{aligned} b^*(\tau,X_t) &= \frac{1}{G(\tau,X_t)}\left(\int_0^\tau \alpha^{\frac{1}{\gamma}}g(s,X_t)b(s)\,ds + (1-\alpha)^{\frac{1}{\gamma}}g(\tau,X_t)b(\tau)\right), \\ B^*(\tau,X_t) &= \frac{1}{G(\tau,X_t)}\left(\int_0^\tau \alpha^{\frac{1}{\gamma}}g(s,X_t)B(s)\,ds + (1-\alpha)^{\frac{1}{\gamma}}g(\tau,X_t)B(\tau)\right). \end{aligned}\tag{4.8}$$

The linear approximation methods I (LM I) and II (LM II) set $(d(\tau),D(\tau)) = (a(\tau),A(\tau))$ where $(a(\tau),A(\tau))$ are solutions used in LLM I and LLM II, respectively. The linear approximation method III (LM III), which is proposed by Kikuchi and Kusuda (2025), sets $(d(\tau),D(\tau)) = (b^*(\tau,0),B^*(\tau,0))$.

To introduce the linear approximation method IV (LM IV), we consider the conditional probability density function of X_t given that $X_{t-1} = 0$ and let $\mathbf{x}_1,\cdots,\mathbf{x}_M$ denote a bootstrapping sample from the conditional probability density function. Then, for each τ, $\dfrac{\tilde{G}_X(\tau,\mathbf{x}_m)}{\tilde{G}(\tau,\mathbf{x}_m)}$ is linearly approximated as the following seemingly unrelated regression (SUR) model, proposed by Zellner (1962).

$$\frac{\tilde{G}_X(\tau, \mathbf{x}_m)}{\tilde{G}(\tau, \mathbf{x}_m)} \approx \beta_0(\tau) + \beta(\tau)'\mathbf{x}_m + \epsilon_m^\tau, \tag{4.9}$$

where ϵ_m^τ is the error term such that $\mathrm{E}[\epsilon_m^\tau(\epsilon_m^\tau)'|\mathbf{x}_m] = \Omega_\tau$. The SUR model can be estimated equation-by-equation using the OLS. It is well known that the OLS estimators of the SUR model is consistent, though not efficient. Let $(\hat{\beta}_0(\tau), \hat{\beta}(\tau))$ denote the OLS estimators. The LM IV sets $(d(\tau), D(\tau)) = (\hat{\beta}_0(\tau), \hat{\beta}(\tau))$.

4.2 Linear Approximate Solutions

Define functions m_2, m_1, and m_0 by

$$\begin{aligned}
m_2(B) &= B^2 - H'B - BH - \frac{\gamma-1}{\gamma(\gamma+\theta)}\bar{\Lambda}'\bar{\Lambda} - \frac{\gamma-1}{\gamma}\bar{\mathcal{R}}, \\
m_1(B,b) &= \Big(B - H'\Big)b - \frac{\gamma+\theta-1}{\gamma+\theta}B\bar{\lambda} - \frac{\gamma-1}{\gamma(\gamma+\theta)}\bar{\Lambda}'\bar{\lambda} - \frac{\gamma-1}{\gamma}\bar{\rho}, \\
m_0(B,b) &= \frac{1}{2}\big(\mathrm{tr}[B] + |b|^2\big) - \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}'b - \frac{\gamma-1}{2\gamma(\gamma+\theta)}|\bar{\lambda}|^2 - \frac{\gamma-1}{\gamma}\bar{\rho}_0 - \frac{\beta}{\gamma},
\end{aligned} \tag{4.10}$$

where H is given by Eq. (3.11).

Kikuchi and Kusuda (2025) demonstrate that the linear approximate optimal consumption and investment for problem (2.20) satisfy Eqs. (4.11) and (4.12), respectively.

