DISCUSSION PAPER SERIES E

Discussion Paper No. E-45

ITERATIVE SCHEME GENERATING METHOD FOR CONTRACTION TYPE MAPPINGS IN BANACH SPACES

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July 2025

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Abstract. We introduce an iterative scheme generating method (ISGM) for contraction type mappings in Banach spaces. This marks the first time ISGM has been demonstrated in a Banach space setting. Our main theorem can apply to contraction mappings, Kannan mappings, etc., enabling to generate infinitely many iterative schemes for finding fixed points. Several corollaries are presented to show a wide variety of iterative methods which can be generated from the main theorems. We also introduce an application to a variational inequality problem (VIP), highlighting how the various iterative schemes covered in this work are directly useful in optimization techniques.

1. Introduction

A mapping T from a metric space (X, d) into itself is called a *contraction mapping* if there exists $a \in (0,1)$ such that

$$d(Tx, Ty) \leq a d(x, y) \quad \text{for all } x, y \in X. \tag{1.1}$$

The following result is known as the Banach contraction principle:

Theorem 1.1 ([2]). *Let X be a complete metric space and let $T : X \to X$ be a contraction mapping. Then, T has a unique fixed point p and a sequence $\{x_n\}$ defined by*

$$x_{n+1} = Tx_n \quad \textit{for all } n \in \mathbb{N} = \{1, 2, \cdots\} \tag{1.2}$$

converges to the fixed point p for any initial point $x_1 \in X$.

The iteration procedure (1.2) is called the *Picard iterative method.* Kannan [13] and Chatterjea [6] each investigated other types of mappings that satisfy the following conditions:

$$d(Tx, Ty) \leq b(d(x, Tx) + d(y, Ty)) \quad \text{for all } x, y \in X, \tag{1.3}$$

$$d(Tx, Ty) \leq c(d(x, Ty) + d(Tx, y)) \quad \text{for all } x, y \in X, \tag{1.4}$$

where $b, c \in \left(0, \frac{1}{2}\right)$. They proved the same conclusion as Theorem 1.1.

Several studies intended to unify these classes of mappings (1.1), (1.3), and (1.4). In 1972, Zamfirescu [38] defined a class of mappings $T : X \to X$

1991 *Mathematics Subject Classification.* 47H09, 47H10.

Key words and phrases. Iterative scheme generating method, contraction type mappings, Zamfirescu mapping, fixed point, Banach space.

characterized by the following condition: there exist $a \in (0,1)$ and $b, c \in \left(0, \frac{1}{2}\right)$ such that for any $x, y \in X$, at least one of the following three conditions holds:

$$\begin{array}{ll} (Z1) & d(Tx,Ty) \leq ad(x,y), \\ (Z2) & d(Tx,Ty) \leq b(d(x,Tx)+d(y,Ty)), \\ (Z3) & d(Tx,Ty) \leq c(d(x,Ty)+d(Tx,y)). \end{array} \tag{1.5}$$

This class of mappings is called *Zamfirescu mappings*. In 1973, Hardy and Rogers [10] considered the following condition: there exist $\alpha, \beta, \gamma, \delta, \varepsilon \in [0,1]$ such that $\alpha + \beta + \gamma + \delta + \varepsilon < 1$ and

$$\begin{aligned} & d(Tx,Ty) \\ & \leq \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \delta d(x,Ty) + \varepsilon d(Tx,y) \end{aligned} \tag{1.6}$$

for all $x, y \in X$. If $\delta = \varepsilon = 0$ in (1.6), the mapping T is a Ćirić-Reich-Rus type [7, 30, 31]. Each class of Zamfirescu mappings (1.5) and Hardy and Rogers mappings (1.6) contains contraction mappings (1.1), Kannan mappings (1.3), and Chatterjea mappings (1.4) simultaneously. Both Zamfirescu [38] and Hardy and Rogers [10] obtained the same conclusions as Theorem 1.1. In other words, the mapping T has a unique fixed point and the Picard iterative scheme is available for finding the fixed point. For more general class of mappings, see Ćirić [8]. Rhoades [33] explored relationships between these classes of mappings. For more recent results concerning the Banach contraction principle, see Berinde [5], Rus *et al.* [32], Agarwal *et al.* [1], Karapınar and Agarwal [14], Micula and Milovanović [26], and Cvetković *et al.* [9].

