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Existence and Uniqueness of General Equilibria in Approximately Complete Security Markets

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Abstract

Jump-diffusion security market models have been extensively studied in finance and economics. Kusuda [32] assumes "approximately complete jump-diffusion security markets," and demonstrates that an "approximate security market equilibrium" in an approximately complete security market economy can be identified with an Arrow–Debreu equilibrium in a corresponding Arrow–Debreu economy. This study posits time-additive utilities and proves the equivalence of Arrow–Debreu and representative agent equilibria. It then demonstrates sufficient conditions for the existence, uniqueness, and local uniqueness of representative agent equilibria.

Keywords Approximately complete markets, Approximate security market equilibrium, Arrow–Debreu equilibrium, Infinite dimensional martingale generator, Jump-diffusion

1 Introduction

Strong evidence[1] suggests that the dynamics of most financial processes, such as equity prices, interest rates, and exchange rates, are better described by jump-diffusion processes than by pure diffusion processes, which are assumed in standard models. Studies[2] also demonstrates that the presence of jumps could significantly influence asset pricing[3] and portfolio choice[4].

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[1] See Akgiray and Booth [2], Andersen, Benzoni, and Lund [3], Bakshi, Cao, and Chen [5], Bates [7, 8, 9], Das [17], Eraker, Johannes, and Polson [23], Jorion [28], and Pan [41].

[2] See Bakshi, Cao, and Chen [5], Bates [9], Duffie, Pan, and Singleton [20], Pan [41], and Rietz [43].

[3] Rietz [43] claims that jump risk premia could be high enough to explain the high equity premia pointed out by Mehra and Prescott [37]. Pan [41] show that jump risk premia is high enough to explain the volatility "smirks" implied by the market quoted prices of options.

[4] For example, see Daglish [13] and Liu, Longstaff, and Pan [33].

Against this extant literature, jump-diffusion security market models have been intensively studied in finance and financial economics, particularly in the context of the capital asset pricing model (CAPM)[5], option pricing[6], and portfolio choice[7]. In most jump-diffusion security market models, the jump magnitude is specified as a continuously distributed random variable at each jump time. In this case, the dimensionality of a martingale generator[8] in the markets, which can be interpreted as "the number of sources of uncertainty," is uncountably infinite, and no finite set of traded securities can complete the markets.

No equilibrium analysis had been conducted in security market economy with an infinite dimensional martingale generator until recently.[9] Kusuda [32] demonstrates that a generalized security market equilibrium, called the "approximate security market" (ASM) equilibrium in an "approximately complete jump-diffusion security market" (Björk *et al.* [11]) economy with an infinite dimensional martingale generator can be identified with an Arrow–Debreu equilibrium in a corresponding Arrow–Debreu economy.

The purpose of this study, in conjunction with Kusuda [32], is to demonstrate sufficient conditions for the existence, uniqueness, and local uniqueness of the general equilibria (GE) in an approximately complete market security market economy. Kusuda [32] shows that an ASM equilibrium in approximately complete markets is identified with an Arrow–Debreu equilibrium in a corresponding Arrow–Debreu economy (Theorem 1). I demonstrate sufficient conditions for the existence, uniqueness, and local uniqueness of the Arrow–Debreu equilibria.

I assume a continuous-time approximately complete security market economy wherein each agent has a time-additive utility. I generalize the Negishi approach (Negishi [40]) adopted by Dana [14, 15] for a static economy to the continuous-time economy with jump-diffusion information. I first

[5]See Ahn and Thompson [1], Back [4], and Madan [?].

[6]See Bakshi, Cao, and Chen [5], Bates [7, 8, 9], Björk *et al.* [10, 11], Duffie, Pan, and Singleton [20], Fujiwara and Miyahara [24], Merton [38], and Naik and Lee [39].

[7]See Daglish [13] and Liu, Longstaff, and Pan [33].

[8]Consider the case wherein the information filtration in security markets is generated by a d-dimensional Wiener process and a d'-dimensional Poisson process. Then, the martingale generator consists of the Wiener process and the compensated Poisson process, and its dimensionality is $d + d'$. In this study, the finite dimensional Poisson process is replaced with "infinite dimensional Poisson process."

[9]Note that, in a security market economy with a finite dimensional martingale generator, many equilibrium analyses have been conducted. In a security market economy wherein information filtration is generated by a finite dimensional Wiener process, Duffie and Zame [21] as well as Huang [26] show sufficient conditions for the existence of equilibria, whereas Karatzas, Lakner, Lehoczky, and Shreve [29] and Karatzas, Lehoczky, and Shreve [30] present sufficient conditions for the existence and uniqueness of equilibria. Dana and Pontier [16] and Duffie [18] present sufficient conditions for the existence of equilibria in a security market economy wherein information filtration is more general than the one generated by finite dimensional Wiener process. However, the martingale generator in their markets is still assumed to be finite dimensional.

present that an Arrow–Debreu equilibrium is identified with a representative agent (RA) equilibrium (Proposition 1). I then demonstrate the following three results: (i) Assume that the aggregate endowment is bounded away from zero. Then an "excess utility" function possesses the properties of a finite dimensional excess demand function (Lemma 3). This result indicates the existence of RA equilibria. (ii) Assume that each agent's relative risk aversion is less than or equal to one. Then the excess utility has a "gross substitute" property (Lemma 4). This result implies the uniqueness of RA equilibria. (iii) Assume that each agent's risk tolerance satisfies an integrability condition and that each agent's endowment process is bounded away from zero (Lemma 5). Then, the local uniqueness of RA equilibria follows (Lemma 6). I thus present sufficient conditions for the existence, uniqueness, and local uniqueness of RA equilibria (Theorem 2).

The remaining paper is organized as follows. Section 2 introduces the approximately complete security market economy and the results on the equivalence of ASM and Arrow–Debreu equilibria. Section 3 shows the equivalence of Arrow–Debreu and RA equilibria. Section 4 demonstrates the sufficient conditions for the existence, uniqueness, and local uniqueness of RA equilibria. Seciton 5 concludes. Appendix introduces the basics of probability theory, approximately complete markets, and the proofs of lemma, propositions, and theorems.

2 Approximately Complete Security Market Economy and Equivalence of ASM and Arrow–Debreu Equilibria

In this section, I introduce approximately complete security market economy and the results on the equivalence of ASM and Arrow–Debreu equilibria (Kusuda [32]).

2.1 Economy

I consider a continuous-time frictionless pure exchange security market economy with time span $\mathbf{T} := [0, T^\dagger]$ for a fixed horizon time $T^\dagger > 0$. The agents' common subjective probability and information structure is modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbf{T}}$ is the natural filtration generated by a d-dimensional Wiener process W and a jump process called *marked point process* $\nu(dt \times dz)$ on a Lusin space $(\mathbb{Z}, \mathcal{Z})$ with the $\mathbb{P}$-intensity kernel $\lambda_t(dz)$ (for marked point process, see Appendix A.1). Note that the aartingale representation theorem (see Chapter III Corollary 4.31 in Jacod and Shiryaev [27]) shows that the martingale generator in this economy is $(W, (\nu(dt \times \{z\}) - \lambda_t(\{z\}))_{z\in\mathbb{Z}})$. If the mark set $\mathbb{Z}$ is infinite, then the martingale generator is infinite dimensional. I use the following notation:

$$\mathbb{R}^N_+ := \{x \in \mathbb{R}^N \,|\, x_n \geq 0 \quad \forall n \in \{1, \cdots, N\}\},$$
$$\mathbb{R}^N_{++} := \{x \in \mathbb{R}^N \,|\, x_n > 0 \quad \forall n \in \{1, \cdots, N\}\}.$$

Consider a single perishable consumption good. The consumption space is a Banach space $\mathbf{L}^\infty := \mathbf{L}^\infty(\Omega \times \mathbf{T}, \mathcal{P}, \mu)$, where $\mathcal{P}$ is the predictable σ-algebra on $\Omega \times \mathbf{T}$, and μ is the product measure of $\mathbb{P}$ and Lebesgue measure on $\mathbf{T}$. There are I agents, and each of them is represented by $(u^i, e^i)_{i \in \mathbf{I} := \{1,2,\cdots,I\}}$, where $e^i \in \mathbf{L}^\infty_+$ is an endowment process and u^i is a time-additive utility with the properties as given below.