$$\tilde{c}_t^* = \frac{\alpha^{\frac{1}{\gamma}} W_t^*}{\alpha^{\frac{1}{\gamma}} \int_0^\tau g(s, X_t)\, ds + (1-\alpha)^{\frac{1}{\gamma}} g(T^* - t, X_{T^*-t})}, \tag{4.11}$$

where g is given by Eq. (4.6), and

$$\tilde{\sigma}_t^* = \frac{1}{\gamma+\theta}\big(\bar{\lambda} + \bar{\Lambda}X_t\big) + \left(1 - \frac{1}{\gamma+\theta}\right)\frac{\gamma}{\gamma-1}\big(b^*(\tau, X_t) + B^*(\tau, X_t)X_t\big), \tag{4.12}$$

where (b^*, B^*) is given by Eq. (4.8), and (B, b, b_0) is a solution of the system of ODEs:

$$\begin{aligned}
\frac{dB}{d\tau} &= m_2(B) + \frac{\theta}{(\gamma-1)(\gamma+\theta)}D(\tau)'B(\tau), \\
\frac{db}{d\tau} &= m_1(B,b) + \frac{\theta}{2(\gamma-1)(\gamma+\theta)}\big(D(\tau)'b(\tau) + B(\tau)'d(\tau)\big), \\
\frac{db_0}{d\tau} &= m_0(B,b) + \frac{\theta}{2(\gamma-1)(\gamma+\theta)}d(\tau)'b(\tau),
\end{aligned} \tag{4.13}$$

with $(B(0), b(0), b_0(0)) = (0, 0, 0)$.

5 Comparison of Approximation Accuracies

Here, we first confirm the optimal portfolio based on a numerical solution of the nonhomogeneous nonlinear PDE (4.1) as the true one and then compare the accuracies of approximate optimal portfolios based on the aforementioned six types of approximate solutions.

5.1 Approximate Optimal Portfolios

From Eqs. (2.22) and (3.14), the loglinear approximate optimal portfolio weights are given by

$$\tilde{\Phi}^*_t = \frac{1}{\gamma+\theta}\Sigma(X_t)'^{-1}\Big(\lambda + \Lambda X_t\Big) + \left(1 - \frac{1}{\gamma+\theta}\right)\Sigma(X_t)'^{-1}\Big(\sigma_p + \Sigma_p X_t\Big) + \left(1 - \frac{1}{\gamma+\theta}\right)\frac{\gamma}{\gamma-1}\Sigma(X_t)'^{-1}\Big(a(\tau) + A(\tau)X_t\Big). \quad (5.1)$$

Similarly, from Eqs. (2.22) and (4.12), the linear approximate optimal portfolio weights are given by

$$\tilde{\Phi}^*_t = \frac{1}{\gamma+\theta}\Sigma(X_t)'^{-1}\Big(\lambda + \Lambda X_t\Big) + \left(1 - \frac{1}{\gamma+\theta}\right)\Sigma(X_t)'^{-1}\Big(\sigma_p + \Sigma_p X_t\Big) + \left(1 - \frac{1}{\gamma+\theta}\right)\frac{\gamma}{\gamma-1}\Sigma(X_t)'^{-1}\Big(b^*(\tau, X_t) + B^*(\tau, X_t)X_t\Big). \quad (5.2)$$

Remark 1. *Note that in Eqs. (5.1) and (5.2), the first term (myopic demand) and the second term (inflation–deflation hedging demand) are identical and evaluated exactly. Conversely, the third term (intertemporal hedging demand) is different and evaluated approximately. Thus, to measure the accuracies of approximate optimal portfolios, measuring the accuracies of approximate optimal intertemporal hedging demands is sufficient.*

5.2 Basic Setup

Consider a long-term investor who has an initial asset $\bar{W}_0$ and plans to invest in the S&P500 and 10-year U.S. TIPS in addition to the money market account over a 35-year period. Then, Φ_t and $\Sigma(X_t)$ are given by

$$\Phi_t = \begin{pmatrix} \Phi_t(10) \\ \Phi_t^1 \end{pmatrix}, \quad \Sigma(X_t) = \begin{pmatrix} (\sigma(10) + \Sigma(10)X_t)' \\ (\sigma_1 + \Sigma_1 X_t)' \end{pmatrix}, \quad (5.3)$$

where $\sigma(10) = \bar{\sigma}(10) + \sigma_p$ and $\Sigma(10) = \bar{\Sigma}(10) + \Sigma_p$. We set $T^* = 35$ and $\alpha = 0.5, \beta = 0.04$. For the parameters in the QSM model, we use them estimated by Batbold *et al.* (2022) (for details, see Appendix C). Since the optimal portfolio depends on the state vector, we use probability-weighted mean absolute errors (PWMAEs) along with the standard mean absolute errors (MAEs) to measure the approximate accuracies. We employ the probability density function of stationary state vector X_∞ as the probability in the PWMAE. Batbold, Kikuchi, and Kusuda (2024) show $X_\infty \sim N(0, \mathbf{\Sigma}_X)$, where $\mathbf{\Sigma}_X$ is the solution of the following standard Lyapunov equation.