In the literature of fixed point theory on Banach and Hilbert spaces, many researchers have explored a number of approximation methods more general than the Picard method (1.2). Berinde [4] demonstrated the following theorem:

Theorem 1.2 ([4]). *Let C be a nonempty, closed, and convex subset of a Banach space E and let T be a Zamfirescu mapping from C into itself. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0,1]$ such that $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$. Define a sequence $\{x_n\}$ in C by the following rule:*

$$\begin{aligned} & x_1 \in C \text{ is given,} \\ & x_{n+1} = \lambda_n x_n + (1-\lambda_n) T x_n \end{aligned} \tag{1.7}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges in norm to a unique fixed point of T.

The iterative scheme (1.7) is called the Mann type [24]. Set $\lambda_n = 0$ for all $n \in \mathbb{N}$ in (1.7). Then, the condition $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$ is satisfied and the iteration rule (1.7) coincides with the Picard type (1.2). In this sense, the Mann iterative method is a generalization of the Picard type.

The following two-step iterative scheme is known as the Ishikawa type [12]:

$$x_1 \in C \text{ is given}, \quad z_n = \nu_n x_n + (1-\nu_n) T x_n, \quad x_{n+1} = \lambda_n x_n + (1-\lambda_n) T z_n \tag{1.8}$$

for all $n \in \mathbb{N}$, where $\lambda_n, \nu_n \in [0,1]$ with certain appropriate conditions. If $\nu_n = 1$, then the Ishikawa iteration (1.8) coincides with the Mann type (1.7). Therefore, Ishikawa iterative method is a generalization of the Mann type. Berinde [3] used the Ishikawa iterative method (1.8) and proved the convergence theorem that approximates a unique fixed point of a Zamfirescu mapping in Banach spaces without any restriction on $\nu_n \in [0,1]$. Although the Ishikawa method is a two-step type, three-step iterative methods have been studied by Noor [28], Dashputre and Diwan [11], Phuengrattana and Suantai [29], and Kondo [17].

Let H be a real Hilbert space with the norm $\|\cdot\|$ induced from an inner product and let C be a nonempty, closed, and convex subset of H. For a mapping $T : C \to H$, we use the notation

$$F(T) = \{x \in C : x = Tx\}$$

to represent the set of fixed points. A mapping $T : C \to H$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping T such that $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if

$$\|Tx - p\| \leq \|x - p\| \quad \text{for all } x \in C \text{ and } p \in F(T). \tag{1.9}$$

A mapping T is called *demiclosed* if

$$x_n - Tx_n \to 0 \text{ and } x_n \rightharpoonup p \Longrightarrow p \in F(T), \tag{1.10}$$

where $x_n \rightharpoonup p$ denotes the weak convergence of the sequence $\{x_n\}$ to a point p. It is often said that $I - T$ is demiclosed iff (1.10) holds, where I is the identity mapping. Nonexpansive mappings are quasi-nonexpansive and demiclosed if they have fixed points. For other types of quasi-nonexpansive and demiclosed mappings, see Appendix in Kondo [22].

Recently, Kondo [22] proved the following theorem:

Theorem 1.3 ([22]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H and let $S, T : C \to C$ be quasi-nonexpansive and demiclosed mappings such that $F(S) \cap F(T) \neq \emptyset$. Denote by $P_{F(S)\cap F(T)}$ the metric projection from H onto $F(S) \cap F(T)$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$. Define a sequence $\{x_n\}$ in C by the following rule:*

$$x_1 \in C \text{ is given}, \quad x_{n+1} = a_n y_n + b_n S z_n + c_n T w_n$$

for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in C that satisfy the following conditions:

$$\|y_n - q\| \leq \|x_n - q\|\,, \ \|z_n - q\| \leq \|x_n - q\|\,, \ \|w_n - q\| \leq \|x_n - q\| \tag{1.11}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$,

$$x_n - y_n \to 0,\ x_n - z_n \to 0,\ \text{and}\ x_n - w_n \to 0\,. \tag{1.12}$$