Assumption 1. *For every agent $i \in \mathbf{I}$, U^i is a time-additive utility functional of the form:*

$$U^i(c) = E\left[\int_0^{T^\dagger} u^i(t, c^i_t)\, dt\right],$$

where the von Neumann-Morgenstern (vNM) utility function $u^i : \mathbf{T} \times \mathbb{R}_+ \to \mathbb{R}$ satisfies the following properties:

1. *$u^i(t, \cdot)$ is strictly increasing and strictly concave on $\mathbb{R}_+$ for every $t \in \mathbf{T}$.*
2. *u^i is $\mathbf{C}^{1,2}(\mathbb{R}^2_+)$.*

Remark 1. In a static economy, Dana [14] assumes that the consumption space is $\mathbf{L}^p_+$ and every agent's endowment is in $\mathbf{L}^p_+$ where $p \in [1, \infty]$, while I specify $p = \infty$. Also, Dana [14] assumes that u^i depends on $\omega \in \Omega$, while I assume that u^i is independent of $\omega \in \Omega$.

The economy is described by a collection

$$\mathbf{E} := ((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), (U^i, e^i)_{i \in \mathbf{I}}).$$

There are markets for the consumption good and securities at every date $t \in \mathbf{T}$. The traded securities are nominal-risk-free security called the *money market account* and a continuum of zero-coupon bonds whose maturity dates are $(0, T^\dagger]$, each of which has \$1 payoff at its maturity date.

Remark 2. In the standard security market model for GE analysis, it is assumed that zero-coupon bonds with a unit payoff of the consumption good at each maturity date, *i.e.*, inflation-linked bonds, are traded. Given that the volume of traded inflation-linked bonds in reality is significantly smaller than that of bonds with a unit payoff of currency, the assumption of trading in inflation-linked bonds contradicts the price-taker assumption, which is the fundamental assumption of the general equilibrium analysis model. Thus, following Kusuda [32], I assume that bonds with \$1 payoff and nominal-risk-free security are traded in lieu of bonds with a unit payoff of the consumption good and risk-free security.

Let p, B, and $(B^T)_{T\in(0,T^\dagger]}$ denote the processes of consumption good price, nominal money market account price, and nominal bond pricees, respectively. Let $\mathbf{B} := (B, (B^T)_{T\in(0,T^\dagger]})$, termed *bond price family.*

2.2 Approximately Complete Security Markets

I briefly review the approximately complete jump-diffusion security market model (Björk *et al.* [10, 11]). Each agent is allowed to hold a portfolio of the money market account and a continuum of bonds. Thus, the portfolio component of the continuum of bonds is set to be a signed finite Borel measure on $[t, T^\dagger]$ for every event $\omega \in \Omega$ and time $t \in \mathbf{T}$.

A *portfolio* is a stochastic process $\vartheta = (\vartheta^0, \vartheta^1(\cdot))$ that satisfies the following:

1. The component ϑ^0 is a real-valued $\mathcal{P}$-measurable process.
2. The component ϑ^1 is such that
   - (i) For every $(\omega, t) \in \Omega \times \mathbf{T}$, the set function $\vartheta^1_t(\omega, \cdot)$ is a signed finite Borel measure on $[t, T^\dagger]$.
   - (ii) For every Borel set A, the process $\vartheta^1(A)$ is $\mathcal{P}$-measurable.

Let $\mathcal{B}$ denote the class of *regular* (for definition, see Appendix B.1) bond price families. The value process of a feasible portfolio is given by

$$\mathcal{V}^{\mathbf{B}}_t(\vartheta) := B_t\, \vartheta^0_t + \int_t^{T^\dagger} B^T_t\, \vartheta^1_t(dT) \qquad \forall t \in \mathbf{T}. \tag{2.1}$$

A *feasible portfolio* (for definition, see Appendix B.2) ϑ is said to be *self-financing at* $\mathbf{B}$ if and only if the following equation holds:

$$\mathcal{V}^{\mathbf{B}}_t(\vartheta) = V^{\mathbf{B}}_0(\vartheta) + \int_0^t \vartheta^0_s\, dB_s + \int_0^t \int_s^{T^\dagger} \vartheta^1_s(dT)\, dB^T_s \qquad \forall t \in \mathbf{T}. \tag{2.2}$$

For a real-valued $\mathcal{P}$-measurable process X, the discounted process is defined by $\tilde{X} := \dfrac{X}{B}$. The collection $(\tilde{B}, (\tilde{B}^T)_{T\in\mathbf{T}})$ of security prices is abbreviated by $\tilde{\mathbf{B}}$.

To eliminate unrealistic portfolios such as a *doubling strategy* (see Chapter 6 in Duffie [19]), the class of feasible portfolios is restricted to that of *admissible portfolios* (introduced by Dybvig and Huang [22]) which are credit-constrained. A self-financing portfolio ϑ at $\mathbf{B}$ is said to be *admissible at* $\mathbf{B}$ if and only if the discounted value process $\tilde{V}^{\mathbf{B}}(\vartheta)$ is bounded below P-a.e. Let $\underline{\Theta}(\tilde{\mathbf{B}})$ denote the class of admissible portfolios at $\mathbf{B}$. The notion of *approximately complete markets* is introduced by Björk *et al.* [10, 11].

Definition 1. Let $\mathbf{B} \in \mathcal{B}$. Markets are *approximately complete at* $\mathbf{B}$ if and only if, for any $T \in (0, T^\dagger]$ and any T-*contingent claim* X_T, there exists a sequence of *replicable claims* $(X_{Tn})_{n\in\mathbb{N}}$ converging to X_T in $\mathbf{L}^\infty(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}^{\mathbf{B}})$, where $\tilde{\mathbb{P}}^{\mathbf{B}}$ is a risk-neutral measure (for definitions of *contingent claim* and *replicable claim*, see Appendix B.3).

2.3 ASM Equilibrium

Kusuda [32] introduces a class of *implementable bond price families.*

Definition 2. Let $\mathbf{B} \in \mathcal{B}$. A family $\mathbf{B}$ of bond prices is *implementable* if and only if there exists a unique risk-neutral measure $\tilde{P}^{\mathbf{B}}$ such that $\tilde{\Lambda}^{\mathbf{B}}$ for $\tilde{P}^{\mathbf{B}}$ is bounded above and bounded away from zero μ-a.e., where $\Lambda^{\mathbf{B}}$ is the *density process for* $\tilde{P}^{\mathbf{B}}$ (for definition, see Appendix A.3.).

Let $\bar{\mathcal{B}}$ denote the class of implementable bond price families. The notion of ASM is proposed by Kusuda [32].

Definition 3. A collection $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B}) \in \prod_{i\in\mathbf{I}} \mathbf{L}^\infty_+ \times \mathbf{L}^\infty_+ \times \bar{\mathcal{B}}$ constitutes an ASM equilibrium for $\mathbf{E}$ if and only if the following conditions hold:

1. For every $i \in \mathbf{I}$, $\hat{c}^i$ solves the problem

$$\max_{c^i \in \bar{\mathcal{C}}^i(p,\mathbf{B})} U^i(c^i)$$

   where

$$\bar{\mathcal{C}}^i(p,\mathbf{B}) = \Big\{ c^i \in \mathbf{L}^\infty_+ \,:\, \exists (\vartheta^i_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \underline{\Theta}(\tilde{\mathbf{B}}) \quad \text{s.t.}$$
$$\mathcal{V}^{\mathbf{B}}_t(\vartheta^i_n) = \int_0^t \vartheta^{i0}_{ns}\, dB_s + \int_0^t \int_s^{T^\dagger} \vartheta^{i1}_{ns}(dT)\, dB^T_s + \int_0^t p_s(e^i_s - c^i_s)\, ds \qquad \forall (n,t) \in \mathbb{N} \times \mathbf{T},$$
$$\lim_{n\to\infty} \mathcal{V}^{\mathbf{B}}_{T^\dagger}(\vartheta^i_n) = 0 \Big\}.$$

2. The good market is cleared: $\sum_{i\in\mathbf{I}} \hat{c}^i_t = \sum_{i\in\mathbf{I}} e^i_t \quad \forall t \in \mathbf{T}$.

3. The security markets are cleared: $\sum_{i\in\mathbf{I}} \hat{\vartheta}^i_n = 0$ for every $n \in \mathbb{N}$ where $(\hat{\vartheta}^i_n)_{n\in\mathbb{N}}$ supports $\hat{c}^i$.