$$-\mathcal{K}\mathbf{\Sigma}_X - \mathbf{\Sigma}_X\mathcal{K}' + I_N = 0. \tag{5.4}$$

Let $\mathcal{C}\mathcal{C}' = \mathbf{\Sigma}_X$ be the Cholesky decomposition and define the standardized stationary state vector $Z = \mathcal{C}^{-1}X_\infty$. Then, $Z \sim N(0, I_N)$. Therefore, it is reasonable to assume that Z takes the following values:

$$Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{5.5}$$

where $z_i = -2.50, -2.25, \cdots, 2.50$ for each $i \in \{1, 2\}$.

5.3 Numerical Solution

A high-precision numerical solution of the PDE (4.1) is required to measure the accuracy of the six approximate solutions. Note that we cannot use the implicit methods in finite difference methods to compute the numerical solution of the PDE because it has no boundary conditions. Thus, we adopt an explicit method. A typical explicit method in the absence of boundary conditions requires one to interpolate the boundary values of the space domain. Therefore, we use a method wherein the boundary values of the space domain are interpolated by the values computed in the previous time step. To measure the approximate accuracy of this numerical solution, we consider the case of $\theta = 0$ in the PDE (4.1), that is,

$$\begin{aligned} G_\tau = \frac{1}{2}\operatorname{tr}[G_{XX}] - \left(\mathcal{K}X_t + \frac{\gamma-1}{\gamma}\left(\bar{\lambda} + \bar{\Lambda}X_t\right)\right)' G_X \\ - \left\{\frac{\gamma-1}{2\gamma^2}|\bar{\lambda} + \bar{\Lambda}X_t|^2 + \frac{\gamma-1}{\gamma}\left(\bar{\rho}_0 + \bar{\rho}'X_t + \frac{1}{2}X_t'\bar{\mathcal{R}}X_t\right) + \frac{\beta}{\gamma}\right\} G - \alpha^{\frac{1}{\gamma}}. \end{aligned} \tag{5.6}$$

Since we obtain a solution of the linear PDE (5.6) up to the system of ODEs, we can compute a high-precision numerical solution. We regard the numerical solution of the system of ODEs as the true solution of the PDE. Subsequently, we evaluate the accuracy of the optimal portfloio based on the numerical solution of the PDE with respect to that based on the numerical solution of the system of ODEs.

We set $\gamma = 4$. Let Δt and $|\Delta X|$ denote the time and space steps, respectively. Then, the explicit method becomes numerically stable and convergent whenever $\Delta t \leq \nu|\Delta X|^2$ where ν is constant and the numerical errors are proportional to Δt and $|\Delta X|^2$. We set $\Delta t = 0.004$ and $|\Delta X| = 0.1 \times \sqrt{2}$.

The MAE and PWMAE between numerical and true optimal portfolio weights to S&P500 and TIPS, when the value of the state vector is in the aforementioned range, are shown in Table 1.

Table 1: The MAE (%) and PWMAE (%) between numerical and true optimal portfolio weights.

Asset	MAE	PWMAE
S&P500	0.312	0.301
TIPS	0.512	0.452
S&P500 & TIPS	0.412	0.376

Both the MAE and PWMAE are negligibly small. Therefore, we regard the numerical solutions of PDE (2.26) as the true solutions. Since the MAE and PWMAE are almost equal, only the PWMAE is used hereafter.

5.4 Comparison of Approximate Optimal Portfolios

We set $\gamma = 4$. Consider high and low cases for relative ambiguity aversion. The high case is set as $\theta = 6$ and the low case as $\theta = 2$. The PWMAE of approximate optimal portfolio weights to S&P500 and TIPS are shown in Tables 2 and 3.

Table 2: The PWMAE (%) between approximate and true optimal portfolio weights in the case of $\theta = 6$.

Asset	LLM I	LLM II	LM I	LM II	LM III	LM IV
S&P500	21.27	16.08	0.42	0.45	0.37	0.37
TIPS	19.93	98.09	0.37	0.66	0.39	0.41
S&P500 & TIPS	20.60	57.08	0.39	0.56	0.38	0.39

Table 3: The PWMAE (%) between approximate and true optimal portfolio weights in the case of $\theta = 2$.