Then, the sequence $\{x_n\}$ converges weakly to a point $\widehat{x} \in F(S) \cap F(T)$, where $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

Theorem 1.3 generates infinite numbers of iterative schemes. For example, consider the following:

$$\begin{aligned} z_n &= \eta_n' x_n + \theta_n' S x_n + \iota_n' T x_n, \\ y_n &= \eta_n z_n + \theta_n S z_n + \iota_n T z_n, \\ x_{n+1} &= a_n y_n + b_n S y_n + c_n T y_n, \end{aligned} \tag{1.13}$$

where an initial point $x_1 \in C$ is given arbitrarily. It is required that the coefficients of convex combinations η_n and η_n' converge to 1. This iterative scheme (1.13) is a three-step type. It can be verified that $\{y_n\}$ in (1.13) satisfies the conditions $\|y_n - q\| \leq \|x_n - q\|$ and $x_n - y_n \to 0$. Consequently, according to Theorem 1.3, the sequence $\{x_n\}$ converges weakly to a common fixed point of S and T. Many other iterative schemes are generated from Theorem 1.3; see, e.g., Section 4 in [22]. Thus, we call this method an *iterative scheme generating method (ISGM)*. This method has been developed to produce various types of iterative schemes in Hilbert spaces; see [16, 18, 19, 20, 23]. However, it has not yet been applied to contraction type mappings in Banach spaces.

In this study, we establish an ISGM for contraction type mappings including Zamfirescu mapping (1.5) and Hardy and Rogers type mappings (2.1) in arbitrary real Banach spaces. Theorem 1.2 is extended by incorporating the ISGM. It is the first attempt to establish the ISGM in Banach spaces. Although required conditions such as (1.12) are discarded in our theorems, the types of mappings are restricted to contraction types rather than nonexpansive types. In the rest of this paper, we prepare some lemmas in Section 2. Section 3 establishes the main theorems. Our main theorems apply to contraction mappings (1.1), Kannan mappings (1.3), and Chatterjea mappings (1.4). In Section 4, some corollaries are presented to show a wide variety of iterative schemes generated from the main theorems in this study. Finally, in Section 5, we introduce an application of this study to a variational inequality problem (VIP). As VIPs directly connect with optimization problems, various iterative schemes addressed in this work are of direct utility in optimization techniques.

2. Lemmas

This section prepares three lemmas. The first addresses Hardy and Rogers type mappings:

Lemma 2.1. *Let T be a self-mapping define on a metric space X such that $F(T) \neq \emptyset$. Assume that there exist $\alpha, \beta, \delta, \varepsilon \in [0,1]$ such that $\alpha+\beta+\delta+\varepsilon < 1$ and*

$$d(Tx, Ty) \leq \alpha d(x,y) + \beta\left(\frac{1}{2}d(x,Tx) + \frac{1}{2}d(y,Ty)\right) + \delta d(x,Ty) + \varepsilon d(Tx,y) \tag{2.1}$$

for all $x, y \in X$. Then, there exists $\rho \in (0,1)$ such that

$$d(Tx,p) \leq \rho d(x,p) \quad \text{for all } x \in X, \text{ where } p \in F(T). \tag{2.2}$$

Proof. A fixed point of the mapping T with the condition (2.1) is uniquely determined. Indeed, if $p, q \in F(T)$, then it follows from (2.1) that

$$\begin{aligned} & d(p,q) \\ & = d(Tp,Tq) \\ & \leq \alpha d(p,q) + \beta\left(\frac{1}{2}d(p,Tp) + \frac{1}{2}d(q,Tq)\right) + \delta d(p,Tq) + \varepsilon d(Tp,q) \\ & = (\alpha+\delta+\varepsilon) d(p,q). \end{aligned}$$

As $\alpha+\delta+\varepsilon < 1$, we obtain $p = q$.