Remark 3. Note that the bond price family $\mathbf{B}$ can be given exogenously, whereas the inflation-linked bond price family is determined endogenously in equilibrium. Therefore, we are free to choose $\mathbf{B}$ from among the implementable bond price families.

The definition of an Arrow–Debreu equilibrium in our economy is as follows. A collection $((\hat{c}^i)_{i\in\mathbf{I}}, \pi) \in \prod_{i\in\mathbf{I}} \mathbf{L}^\infty_+ \times \mathbf{L}^\infty_+$ constitutes an *Arrow–Debreu equilibrium for* $\mathbf{E}$ if and only if the following conditions hold:

1. For every $i \in \mathbf{I}$, $\hat{c}^i$ solves the problem

$$\max_{c^i \in \mathcal{C}^i(\pi)} U^i(c^i)$$

where

$$\mathcal{C}^i(\pi) = \Big\{ c^i \in \mathbf{L}^\infty_+, : \; E\Big[\int_0^{T^\dagger} \pi_s c^i_s \, ds\Big] = E\Big[\int_0^{T^\dagger} \pi_s e^i_s \, ds\Big]\Big\}.$$

2. The good market is cleared: $\sum_{i\in\mathbf{I}} \hat{c}^i_t = \sum_{i\in\mathbf{I}} e^i_t \quad \forall t \in \mathbf{T}$.

2.4 Equivalence of ASM and Arrow–Debreu Equilibria

Kusuda [32] proves that, for every implementable family of bond price family, an ASM equilibrium is identified with an Arrow–Debreu equilibrium.

Theorem 1. *Let* $\mathbf{B} \in \bar{\mathcal{B}}$. *Under Assumption 1, the following holds:*

1. *If* $((\hat{c}^i)_{i\in\mathbf{I}}, \pi)$ *satisfying* $\pi \in \mathbf{L}^\infty_{++}$ *is an Arrow–Debreu equilibrium for* $\mathbf{E}$, *then* $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ *is an ASM equilibrium for* $\mathbf{E}$, *where* $p = (\tilde{\Lambda}^{\mathbf{B}})^{-1}\pi$.

2. *If* $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ *satisfying* $p \in \mathbf{L}^\infty_{++}$ *is an ASM equilibrium for* $\mathbf{E}$, *then* $((\hat{c}^i)_{i\in\mathbf{I}}, \pi)$ *is an Arrow–Debreu equilibrium for* $\mathbf{E}$, *where* $\pi = \tilde{\Lambda}^{\mathbf{B}} p$.

Now my task is reduced to present sufficient conditions for the existence, uniqueness, and local uniqueness of Arrow–Debreu equilibria in a corresponding Arrow–Debreu economy.

3 Equivalence of Arrow–Debreu and RA Equilibria

In this section, I show that, under certain regularity conditions, an Arrow–Debreu equilibrium is identified with an RA equilibrium.

3.1 Representation and Properties of Aggregate Utility

I introduce the aggregate utility to exploit the Negishi approach. Let $\alpha \in \Delta^I_+$, where $\Delta^I_+ = \{\alpha \in \mathbb{R}^I_+ \mid \sum_{i\in\mathbf{I}} \alpha_i = 1\}$, and I define the *aggregate utility* $U^\alpha : \mathbf{L}^\infty_+ \to \mathbb{R}$ by

$$U^\alpha(c) = \max_{(c^1, c^2, \cdots, c^I) \in \prod_{i\in\mathbf{I}} \mathbf{L}^\infty_+} \sum_{i\in\mathbf{I}} \alpha_i U^i(c^i) \quad \text{s.t.} \quad \sum_{i\in\mathbf{I}} c^i \le c.$$

Then, the *demand function* $c^* : \mathbf{T} \times \mathbb{R}_+ \times \mathbb{R}_+^I \to \mathbb{R}_+^I$ is defined by

$$(c_i^*(t,x,\alpha))_{i\in\mathbf{I}} = \operatorname{argmax}_{\{(x_1,\cdots,x_I)\in\mathbb{R}_+^I : \sum_{i\in\mathbf{I}} x_i \le x\}} \sum_{i\in\mathbf{I}} \alpha_i u^i(t,x_i).$$

Let $u_c(\,\cdot\,,x,\,\cdot\,)$ denote the first partial derivative of u with respect to x. The subsequent lemma immediately follows from Propositions 2.1 and 2.3 in Dana [14]. It demonstrates that the aggregate utility U^α has the expected utility representation and the properties of the vNM aggregate utility function and the demand function.

Lemma 1. *Under Assumption 1, the aggregate utility U^α is a time-additive utility of the form*

$$U^\alpha(c) = E\left[\int_0^{T^\dagger} u(t,c_t,\alpha)\,dt\right] \quad \text{where} \quad u(t,x,\alpha) = \sum_{i\in\mathbf{I}} \alpha_i u^i(t,c_i^*(t,x,\alpha)). \tag{3.1}$$

Moreover, u and $(c_i^*)_{i\in\mathbf{I}}$ satisfy the following conditions:

1. *(i) The function u is a real-valued $\mathbf{C}^{1,1,0}$-function on $\mathbf{T}\times\mathbb{R}_+\times\mathbb{R}_+^I$ such that $u(t,\,\cdot\,,\alpha)$ is strictly increasing and strictly concave on $\mathbb{R}_+$ for every $(t,\alpha)\in\mathbf{T}\times\mathbb{R}_+^I$.*

   (ii) For every $(t,x,\alpha)\in\mathbf{T}\times\mathbb{R}_+\times\mathbb{R}_+^I$ such that $c_i^*(t,x,\alpha)>0$,
   $$u_c(t,x,\alpha) = \alpha_i u_c^i(t,c_i^*(t,x,\alpha)). \tag{3.2}$$

2. *Let $i\in\mathbf{I}$.*

   (i) The function c_i^* is continuous on $\mathbf{T}\times\mathbb{R}_+\times\mathbb{R}_+^I$.

   (ii) For every $(t,x)\in\mathbf{T}\times\mathbb{R}_{++}$, the function $c_i^*(t,x,\,\cdot\,)$ is homogeneous of degree zero.

   (iii) For every $(t,\alpha)\in\mathbf{T}\times\mathbb{R}_+^I$, $c_i^*(t,0,\alpha)=0$.

3. *(i) The functions u_c and c_i^* for every $i\in\mathbf{I}$ are differentiable off the set $\mathcal{D}$ of Lebesgue measure zero:*
   $$\mathcal{D} = \{\,(t,x,\alpha)\in\mathbf{T}\times\mathbb{R}_{++}\times\mathbb{R}_{++}^I : u_c(t,x,\alpha) = \alpha_i u_c^i(t,c_i^*(t,0,\alpha)) \text{ for some } i\in\mathbf{I}\,\}.$$

   (ii) For every $t\in\mathbf{T}$, the functions $u_c(t,\,\cdot\,,\,\cdot\,)$ and $c_i^*(t,\,\cdot\,,\,\cdot\,)$ for every $i\in\mathbf{I}$ are Lipschitz continuous on compact subsets of $\mathbb{R}_+\times\mathbb{R}_+^I$.

   (iii) Let $(t,x,\alpha)\in\mathcal{D}^c$. Assume that $c_i^*(t,x,\alpha)>0$ for every $i\in\mathbf{I}$. Then, it follows that, for every $i,j\in\mathbf{I}$,
   $$\frac{\partial c_i^*}{\partial\alpha_j}(t,x,\alpha) = \frac{u_c^j(t,c_j^*(t,x,\alpha))}{\alpha_i\alpha_j u_{cc}^i(t,c_i^*(t,x,\alpha))u_{cc}^j(t,c_j^*(t,x,\alpha))\eta(t,x,\alpha)} \tag{3.3}$$

where

$$\eta(t,x,\alpha) = \sum_{i\in\mathbf{I}} \frac{1}{\alpha_i u^i_{cc}(t, c^*_i(t,x,\alpha))}.$$

Proof. Proofs of 1(i) and 2(iii) that are obvious are omitted. For proofs of 1(ii) and 2(i)(ii), see the proofs of Proposition 2.1(ii) and 2.1(1) in Dana [14], respectively. For proofs of 3(i)(ii), see the proofs of Proposition 2.3(a) in Dana [14]. 3(iii) is obtained by differentiating the first-order condition

$$\alpha_1 u^i_c(t, c^*_i(t,x,\alpha)) = \alpha_2 u^2_c(t, c^*_2(t,x,\alpha)) = \cdots = \alpha_I u^I_c(t, c^*_I(t,x,\alpha))$$

and the relation $\sum_{i\in\mathbf{I}} c^*_i(t,x,\alpha) = x$ with respect to α_j. □

3.2 Equivalence of Arrow–Debreu and RA Equilibria

The notion of an *RA equilibrium* $\hat{\alpha} \in \Delta^I_{++}$ *for* $\mathbf{E}$ is introduced, which is characterized by the Pareto optimal allocation $(c^*_i(t, e_t, \hat{\alpha}))$ without transfer payments under the supporting state price $u_c(s, e_s, \hat{\alpha})$.