Asset	LLM I	LLM II	LM I	LM II	LM III	LM IV
S&P500	18.44	11.40	0.35	0.36	0.33	0.33
TIPS	17.65	86.58	0.46	0.45	0.37	0.38
S&P500 & TIPS	18.04	48.99	0.35	0.40	0.35	0.35

The results show that the PWMAEs of both loglinear approximate optimal portfolio weights are very large, whereas those of all linear approximate optimal portfolio weights are negligibly small. Therefore, we focus on the results of the linear approximation methods and analyze them in detail. First, the approximation accuracy with high ambiguity aversion is slightly lower, but it remains high. Second, the accuracy of TIPS is slightly lower than that of the S&P500, but it remains high. Although all the linear approximation methods are highly accurate, LM I, III and IV appear to be slightly more accurate and stable than the rest.

6 Conclusion

We considered a finite-time consumption–investment problem for homothetic robust utility under a QSM model. Since the PDE for indirect utility is nonlinear and nonhomogeneous, we introduced two types of loglinear approximation methods and four types of linear approximation methods, including the proposed method by Kikuchi and Kusuda (2025). We derived loglinear approximate solutions and presented linear approximate solutions derived by Kikuchi and Kusuda (2025). We confirmed that the optimal portfolio with respect to the numerical solution of the PDE can be regarded as the true optimal portfolio. Subsequently, we compared the accuracies of approximate optimal portfolios based on these six types of solutions. The numerical analysis showed that the PWMAEs of both loglinear approximate optimal portfolios are very large, whereas those of all linear approximate optimal portfolios are negligibly small. Among the high-precision linear approximation methods, LM I, III, and IV appear to demonstrate slightly higher accuracy and stability than the remaining method. Among them, LM III is recommended for adoption due to its simplicity in implementation.

For infinite-time consumption-investment problems, loglinear approximation methods have been exclusively used. However, the findings strongly indicate that linear approximation methods should be used instead of loglinear approximation methods for finite-time problems.

Finally, the poor performance of loglinear approximation methods for finite-time consumption-investment problems also raises questions about the reliability of loglinear approximation methods for infinite-time problems. Consequently, it is imperative for future studies to ascertain the accuracy of these methods for infinite-time problems.

A Coefficients of Volatility Functions

1. $(\Sigma(\tilde{\tau}), \sigma(\tilde{\tau}))$ is a solution to the following system of ODEs.

$$\frac{d\bar{\Sigma}(\tilde{\tau})}{d\tilde{\tau}} = (\mathcal{K} + \bar{\Lambda})'\bar{\Sigma}(\tilde{\tau}) + \bar{\Sigma}(\tilde{\tau})(\mathcal{K} + \bar{\Lambda}) - \bar{\Sigma}(\tilde{\tau})^2 + \bar{\mathcal{R}}, \tag{A.1}$$

$$\frac{d\bar{\sigma}(\tilde{\tau})}{d\tilde{\tau}} = -(\mathcal{K} + \bar{\Lambda} - \bar{\Sigma}(\tilde{\tau}))'\bar{\sigma}(\tilde{\tau}) - (\bar{\Sigma}(\tilde{\tau})\bar{\lambda} + \bar{\rho}), \tag{A.2}$$

with $(\bar{\Sigma}, \bar{\sigma})(0) = (0, 0)$.

2. Σ_k is a solution to Eq. (A.3) and σ_k is given by Eq. (A.4).

$$0 = (\mathcal{K} + \Lambda)'\Sigma_k + \Sigma_k(\mathcal{K} + \Lambda) - \Sigma_k^2 + \mathcal{R} - \Delta_k, \tag{A.3}$$

$$\sigma_k = (\mathcal{K} + \Lambda - \Sigma_k)'^{-1}(\delta_k - \rho - \Sigma_k\lambda), \tag{A.4}$$

B Proofs

B.1 Proof of Lemma 1

This proof is a time-dependent version of that shown by Batbold *et al.* (2019). X_t is expressed as the solution to linear SDE (2.1) as follows:

$$X_t = \mathcal{Q}e^{-tL}\mathcal{Q}^{-1}X_0 + \mathcal{Q}\int_0^t e^{-(t-s)L}\mathcal{Q}^{-1}\, dB_s.$$