For $x \in X$ and $p \in F(T)$, it holds that

$$\begin{aligned} & d(Tx,p) \\ & = d(Tx,Tp) \\ & \leq \alpha d(x,p) + \beta\left(\frac{1}{2}d(x,Tx) + \frac{1}{2}d(p,Tp)\right) + \delta d(x,Tp) + \varepsilon d(Tx,p) \\ & = \alpha d(x,p) + \beta\left(\frac{1}{2}d(x,Tx)\right) + \delta d(x,p) + \varepsilon d(Tx,p) \\ & \leq \alpha d(x,p) + \beta\left(\frac{1}{2}d(x,p) + \frac{1}{2}d(p,Tx)\right) + \delta d(x,p) + \varepsilon d(Tx,p). \end{aligned}$$

From this,

$$\left(1 - \frac{\beta}{2} - \varepsilon\right) d(Tx,p) \leq \left(\alpha + \frac{\beta}{2} + \delta\right) d(x,p).$$

Consequently,

$$d(Tx,p) \leq \frac{\alpha + \frac{\beta}{2} + \delta}{1 - \frac{\beta}{2} - \varepsilon} d(x,p).$$

Defining

$$\rho = \frac{\alpha + \frac{\beta}{2} + \delta}{1 - \frac{\beta}{2} - \varepsilon} \in (0,1),$$

we obtain the desired result. $\square$

Remark 2.1. *Three remarks are provided here:*

(1) *A mapping with* (2.1) *satisfies the condition* (1.6) *for the Hardy and Rogers mappings and hence, it is a particular case of the Hardy and Rogers mappings.*

(2) *Contraction mappings, Kannan mappings, and Chatterjea mappings satisfy the condition* (2.1) *and therefore, these types of mappings possess the property* (2.2)*.*

(3) *We compare the condition* (2.2) *with* (1.9)*. The condition* (2.2) *can be interpreted as the contraction version of the condition for quasi-nonexpansive mappings.*

Next lemma shows that Zamfirescu mappings (1.5) also fulfill the condition (2.2):

Lemma 2.2. *Let $T : X \to X$ be a Zamfirescu mapping with a fixed point $p \in F(T)$, where X is a metric space. Then, there exists $\rho \in (0, 1)$ that satisfies the condition* (2.2)*.*

Proof. As T is a Zamfirescu mapping (1.5), there are $a \in (0, 1)$ and $b, c \in \left(0, \frac{1}{2}\right)$ such that either of the three conditions $(Z1)$–$(Z3)$ holds. Define

$$\rho = \max\left\{a,\ \frac{b}{1-b},\ \frac{c}{1-c}\right\} \in (0, 1).$$

Choose $x \in X$ arbitrarily. First, assume that $(Z1)$ holds for $x \in X$ and the unique fixed point p of T. Then, (2.2) can be ascertained as follows:

$$d(Tx, p) = d(Tx, Tp) \leq ad(x, p) \leq \rho d(x, p).$$

Next, assume that $(Z2)$ holds for $x \in X$ and $p \in F(T)$. Then, from the condition $(Z2)$, it follows that

$$\begin{aligned} &d(Tx, p) = d(Tx, Tp) \\ &\leq b\left(d(x, Tx) + d(p, Tp)\right) = bd(x, Tx) \leq b\left(d(x, p) + d(p, Tx)\right). \end{aligned}$$

Consequently, we have

$$d(Tx, p) \leq \frac{b}{1-b} d(x, p) \leq \rho d(x, p).$$

Finally, assume that $(Z3)$ holds for $x \in X$ and $p \in F(T)$. Using the condition $(Z3)$ yields

$$\begin{aligned} &d(Tx, p) = d(Tx, Tp) \\ &\leq c\left(d(x, Tp) + d(Tx, p)\right) = c\left(d(x, p) + d(Tx, p)\right). \end{aligned}$$

Thus,

$$d(Tx, p) \leq \frac{c}{1-c} d(x, p) \leq \rho d(x, p).$$

This completes the proof. □

The next lemma is known in the literature; see, e.g., Lemma 2.3 in Kondo [17]. However, we provide a proof for completeness.