Definition 4. A utility weight $\hat{\alpha} \in \Delta^I_{++}$ constitutes an *RA equilibrium for* $\mathbf{E}$ if and only if $\hat{\alpha}$ is a solution of the equation $\xi(\hat{\alpha}) = 0$, where $\xi : \mathbb{R}^I_{++} \to \mathbb{R}^I$ is the *excess utility function* defined by

$$\xi_i(\alpha) = \frac{1}{\alpha_i} E\left[\int_0^{T^\dagger} u_c(s, e_s, \alpha)(c^*_i(s, e_s, \alpha) - e^i_s)\, ds\right] \qquad \forall i \in \mathbf{I}.$$

To show that an RA equilibrium is identified with an Arrow–Debreu equilibrium, I use the following lemma, which is a direct generalization of Proposition 2.6 in Dana [14].

Lemma 2. *Under Assumption 1, an allocation* $(c^i)_{i\in\mathbf{I}} \in \prod_{i\in\mathbf{I}} \mathbf{L}^\infty_+$ *is Pareto optimal for* $\mathbf{E}$ *if and only if there exists* $\hat{\alpha} \in \Delta^I_{++}$ *such that* $c^*(t, e_t(\omega), \hat{\alpha}) = (c^i_t(\omega))_{i\in\mathbf{I}}$ μ*-a.e.*

Proof. See the proof of Proposition 2.6 in Dana [14]. □

Proposition 1. *Under Assumption 1, it follows that:*

1. *Assume* $\hat{\alpha}$ *is an RA equilibrium for* $\mathbf{E}$*. Define* $((\hat{c}^i)_{i\in\mathbf{I}}, \pi)$ *by* $(\hat{c}^i_t(\omega))_{i\in\mathbf{I}} = c^*(t, e_t(\omega), \hat{\alpha})$ *and* $\pi_t = u_c(t, e_t(\omega), \hat{\alpha})$ *for every* $(\omega, t) \in \Omega \times \mathbf{T}$*. Then* $\pi \in \mathbf{L}^\infty_{++}$ *and* $((\hat{c}^i)_{i\in\mathbf{I}}, \pi)$ *is an Arrow–Debreu equilibrium for* $\mathbf{E}$*.*

2. *Assume* $((\hat{c}^i)_{i\in\mathbf{I}}, \pi)$ *is an Arrow–Debreu equilibrium for* $\mathbf{E}$*. Then, there exists* $\hat{\alpha} \in \Delta^I_{++}$ *such that* $c^*(t, e_t(\omega), \hat{\alpha}) = (\hat{c}^i_t(\omega))_{i\in\mathbf{I}}$ μ*-a.e., and* $\hat{\alpha}$ *is an RA equilibrium for* $\mathbf{E}$*.*

Proof. See Appendix C.1. □

Now my task is reduced to show sufficient conditions for the existence, uniqueness, and local uniqueness of RA equilibria.

4 Existence, Uniqueness, and Local Uniqueness of ASM Equilibria

In this section, by showing sufficient conditions for the existence, uniqueness, and local uniqueness of RA equilibria, I show sufficient conditions for the existence, uniqueness, and local uniqueness of ASM equilibria.

4.1 Existence of RA Equilibria

To prove the existence of RA equilibria, I impose the following assumption on the aggregate endowment.

Assumption 2. *The aggregate endowment is bounded away from zero, i.e., there exists a positive constant $\underline{\delta}$ such that $e_t \geq \underline{\delta}$ μ-a.e. for every $t \in \mathbf{T}$.*

Note that this assumption implies that $u_c(t, e_t(\omega), \alpha) \in \mathbf{L}^{\infty}_{+}$ because

$$0 < u_c(t, e_t, \alpha) \leq \max_{(t,\alpha) \in \mathbf{T} \times \Delta^I_+} u_c(t, \underline{\delta}, \alpha) \quad \mu\text{-a.e.},$$

and $u_c(\,\cdot\,, \underline{\delta}, \,\cdot\,)$ is continuous on $\mathbf{T} \times \Delta^I_+$. Then, the excess utility function satisfies the following desired properties for proving the existence of RA equilibria.

Lemma 3. *Under Assumptions 1 and 2,*

1. *The excess utility function ξ is homogeneous of degree zero, and satisfies $\alpha \cdot \xi(\alpha) = 0$ for every $\alpha \in \mathbb{R}^I_+$, and bounded above on $\mathbb{R}^I_+$.*
2. *The excess utility function ξ is continuous on $\mathbb{R}^I_{++}$, and $\xi_i(\alpha) \to -\infty$ whenever $\alpha_i \to 0$ for some $i \in \mathbf{I}$.*

Proof. Note that there exists a positive constant $\bar{\delta}$ such that $e_t(\omega) \leq \bar{\delta}$ μ-a.e. because $e \in \mathbf{L}^{\infty}_{+}$

Step 1 – (1): It is obvious that ξ is homogeneous of degree zero, and satisfies $\alpha \cdot \xi(\alpha) = 0$ for every $\alpha \in \mathbb{R}^I_+$. Therefore, it is proven that ξ is bounded above on $\mathbb{R}^I_+$. Let $i \in \mathbf{I}$ and $\alpha^0 \in \Delta^I_+$ be such that $\alpha^0_i = 0$. It is sufficient to show that $\xi_i(\alpha)$ is bounded above, as $\alpha \in \Delta^I_+$ tends to α^0. It follows from the Lipschitz continuity of $c^*_i(t, \,\cdot\,, \,\cdot\,)$ and $c^*_i(t, e_i(\omega), \alpha^0) = 0$ that there exists a K such that

$$c^*_i(t, e_t(\omega), \alpha) \leq \max_{t \in \mathbf{T}} c^*_i(t, \bar{\delta}, \alpha) \leq K\|\alpha - \alpha^0\| \quad \mu\text{-a.e.}$$

Thus, it follows that

$$\frac{1}{\alpha_i} u_c\left(t, e_t(\omega), \alpha\right)\{c^*_i(t, e_t(\omega), \alpha) - e^i_t(\omega)\} \leq \frac{\|\alpha - \alpha^0\|}{\alpha_i} K \max_{(t,\alpha') \in \mathbf{T} \times \Delta^I_+} \{u_c(t, \underline{\delta}, \alpha')\} \quad \mu\text{-a.e.}$$

The right-hand side of the above equation converges to $K \max_{(t,\alpha')\in \mathbf{T}\times\Delta^I_+}\{u_c(t,\underline{\delta},\alpha')\}$ as α tends to α^0. Therefore, it follows from Lebesgue's dominated convergence theorem that $\xi(\alpha)$ is bounded above as α tends to α^0.

Step 2 – (2) Continuity on $\mathbb{R}^I_{++}$: It is enough to present the continuity of ξ on a compact subset S of $\mathbb{R}^I_{++}$ bounded away from the boundary. Since ξ is homogeneous of degree zero on α, it follows that for every $i \in \mathbf{I}$,

$$\begin{aligned}
&\left|\frac{1}{\alpha_i}u_c\left(t,e_t(\omega),\alpha\right)\{c_i^*(t,e_t(\omega),\alpha)-e^i_t(\omega)\}\right| \\
&\qquad = \frac{\sum_{j\in\mathbf{I}}\alpha_j}{\alpha_i}\left|u_c\left(t,e_t(\omega),\frac{\alpha}{\sum_{j\in\mathbf{I}}\alpha_j}\right)\left\{c_i^*\left(t,e_t(\omega),\frac{\alpha}{\sum_{j\in\mathbf{I}}\alpha_j}\right)-e^i_t(\omega)\right\}\right| \\
&\qquad \le \frac{\sqrt{I}\,\|\alpha\|}{\alpha_i}\max_{(t,\alpha')\in\mathbf{T}\times\Delta^I_+}\{u_c(t,\underline{\delta},\alpha')\}\,\bar{\delta} \quad \mu\text{-a.e.}
\end{aligned}$$

Thus, the continuity of ξ on S follows from Lebesgue's dominated convergence theorem.