Thus, as $\lim_{t\to\infty} e^{-tL} = 0$, $\mathrm{E}[\lim_{t\to\infty} X_t] = 0$ holds. Next, the following equation holds:

$$X_t'A(\tau)X_t = \left\{\mathcal{Q}e^{-tL}\mathcal{Q}^{-1}X_0 + \mathcal{Q}\int_0^t e^{-(t-s)L}\mathcal{Q}^{-1}\Sigma\, dB_s\right\}' A(\tau)\left\{\mathcal{Q}e^{-tL}\mathcal{Q}^{-1}X_0 + \mathcal{Q}\int_0^t e^{-(t-s)L}\mathcal{Q}^{-1}\, dB_s\right\}.$$

Because $\mathrm{E}[dB_s dB_t'] = \delta_{st} I_N ds$[7], the following equation holds:

$$\mathrm{E}[\lim_{t\to\infty} X_t'A(\tau)X_t] = \lim_{t\to\infty}\int_0^t \mathrm{tr}\left[(\mathcal{Q}^{-1})'e^{-(t-s)L}\mathcal{Q}'A(\tau)\mathcal{Q}e^{-(t-s)L}\mathcal{Q}^{-1}\right] ds$$
$$= \mathrm{tr}\left[(\mathcal{Q}^{-1})'\lim_{t\to\infty}\int_0^t e^{-(t-s)L}\mathcal{Q}'A(\tau)\mathcal{Q}e^{-(t-s)L}ds\,\mathcal{Q}^{-1}\right] = \mathrm{tr}\left[(\mathcal{Q}^{-1})'\mathcal{M}(\tau)\mathcal{Q}^{-1}\right]. \tag{B.1}$$

Therefore, Eq.(3.8) holds.

[7] δ_{st} is the Kronecker's delta.

B.2 Proof of Proposition 1

The proof establised by Kikuchi and Kusuda (2025) is as follows. Substituting G and its drivatives into Eqs. (2.24), (2.25), and the PDE (3.3), we obtain Eqs. (3.13), (3.14), and

$$\frac{da_0}{d\tau}+X'\frac{da}{d\tau}+\frac{1}{2}X'\frac{dA}{d\tau}X = h(X_t)+\alpha^{\frac{1}{\gamma}}k(1-\log k)-\alpha^{\frac{1}{\gamma}}k\left(\frac{1}{\gamma}\log(1-\alpha)+a_0+a'X_t+\frac{1}{2}X_t'AX_t\right), \tag{B.2}$$

where $h(X_t)$ is given by

$$\begin{aligned} h(X_t) &= \frac{1}{2}\mathrm{tr}[A]+\frac{1}{2}\left(1+\frac{\theta}{(\gamma-1)(\gamma+\theta)}\right)(|a|^2+2X_t'Aa+X_t'A^2X_t) \\ &-\left\{\frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}+\left(\mathcal{K}+\frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)X_t\right\}'a-\frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}'AX_t \\ &-\frac{1}{2}X_t'\left(\mathcal{K}+\frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)'AX_t-\frac{1}{2}X_t'A\left(\mathcal{K}+\frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)X_t \\ -&\frac{\gamma-1}{2\gamma(\gamma+\theta)}(|\bar{\lambda}|^2+2\bar{\lambda}'\bar{\Lambda}X_t+X_t'\bar{\Lambda}'\bar{\Lambda}X_t)-\frac{\gamma-1}{\gamma}\left(\bar{\rho}_0+\bar{\rho}'X_t+\frac{1}{2}X_t'\bar{\mathcal{R}}X_t\right)-\frac{\beta}{\gamma}. \end{aligned} \tag{B.3}$$

As Eq. (B.2) is identical on X, we have the system of ODEs (3.15), (3.16), and (B.4).

$$\frac{da_0}{d\tau} = h_0(A,a)+\alpha^{\frac{1}{\gamma}}k\left(1-\log g-\frac{1}{\gamma}\log(1-\alpha)-a_0\right) = h_0(A,a)+\alpha^{\frac{1}{\gamma}}k, \quad a_0(0)=0. \tag{B.4}$$

Differentiating both sides of Eq. (3.5), we obtain

$$\frac{dg}{d\tau} = -k\frac{da_0}{d\tau}. \tag{B.5}$$

Substituting Eqs. (3.15) and (B.4) into the above equation, we obtain Eq. (3.17).