Step 3 – (2) Boundary condition: Let $i \in \mathbf{I}$ and $\alpha^0 \in \Delta^I_+$ be such that $\alpha^0_i = 0$. It suffices to show that $\xi_i(\alpha)$ tends to $-\infty$ as $\alpha \in \Delta^I_+$ tends to α^0. Note that there exists $\mathbf{A} \in \mathcal{P}$ such that $\mu(\mathbf{A}) > 0$ and $e^i_t(\omega) > 0$ for every $(\omega, t) \in \mathbf{A}$ since every agent's endowment process is assumed to be nonzero. Then it follows that

$$\begin{aligned}
\xi_i(\alpha) &\le \frac{1}{\alpha_i}E\left[\int_0^{T^\dagger} u_c(s,e_s(\omega),\alpha)\,e_i^*(s,e_s,\alpha)\,ds\right]-\frac{1}{\alpha_i}\int_{\mathbf{A}}u_c(s,e_s(\omega),\alpha)\,e^i_s(\omega)\,\mu(d\omega\times ds) \\
&\le \frac{\|\alpha-\alpha^0\|}{\alpha_i}KT^\dagger\max_{(t,\alpha')\in\mathbf{T}\times\Delta^I_+}\{u_c(t,\underline{\delta},\alpha')\}-\frac{1}{\alpha_i}\min_{(t,\alpha')\in\mathbf{T}\times\Delta^I_+}\{u_c(t,\bar{\delta},\alpha')\}\int_{\mathbf{A}}e^i_s(\omega)\,\mu(d\omega\times ds),
\end{aligned}$$

which tends to $-\infty$ as α tends to α^0. □

4.2 Uniqueness of RA Equilibria

To prove the uniqueness of the RA equilibria, I make the following two assumptions.

Assumption 3. *1. For every $i \in \mathbf{I}$, agent i's relative risk aversion coefficient satisfies*

$$\gamma^i(t,x) := -\frac{x\,u^i_{cc}(t,x)}{u^i_c(t,x)} \le 1 \qquad \forall(t,x)\in\mathbf{T}\times\mathbb{R}_+.$$

2. Either of the following two conditions is satisfied:

(i) Every agent's endowment is positive μ-a.e., i.e. $e^i > 0$ μ-a.e. for every $i \in \mathbf{I}$.

(ii) Every agent's utility satisfies the Inada condition, i.e. $\lim_{x \downarrow 0} u_c^i(t,x) = \infty$ *for every* $i \in \mathbf{I}$.

Then, the excess utility function is *strongly gross substitute.*

Lemma 4. *Under Assumptions 1-3,* ξ *is strongly gross substitute, i.e.:*

1. *For every* (i,j) *such that* $i \neq j$, $\xi_i(\alpha_1, \cdots, \alpha_{j-1}, \cdot, \alpha_{j+1}, \cdots, \alpha_I)$ *is non-increasing, and for every* i, $\xi_i(\alpha_1, \cdots, \alpha_{i-1}, \cdot, \alpha_{i+1}, \cdots, \alpha_I)$ *is non-decreasing.*

2. *If* $c_i^*(t, e_t(\omega), \alpha) > 0$ *on some* $\mathbf{A} \in \mathcal{P}$ *with* $\mu(\mathbf{A}) > 0$, *then, for every* $j \neq i$, $\xi_i(\alpha_1, \cdots, \alpha_{j-1}, \cdot, \alpha_{j+1}, \cdots, \alpha_I)$ *is strictly decreasing on a neighborhood of* α.

Proof. See the proof of Theorem 3.1 in Dana [14]. □

4.3 Local Uniqueness of RA Equilibria

Unfortunately, there is no strong evidence that supports Assumption 3. Therefore, I show that, under more reasonable assumptions, the local uniqueness of equilibria, or equivalently the finiteness of the number of equilibria, is a generic property of the economy $\mathbf{E}$ using the Negishi approach given by Dana [14] for a static economy.

The space of economies is parameterized by keeping agents' common subjective probability and information structure $(\Omega, \mathcal{F}, \mathbb{F}^{W,\nu}, P)$, utilities $(u^i)_{i \in \mathbf{I}}$, and the aggregate endowment e fixed, and varying the distribution of individual endowments. I impose the following assumptions on utilities and endowments.

Assumption 4. *For every* $i \in \mathbf{I}$, *the vNM utility function satisfies*

$$-\frac{u_c^i(t,x)}{u_{cc}^i(t,x)} \leq \beta_1^i x + \beta_2^i \qquad \forall (t,x) \in \mathbf{T} \times \mathbb{R}_+$$

for some $(\beta_1^i, \beta_2^i) \in \mathbb{R}_+^2$.

Assumption 5. *There exists* $\delta \in \mathbb{R}_{++}^{\mathbf{I}}$ *such that* $e_t^i > \delta_i$ μ*-a.e. on* $\mathbf{T} \times \Omega$ *for every* $i \in \mathbf{I}$.

I introduce the following space of economies wherein each economy is characterized by the distribution of individual endowments.

$$\mathcal{E}_\delta = \Big\{ \mathbf{E} = ((\Omega, \mathcal{F}, \mathbb{F}^{W,\nu}, P), (u^i, e^i)_{i \in \mathbf{I}}) \Big| \\ (e^i)_{i \in \mathbf{I}} \in \prod_{i \in \mathbf{I}} \mathbf{L}_+^\infty, \quad \sum_{i \in \mathbf{I}} e^i = e, \text{ and } (e^i)_{i \in \mathbf{I}} \text{ satisfies Assumption 5 for } \delta \Big\}$$

A function $\hat{\xi} : \Delta^I_+ \times \mathcal{E}_\delta \to \mathbb{R}^I$ is defined by

$$\hat{\xi}_i(\alpha, \mathbf{E}) = \frac{1}{\alpha_i} E\left[\int_0^{T^\dagger} u_c(s, e_s, \alpha)(c_i^*(s, e_s, \alpha) - e_s^i)\, ds\right] \qquad \forall i \in \mathbf{I}.$$

The continuity of $\hat{\xi}$ follows from dominated convergence theorem. The differentiability of $\hat{\xi}$ with respect to α and the continuity of the derivative can also be shown.

Lemma 5. *Under Assumptions 1, 2, 4, and 5, $\hat{\xi}$ is differentiable with respect to α on $\Delta^{\mathbf{I}}_{++}$ and its derivative is continuous on $\Delta^I_{++} \times \mathcal{E}_\delta$.*

Proof. See Appendix C.2. □

Since $\sum_{i \in \mathbf{I}} \hat{\xi}_i(\alpha, \mathbf{E}) = 0$ for every $\alpha \in \Delta^I_+$, it follows that rank $D_\alpha \hat{\xi}(\alpha, \mathbf{E}) \leq I - 1$. I say that the economy $\mathbf{E}$ is *regular* if and only if $\hat{\xi}(\hat{\alpha}, \mathbf{E}) = 0$ implies rank $D_\alpha \hat{\xi}(\alpha, \mathbf{E}) = I - 1$. Let $\mathcal{R}_\delta$ denote the set of all regular economies in $\mathcal{R}_\delta$. It is well known that any regular economy can only have a finite number of equilibria (see Proposition 17.D.1 in Mas-Collel, Whinston, and Green [35]). Thus, to see that the number of equilibria is generically finite, it is enough to show that the set of regular economies $\mathcal{R}_\delta$ is open and dense in $\mathcal{E}_\delta$. In order to do so, a correspondence $\{\hat{\alpha}\}(\mathbf{E}) : \mathcal{E}_\delta \to \Delta^I_+$ is defined by

$$\{\hat{\alpha}\}(\mathbf{E}) = \{\, \alpha \in \Delta^I_+ \;:\; \hat{\xi}(\alpha, \mathbf{E}) = 0 \,\},$$

I show the following lemma. For *upper hemicontinuity*, see Definition 3.AA.1 in Mas-Collel *et al.* [35].

Lemma 6. *Under Assumptions 1, 2, 4, and 5, it follows that:*

1. *The correspondence $\{\hat{\alpha}\}$ is upper hemicontinuous (u.h.c.), and for every $\mathbf{E} \in \mathcal{E}_\delta$, $\{\hat{\alpha}\}(\mathbf{E})$ is compact.*
2. *If $\mathbf{E}$ is regular then $\{\hat{\alpha}\}(\mathbf{E})$ is finite.*

Proof. Proof of 1 immediately follows from the continuity of $\hat{\xi}$. Let $\mathbf{E}$ be a regular economy. Suppose $\{\hat{\alpha}\}(\mathbf{E})$ is infinite. Then, since $\{\hat{\alpha}\}(\mathbf{E})$ is compact, it has an accumulation point $\hat{\alpha} \in \{\hat{\alpha}\}(\mathbf{E})$. This implies that $\hat{\alpha}$ is not locally unique. This is a contradiction. □

4.4 Existence, Uniqueness, and Local Uniqueness of Equilibria

Now I prove the existence, uniqueness, and local uniqueness of ASM equilibria.

Theorem 2. *Under Assumptions 1 and 2, it follows that, for every $\mathbf{B} \in \bar{\mathcal{B}}$.*

1. *There exists an ASM equilibrium* $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ *for* $\mathbf{E}$*. In particular, if the mark set* $\mathbb{Z}$ *is finite, then* $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ *is a security market equilibrium for* $\mathbf{E}$*. The equilibrium* $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ *is characterized by the corresponding representative agent equilibrium* $\hat{\alpha}$ *for* $\mathbf{E}$*, i.e.* $((\hat{c}^i)_{i\in\mathbf{I}}, p)$ *satisfies*

$$\begin{aligned} (\hat{c}^i_t(\omega))_{i\in\mathbf{I}} &= c^*(t, e_t(\omega), \hat{\alpha}), \\ p_t(\omega) &= \frac{B_t(\omega)}{\Lambda^{\mathbf{B}}_t(\omega)} u_c(t, e_t(\omega), \hat{\alpha}) > 0 \end{aligned} \tag{4.1}$$

for almost every $(\omega, t) \in \Omega \times \mathbf{T}$*. Moreover, the allocation* $(\hat{c}^i)_{i\in\mathbf{I}}$ *is Pareto optimal.*

2. *Under Assumption 3, the ASM equilibrium is unique.*

3. *Under Assumptions 4 and 5, the set of regular economies* $\mathcal{R}_\delta$ *is open and dense in* $\mathcal{E}_\delta$*.*

Proof. 1: It follows from Lemma 3 and Kakutani's fixed-point theorem that there exists an $\hat{\alpha} \in \Delta^I_{++}$ such that $\xi(\hat{\alpha}) = 0$, *i.e.* there exists an RA equilibrium $\hat{\alpha}$ for $\mathbf{E}$ (see the proof of Proposition 17.C. 1 in Mas-Collel *et al.* [35]). Define $(\hat{c}^i)_{i\in\mathbf{I}}$ and p by $(\hat{c}^i_t(\omega))_{i\in\mathbf{I}} = c^*(t, e_t(\omega), \hat{\alpha})$ and $p_t(\omega) = (\tilde{\Lambda}^{\mathbf{B}}_t(\omega))^{-1} u_c(t, e_t(\omega), \hat{\alpha})$ for every $(\omega, t) \in \Omega \times \mathbf{T}$, respectively. Then, by Proposition 1.1 and Theorem 1.1, $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ is an ASM equilibrium for $\mathbf{E}$, and $(\hat{c}^i)_{i\in\mathbf{I}}$ is a Pareto optimal allocation. Suppose that the mark set $\mathbb{Z}$ is finite. It then follows from Theorem 1.2 that $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ constitutes a security market equilibrium for $\mathbf{E}$.

2: By Theorem 1 and Proposition 1, it is sufficient to show that the RA equilibrium is unique. I use the proof of Dana [14]. Assume that there exist two non-collinear solutions for $\xi(\alpha) = 0$ and let them be $\hat{\alpha}$ and $\breve{\alpha}$. Since E is homogeneous of degree zero by Lemma 3, let w.l.o.g. $\hat{\alpha} < \breve{\alpha}$ with $\hat{\alpha}_i = \breve{\alpha}_i$ for some $i \in \mathbf{I}$. As $\breve{\alpha}$ is a solution for $\xi(\alpha) = 0$, $c^*_i(t, e_t(\omega), \breve{\alpha}) \neq 0$ for every j. Therefore, ξ_i is strictly increasing at $\breve{\alpha}$. Let $\hat{\alpha} < \alpha < \breve{\alpha}$. Then, $0 = \xi_i(\hat{\alpha}) < \xi_i(\alpha) < \xi_i(\breve{\alpha}) = 0$, which is a contradiction.

3: See Appendix C.3.

□

5 Future Research Direction

This study, in conjunction with Kusuda [32], analyzes ASM equiribria in an approximately complete security market economy assuming time-additive utility. The class of stochastic differential utilities includes various promising utilities, such as robust utility (Hansen and Sargent [25]), homothetic robust utility (Maenhout [34]), and age-dependent robust utility (Kikuchi and Kusuda [31]). I thus analyze ASM equiribira assuming stochastic differential utility.

Various studies consider consumption–investment problems with complete security market models under duffusion infrmation; some, including Batbold, Kikuchi and Kusuda [6], derive analytical expressions of the optimal control. In future, I seek to consider the consumption–investment problem with the approximately complete security market model under jump-diffusion information and derive the optimal control.

A Basics of Probability Theory

Let $\mathcal{L}^n$ denote the set of real-valued $\mathcal{P}$-measurable process X satisfying the integrability condition $\int_0^{T^\dagger} |X_s|^n \, ds < \infty$ $\mathbb{P}$-almost surely. Also, let $\mathcal{L}^n(\lambda_t(dz) \times dt)$ denote the set of real-valued $\mathcal{P} \otimes \mathcal{Z}$-measurable process H satisfying the integrability condition $\int_0^{T^\dagger} \int_{\mathbb{Z}} |H_s(z)|^n \, \lambda_s(dz) \, ds < \infty$ $\mathbb{P}$-a.s.

A.1 Marked Point Process

I consider a double sequence $(s_n, Z_n)_{n \in \mathbb{N}}$, where s_n is the occurrence time of an nth jump and Z_n is a random variable taking its values on a measurable space $(\mathbb{Z}, \mathcal{Z})$ at time s_n. Define a random counting measure $\nu(dt \times dz)$ by

$$\nu([0,t] \times A) = \sum_{n \in \mathbb{N}} 1_{\{s_n \leq t, \, Z_n \in A\}} \qquad \forall (t, A) \in [0, T^\dagger] \times \mathcal{Z}.$$

This counting measure $\nu(dt \times dz)$ is called the $\mathbb{Z}$*-marked point process.* Let λ be such that

1. For every $(\omega, t) \in \Omega \times (0, T^\dagger]$, the set function $\lambda_t(\omega, \cdot)$ is a finite Borel measure on $\mathbb{Z}$.
2. For every $A \in \mathcal{Z}$, the process $\lambda(A)$ is $\mathcal{P}$-measurable and satisfies $\lambda(A) \in \mathcal{L}^1$.

The marked point process $\nu(dt \times dz)$ is said to have the P*-intensity kernel* $\lambda_t(dz)$ if and only if

$$E\left[\int_0^{T^\dagger} Y_s \, \nu(ds \times A)\right] = E\left[\int_0^{T^\dagger} Y_s \lambda_s(A) \, ds\right] \qquad \forall A \in \mathcal{Z}$$

holds for any nonnegative $\mathcal{P}$-measurable process Y. Then, the marked point process $\nu(dt \times dz)$ is said to have the P*-intensity kernel* $\lambda_t(dz)$.

Let $\nu(dt \times dz)$ be a $\mathbb{Z}$-marked point process with the P-intensity kernel $\lambda_t(dz)$. Let H be a $\mathcal{P} \otimes \mathcal{Z}$-measurable function. It follows that:

1. If the following integrability condition

$$E\left[\int_0^{T^\dagger}\int_{\mathbb{Z}} |H_s(z)|\lambda_s(z)\,ds\right] < \infty$$

holds, then the process $\int_0^t \int_{\mathbb{Z}} H_s(z)\{\,\nu(ds\times dz) - \lambda_s(dz)\,ds\,\}$ is a P-martingale.