C Estimated Parameters in the QSM Model

Batbold *et al.* (2022) estimate the QSM model using the quasi-maximum likelihood method with unscented Kalman filter on 262 month-end data from January 1999 to October 2020, observed in the US security markets. The time-series data used for estimation are 6-month, 5-year, and 10-year treasury spot rates, To reduce the estimation burden, they assume $\mathcal{I} = 0$.

The estimation results are as follows.

$$dX_t = -\mathcal{K}X_t dt + I_2 dB_t = -\begin{pmatrix} 0.002356 & 0 \\ -0.01797 & 0.005239 \end{pmatrix}\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} dt + I_2 dB_t,$$

$$\lambda_t = \lambda + \Lambda X_t = \begin{pmatrix} 0.2515 \\ 0.3235 \end{pmatrix} + \begin{pmatrix} 0.01744 & 0 \\ 0.001100 & 1.670 \times 10^{-6} \end{pmatrix}\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix},$$

$$i_t = \iota_0 + \iota' X_t = 0.01844 + \begin{pmatrix} 6.179 \times 10^{-4} \\ 5.030 \times 10^{-5} \end{pmatrix}'\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix},$$

$$\begin{aligned} r_t = \rho_0 + \rho' X_t + \frac{1}{2}X_t' \mathcal{R} X_t &= 0.03146 + \begin{pmatrix} -0.01227 \\ 0.007496 \end{pmatrix}'\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} \\ &+ \frac{1}{2}\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix}'\begin{pmatrix} 0.002460 & -0.001156 \\ -0.001156 & 0.002308 \end{pmatrix}\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix}, \end{aligned}$$

$$\sigma_t^p = \sigma_p + \Sigma_p X_t = \begin{pmatrix} 0.04742 \\ -0.03516 \end{pmatrix} + \begin{pmatrix} 1.953 \times 10^{-4} & 0 \\ 0 & 6.656 \times 10^{-4} \end{pmatrix}\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix},$$

$$\bar{\lambda} = \lambda - \sigma_p = \begin{pmatrix} 0.2041 \\ 0.3587 \end{pmatrix}, \quad \bar{\Lambda} = \Lambda - \Sigma_p = \begin{pmatrix} 0.01725 & 0 \\ 0.001097 & -6.639 \times 10^{-4} \end{pmatrix},$$

$$\bar{\rho}_0 = \rho_0 - \iota_0 + \sigma_p' \lambda = 0.01357, \quad \bar{\rho} = \rho - \iota + \Lambda' \sigma_p + \Sigma_p' \lambda = \begin{pmatrix} -0.01205 \\ 0.007661 \end{pmatrix},$$

$$\begin{aligned} \frac{D_t}{S_t} = \delta_0 + \delta' X_t + \frac{1}{2}X_t' \Delta X_t &= 0.01441 + \begin{pmatrix} -0.003598 \\ 0.008137 \end{pmatrix}'\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} \\ &+ \frac{1}{2}\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix}'\begin{pmatrix} 0.002427 & -0.001154 \\ -0.001154 & 0.002308 \end{pmatrix}\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix}. \end{aligned}$$

Acknowledgments

We thank Japan Society for the Promotion of Science Grant-in-Aid for Scientific Research(C) and by Ishii Memorial Securities Research Promotion Foundation for their generous financial assistance. We would also like to thank Editage [https://editage.com/] for editing and reviewing the manuscript for English language.

Statements and Declarations

Author Contribution

Kusuda proposed several approximation methods, presented Proposition 1, and wrote the paper. Kikuchi proposed several approximation methods and performed numerical analysis.

Competing Interests

The authors have no competing interests to declare that are relevant to the content of this article.

Funding

A part of this study was supported by the Japan Society for the Promotion of Science Grant-in-Aid for Scientific Research(C) Grant Number JP20K01768 and by the Ishii Memorial Securities Research Promotion Foundation.

Data Availability

While the datasets used for estimation during the current study are publicly available, the datasets generated and analyzed during the current study are not publicly available but are available from the corresponding author on reasonable request.