2. If $H \in \mathcal{L}^1(\lambda_t(dz)\times dt)$, then the process $\int_0^t \int_{\mathbb{Z}} H_s(z)\{\,\nu(ds\times dz) - \lambda_s(dz)\,ds\,\}$ is a local P-martingale.

Proof. See p. 235 in Brémaud [12]. □

A.2 Ito's Formula

Let $X = (X^1, ..., X^d)'$ be a d-dimensional semimartingale, and g be a real-valued $\mathbf{C}^2$ function on $\mathbb{R}^d$. Then, $g(X)$ is a semimartingale of the form:

$$g(X_t) = g(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x_i} g(X_{s-})\,dX_s^i + \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} g(X_{s-})\,d\langle X^{ic}, X^{jc}\rangle$$

$$+ \sum_{0\le s\le t}\left\{g(X_s) - g(X_{s-}) - \sum_{i=1}^d \frac{\partial}{\partial x_i} g(X_{s-})\,\Delta X_s^i\right\} \quad \text{(A.1)}$$

where X^{ic} is the continuous part of X^{ic} and $\langle X^{ic}, X^{jc}\rangle$ is the quadratic covariation of X^{ic} and X^{jc}.

A.3 Girsanov's Theorem

1. Let $\upsilon \in \prod_{j=1}^d \mathcal{L}^2$ and $H \in \mathcal{L}^1(\lambda_t(dz)\times dt)$. Define a process Λ by

$$\frac{d\Lambda_t}{\Lambda_{t-}} = -\upsilon_t \cdot dW_t - \int_{\mathbb{Z}} H_t(z)\,\{\,\nu(dt\times dz) - \lambda_t(dz)\,dt\} \qquad \forall t \in [0, T^\dagger)$$

with $\Lambda_0 = 1$ and $E\left[\Lambda_{T^\dagger}\right] = 1$. Then, there exists a probability measure $\tilde{P}$ on $(\Omega, \mathcal{F}, \mathbb{F})$ defined by[10]

$$E_t\left(\frac{d\tilde{P}}{dP}\right) = \Lambda_t, \quad \text{(A.2)}$$

such that:

[10] Here, $\frac{d\tilde{P}}{dP}$ is the Radon-Nikodym derivative of $\tilde{P}$ with respect to P, and Λ_t is the *density process* for $\tilde{P}$.

(i) The measure $\tilde{P}$ is equivalent to P.

(ii) The process given by

$$\tilde{W}_t = W_t + \int_0^t v_s \, ds \qquad \forall t \in \mathbf{T}$$

is a $\tilde{P}$-Wiener process.

(iii) The marked point process $\nu(dt \times dz)$ has the $\tilde{P}$-intensity kernel such that

$$\tilde{\lambda}_t(dz) = (1 - H_t(z))\lambda_t(dz) \qquad \forall (t, z) \in \mathbf{T} \times \mathbb{Z}.$$

2. Every probability measure equivalent to P has the structure above.

B Definitions on Approximately Complete Security Markets

I review the notion of approximately complete markets given by Björk *et al.* [10, 11], and introduce a class of families of bond prices such that, for every family of bond prices in this class, an ASM equilibrium can be identified with an Arrow–Debreu equilibrium.

B.1 Regular Bond Price Family

A bond price family $\mathbf{B}$ is *regular* if and only if the following conditions hold:

1. For every $T \in (0, T^\dagger]$, the dynamics of nominal bond price process B^T satisfies the following stochastic differential–difference equation (SDDE)

$$\frac{dB_t^T}{B_{t-}^T} = r_t^T dt + v_t^T \cdot dW_t + \int_{\mathbb{Z}} H_t^T(z) \{ \nu(dt \times dz) - \lambda_t(dz) \, dt \} \quad \forall t \in [0, T) \tag{B.1}$$

with $B_T^T = 1$ and $B_t^T = 0$ for every $t \in (T, T^\dagger]$ for some $r^T \in \mathcal{L}^1$, $v^T \in \mathcal{L}^2$, and $H^T \in \mathcal{L}^1(\lambda_t(dz) \times dt)$. Moreover, it follows that:

(i) For every $(\omega, t) \in \Omega \times \mathbf{T}$, $r_t^{\cdot}(\omega), v_t^{\cdot}(\omega) \in \mathbf{C}^1((t, T^\dagger])$ and for every $(\omega, t, z) \in \Omega \times \mathbf{T} \times \mathbb{Z}$, $H_t^{\cdot}(\omega, z) \in \mathbf{C}^1((t, T^\dagger])$.

(ii) For every $T \in (0, T^\dagger]$, $H_t^T(\omega, z)$ is bounded.

(iii) The processes $(B^T)_{T \in \mathbf{T}}$ are regular enough to allow for differentiation under the integral sign and interchange of integration order.[11]

[11] For the integrals of the marked point process, the ordinary Fubini theorem can be applied, and for the interchange of integration with respect to dW_t and dt, the stochastic Fubini theorem holds (see Protter [42]).

2. The dynamics of nominal money market account price process B satisfies the following SDDE

$$\frac{dB_t}{B_t} = r_t^B \, dt \qquad \forall t \in \mathbf{T} \tag{B.2}$$

with $B_0 = 1$, where $r_t^B = -\left.\frac{\partial \ln B_t^T}{\partial T}\right|_{T=t}$.

B.2 Feasible Portfolio

Let $\mathbf{B} \in \mathcal{B}$. A portfolio ϑ is a *feasible portfolio at* $\mathbf{B}$ if and only if it follows that

$$\begin{aligned}
&\int_t^{T^\dagger} |B_t^T| \, |\vartheta_t^1(dT)| < \infty \qquad \mathbb{P}\text{-a.s.} \qquad \forall t \in \mathbf{T}, \\
&B_t r_t^B \vartheta_t^0, \int_t^{T^\dagger} |B_t^T r_t^T| \, |\vartheta_t^1(dT)| \in \mathcal{L}^1, \qquad \int_t^{T^\dagger} \|B_t^T v_t^T\| |\vartheta_t^1(dT)| \in \mathcal{L}^2, \\
&\int_t^{T^\dagger} |B_t^T H_t^T(z)| \, |\vartheta_t^1(dT)| \in \mathcal{L}^1(\lambda_t(dz) \times dt).
\end{aligned} \tag{B.3}$$

B.3 Contingent Claim and Replicable Claim

Let $\mathbf{B} \in \mathcal{B}$.

1. For every $T \in (0, T^\dagger]$, a *contingent T-claim at* $\mathbf{B}$ is a $\mathcal{F}_T$-measurable random variable X_T such that $\tilde{X}_T \in \mathbf{L}_+^\infty(\Omega, \mathcal{F}_T)$, where $\mathbf{L}^\infty(\Omega, \mathcal{F}_T)$ is the space of almost surely bounded $\mathcal{F}_T$-measurable random variables.

2. A contingent T-claim X_T is *replicable at* $\mathbf{B}$ if and only if there exists an admissible self-financing portfolio $\vartheta \in \underline{\Theta}(\tilde{\mathbf{B}})$ such that its value process satisfies $\mathcal{V}_T^{\mathbf{B}}(\vartheta) = X_T$.

C Proofs

C.1 Proof of Proposition 1

Proof of 1. Let $\hat{\alpha}$ be an RA equilibrium for $\mathbf{E}$. Define $((\hat{c}^i)_{i\in\mathbf{I}}, \pi)$ by $(\hat{c}_t^i(\omega))_{i\in\mathbf{I}} = c^*(t, e_t(\omega), \hat{\alpha})$ and $\pi_t = u_c(t, e_t(\omega), \hat{\alpha})$ for every $(\omega, t) \in \Omega \times \mathbf{T}$. Then, $\hat{c}^i \in \mathbf{L}_+^\infty$ for every $i \in \mathbf{I}$ and $\pi \in \mathbf{L}_{++}^\infty$. It also follows that $\sum_{i\in\mathbf{I}} \hat{c}^i = \sum_{i\in\mathbf{I}} e^i$ by definition of c^* and that $\hat{c}_t^i$ satisfies the necessary and sufficient condition for every agent's optimality $u_c^i(t, \hat{c}_t^i) = \frac{1}{\hat{\alpha}_i}\pi_t$ for every $i \in \mathbf{I}$.