References

Ahn, D.H., Dittmar, R.F., Gallant, A.R. (2002). Quadratic term structure models: theory and evidence. Rev. Financ. Stud., 15(1), 243–288.

Batbold, B., Kikuchi, K., Kusuda, K. (2019). Approximate analytical solution to consumption and long-term security investment optimization problem with homothetic robust utility (in Japanese), Trans. Oper. Res. Soc. Jpn., 62, 71–89.

Batbold, B., Kikuchi, K., Kusuda, K. (2022). Semi-analytical solution for consumption and investment problem under quadratic security market model with inflation risk. Math. Financ. Econ., 16(3), 509–537.

Batbold, B., Kikuchi, K., Kusuda, K. (2024). Worst-case premiums and identification of homothetic robust Epstein-Zin utility under a quadratic model. preprint. https://ssrn.com/abstract=5046600

Batbold, B., Kikuchi, K., Kusuda, K. (2025). Strategic international asset allocation under a quadratic model with exchange rate and inflation-deflation risks. Decis. Econ. Finance. https://link.springer.com/article/10.1007/s10203-025-00527-8

Boyarchenko, N., Levendorskii, S. (2007). The eigenfunction expansion method in multi-factor quadratic term structure models. Math. Financ. 17(4), 503–539.

Branger, N., Larsen, L.S., Munk, C. (2013). Robust portfolio choice with ambiguity and learning about return predictability. J. Bank. Financ., 37(5), 1397–1411.

Campbell, J.Y. (1993). Intertemporal asset pricing without consumption data. Amer. Econ. Rev., 83(3), 487–512.

Campbell, J.Y., Viceira, L.M. (2001). Stock market mean reversion and the optimal equity allocation of a long-lived investor. Rev. Financ., 5(3), 269-292.

Campbell, J.Y., Viceira, L.M. (2002). Strategic Asset Allocation. Oxford University Press, New York.

Chen, L., Filipović, D., Poor, H.V. (2004). Quadratic term structure models for risk-free and defaultable rates. Math. Financ., 14(4), 515-536.

Filipović, D., Gourier, E., Mancini, L. (2016). Quadratic variance swap models. J. Financ. Econ., 119(1), 44-68.

Hansen, L.P., Sargent, T.J. (2001). Robust control and model uncertainty. Amer. Econ. Rev., 91(2), 60–66.

Kikuchi, K. (2024). A term structure interest rate model with the Brownian bridge lower bound. Annal. Finance 20(3), 301–328.

Kikuchi, K. Kusuda, K. (2024). Age-dependent robust strategic asset allocation with inflation–deflation hedging demand. Math. Financ. Econ. 18(4), 641–670.

Kikuchi, K., & Kusuda, K. (2025). A Linear approximate robust strategic asset allocation with inflation-deflation hedging demand. preprint, DOI: 10.21203/rs.3.rs-3012011/v3

Kim, D.H., Singleton, K.J. (2012). Term structure models and the zero bound: An empirical investigation of Japanese yields. J. Econome., 170(1), 32–49.

Leippold, M. Wu, L. (2002). Asset pricing under the quadratic class. J. Financ. Quant. Anal., 37(2), 271–295.

Leippold, M., Wu, L. (2007). Design and estimation of multi-currency quadratic models. Rev. Finance 11(2), 167–207.

Liu, H. (2010). Robust consumption and portfolio choice for time varying investment opportunities. Annal. Financ., 6(4), 435–454.

Maenhout, P.J. (2004). Robust portfolio rules and asset pricing. Rev. Financ. Stud., 17(4), 951–983.

Maenhout, P.J. (2006). Robust portfolio rules and detection–error probabilities for a mean-reverting risk premium. J. Econ. Theor., 128(1), 136–163.

Munk, C., Rubtsov, A.V. (2014). Portfolio management with stochastic interest rates and inflation ambiguity. Annal. Financ., 10(3), 419–455.

Skiadas, C. (2003). Robust control and recursive utility. Financ. Stoch., 7(4), 475–489.

Yi, B., Viens, F., Law, B., Li, Z. (2015). Dynamic portfolio selection with mispricing and model ambiguity. Annal. Financ., 11(1), 37–75.

Zellner, A. (1962). An efficient method of estimating seemingly unrelated regression equations and tests for aggregation bias. J Amer. Stat. Assoc., 57(298), 348—368.