Proof of 2. Let $((\hat{c}^i)_{i\in\mathbf{I}}, \pi)$ be an Arrow–Debreu equilibrium for $\mathbf{E}$. Since $(u^i)_{i\in\mathbf{I}}$ are strictly increasing by Assumption 1, the allocation $(\hat{c}^i)_{i\in\mathbf{I}}$ is Pareto optimal by first welfare theorem (see Mas-Collel and Zame [36]). Thus, by Lemma 2, there exists $\hat{\alpha} \in \Delta^I_{++}$ such that

$$c^*(t, e_t(\omega), \hat{\alpha}) = (\hat{c}^i_t(\omega))_{i\in\mathbf{I}} \qquad \mu\text{-a.e.} \tag{C.1}$$

Combining (3.2) with (C.1) yields

$$u_c(t, e_t(\omega), \hat{\alpha}) = \hat{\alpha}_i u^i_c(t, \hat{c}^i_t(\omega)) \qquad \mu\text{-a.e.} \tag{C.2}$$

for every $i \in \mathbf{I}$. The optimality of consumption plans implies that there exists a rescaled Lagrange multiplier $\hat{\alpha}^- \in \{\alpha^- \in \mathbb{R}_{++} \mid \sum_{i\in\mathbf{I}} \frac{1}{\alpha^-_i} = 1\}$ such that, for every $i \in \mathbf{I}$ and

$$u^i_c(t, \hat{c}^i_t) = \hat{\alpha}^-_i \pi_t \qquad \mu\text{-a.e.} \tag{C.3}$$

Comparing (C.2) with (C.3) yields $u_c(t, e_t(\omega), \hat{\alpha}) = \pi_t(\omega)$, which implies $\xi(\hat{\alpha}) = 0$.

C.2 Proof of Lemma 5

The proof of Dana [14] is exploited. Let S be a compact subset of Δ^I_+ bounded away from the boundary. It suffices to prove the differentiability of $\hat{\xi}$ with respect to α on S. Define a function $\zeta : \mathbf{T} \times \mathbb{R}_+ \times S \to \mathbb{R}^I$ by

$$\zeta_i(t, e_t, \alpha) = \frac{1}{\alpha_i} u_c(t, e_t, \alpha)(c^*_i(t, e_t, \alpha) - e^i_t).$$

Partially differentiating ζ with respect to α_j yields

$$\frac{\partial \zeta_i}{\partial \alpha_j}(t, e_t, \alpha) = \frac{1}{\alpha_i}\frac{\partial c^*_i}{\partial \alpha_j}(t, e_t, \alpha)\{u^i_{cc}(t, c^*_i)(c^*_i(t, e_t, \alpha) - e^i_t) + u^i_c(t, c^*_i)\}. \tag{C.4}$$

Meanwhile, it follows from (3.3) that

$$\frac{\partial c^*_i}{\partial \alpha_j}(t, e_t, \alpha)\, u^i_{cc}(t, c^*_i) \le \frac{1}{\alpha_i} u^j_c(t, c^*_j) = \frac{1}{\alpha_i \alpha_j} u_c(t, e_t, \alpha). \tag{C.5}$$

It follows from (C.4),(C.5), and Assumptions 4 and 5 that

$$\left|\frac{\partial \zeta_i}{\partial \alpha_j}(t, e_t, \alpha)\right| \le \frac{1}{\alpha_i \alpha_j} \max_{(\alpha')\in\Delta^I} [u_c(t, e_t(\omega), \alpha')]\{(\beta^i_1 + 2)e_t + \beta^i_2\}. \tag{C.6}$$

Thus, by Lebesgue's dominated convergence theorem, $\hat{\xi}$ is differentiable with respect to α on S, and its derivative is

$$\frac{\partial \hat{\xi}_i}{\partial \alpha_j}(\alpha, \mathbf{E}) = E\left[\int_0^{T^\dagger} \frac{\partial c^*_i}{\partial \alpha_j}(s, e_s, \alpha)\left\{u^i_{cc}(s, c^*_i(s, e_s, \alpha)(c^*_i(s, e_s, \alpha) - e^i_s) + u^i_c(t, c^*_i(s, e_s, \alpha)\right\} ds\right].$$

Since e is fixed, $\left|\dfrac{\partial F_i}{\partial \alpha_j}\right|$ are bounded independently of $(\alpha, \mathbf{E})$ on S. Therefore, $\dfrac{\partial \hat{\xi}_i}{\partial \alpha_j}$ is continuous on $\Delta^I_{++} \times \mathcal{E}_\delta$.

C.3 Proof of Theorem 2.3

The proof of Dana [14] is generalized. First, the openness of $\mathcal{R}_\delta$ is shown. Let $\mathbf{E}_0 \in \mathcal{R}_\delta$. Then, for any $\alpha_0 \in \delta^I$ such that $\hat{\xi}(\alpha_0, \mathbf{E}_0) = 0$, and that rank $D_\alpha \hat{\xi}(\alpha_0, \mathbf{E}_0) = I - 1$. Since $\{\hat{\alpha}\}(\mathbf{E})$ is compact and $D_\alpha \hat{\xi}$ is continuous, there exists neighborhoods $\mathcal{V} \subset \mathcal{E}_\delta$ of $\mathbf{E}_0$ and $V \subset \Delta^I_+$ of α_0 such that $D_\alpha \hat{\xi}(\alpha, \mathbf{E}) = I - 1$ for every $(\alpha, \mathbf{E}) \in V \times \mathcal{V}$. Since $\{\hat{\alpha}\}$ is u.h.c., there exists $\mathcal{V}' \subset \mathcal{V}$ such that $\{\hat{\alpha}\}(\mathcal{V}') \subset \mathcal{V}$. Thus, if $\mathbf{E} \in \mathcal{V}'$, then rank $D_\alpha \hat{\xi}(\alpha_0, \mathbf{E}_0) = I - 1$ for every $\alpha \in \{\hat{\alpha}\}(\mathbf{E})$. Therefore, $\mathcal{V}' \subset \mathcal{R}_\delta$ and $\mathcal{R}_\delta$ is open in $\mathcal{E}_\delta$. Next, the denseness of $\mathcal{R}_\delta$ is proven. Let $\mathbf{E} \in \mathcal{E}_\delta$ and let $\varepsilon > 0$ such that $e^i - \varepsilon > \delta_i$ μ-a.e. for every $i \in \{1, 2, \cdots, I-1\}$. Moreover, let $(X^\varepsilon_i)_{i \in \{1,2,\cdots,I-1\}}$ such that $\max\{\|X_i\|_{\mathbf{L}^\infty}, \|X_i\|_{\mathbf{L}^\infty}\} \leq \varepsilon \quad \forall i \in \{1, 2, \cdots, I-1\}$, and let $A = \{(a_i)_{i\in\{1,2,\cdots,I-1\}} \in \mathbb{R}^{I-1} : 0 \leq a_i \leq 1 \quad \forall i \in \{1, 2, \cdots, I-1\}\}$. Define a function $h : \Delta^I \times A \to \mathbb{R}^I$ by

$$h_i(\alpha, a) = E\left[\int_0^{T^\dagger} u_c(s, e_s, \alpha)(c^*_i(s, e_s, \alpha) - e^i_s - a_i X^\varepsilon_{is})\, ds\right] \qquad \forall i \in \{1, 2, \cdots, I-1\},$$

and

$$h_I(\alpha, a) = E\left[\int_0^{T^\dagger} u_c(s, e_s, \alpha)(c^*_I(s, e_s, \alpha) - e^I_s + \sum_{i=1}^{I-1} a_i X^\varepsilon_{is})\, ds\right].$$

One can easily check that rank $D_a g(\alpha, a) = I - 1$. By transversality theorem, there exists $a \in A$ such that 0 is a regular value of $h(\cdot\, a)$ that is 0 is a regular value of the economy in $\mathcal{E}$, $(e^1 + a_1 X^\varepsilon_1, e^2 + a_2 X^\varepsilon_2, \cdots, e^{I-1} + a_{I-1} X^\varepsilon_{I-1}, e^I - \sum_{i=1}^{I-1} a_i X^\varepsilon_i)$, arbitrarily close to $\mathbf{E}$, since ε can be chosen arbitrarily close to zero.

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Competing Interests

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