Lemma 2.3. *Let $\Lambda > 0$ and let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0,1]$. Suppose that $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$ and $\Lambda(1-\lambda_n) < 1$ for all $n \in \mathbb{N}$. Then, $\prod_{n=1}^{\infty}(1-\Lambda(1-\lambda_n)) = 0$.*

Proof. Define $P_n = \prod_{i=1}^{n}(1-\Lambda(1-\lambda_i))$. As $\Lambda(1-\lambda_i) < 1$, it holds that $P_n > 0$. We aim to show that $P_n \to 0$. Note that the following inequality holds in general: $\log(1-x) \leq -x$ for all $x < 1$. Using this for $x = \Lambda(1-\lambda_i)\,(< 1)$ yields

$$\begin{aligned}\log P_n &= \sum_{i=1}^{n} \log(1-\Lambda(1-\lambda_i)) \\ &\leq \sum_{i=1}^{n}\{-\Lambda(1-\lambda_i)\} = -\Lambda\sum_{i=1}^{n}(1-\lambda_i).\end{aligned}$$

Therefore, we have $0 < P_n \leq \exp(-\Lambda\sum_{i=1}^{n}(1-\lambda_i))$. From the hypotheses $\sum_{i=1}^{\infty}(1-\lambda_i) = \infty$ and $\Lambda > 0$, we obtain $\exp(-\Lambda\sum_{i=1}^{n}(1-\lambda_i)) \to 0$ as $n \to \infty$. Consequently, we obtain $P_n \to 0$ as $n \to \infty$. This completes the proof. □

3. Main result

In this section, we prove convergence theorems for contraction type mappings with the condition (2.2). One result in this section generalizes Theorem 1.2; see Remark 3.2. The Picard iterative method (1.2) in Theorem 1.1 is generalized. Also, main theorems in this section complement Theorem 1.3, which is a result for nonexpansive type mappings in the setup of Hilbert spaces.

Theorem 3.1. *Let C be a nonempty, closed, and convex subset of a real Banach space E. Let T be a self-mapping defined on C such that $F(T) = \{p\}$. Suppose that the mapping T satisfies the condition (2.2) with $\rho \in (0,1)$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0,1]$ such that $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$. Define a sequence $\{x_n\}$ in C by the following rule:*

$$\begin{aligned}&x_1 \in C \text{ is given,}\\ &x_{n+1} = \lambda_n y_n + (1-\lambda_n)Tz_n\end{aligned}$$

for all $n \in \mathbb{N}$, where $\{y_n\}$ and $\{z_n\}$ are sequences in C that satisfy

$$\|y_n - p\| \leq \|x_n - p\| \text{ and } \|z_n - p\| \leq \|x_n - p\| \tag{3.1}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges in norm to the unique fixed point $p \in F(T)$.

Proof. As $\{z_n\}$ is a sequence in C and $T : C \to C$, it holds that $Tz_n \in C$. As $\{y_n\} \subset C$ and C is convex, $\{x_n\}$ is properly defined as a sequence in C.

From (2.2) and (3.1), it follows that

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&= \|\lambda_n y_n + (1-\lambda_n) T z_n - p\| \\
&\le \lambda_n \|y_n - p\| + (1-\lambda_n) \|T z_n - p\| \\
&\le \lambda_n \|y_n - p\| + (1-\lambda_n) \rho \|z_n - p\| \\
&\le \lambda_n \|x_n - p\| + (1-\lambda_n) \rho \|x_n - p\| \\
&= \{\lambda_n + (1-\lambda_n) \rho\} \|x_n - p\| \\
&= \{\lambda_n + 1 - 1 + (1-\lambda_n) \rho\} \|x_n - p\| \\
&= \{1 - (1-\rho)(1-\lambda_n)\} \|x_n - p\| \\
&\le \{1 - (1-\rho)(1-\lambda_n)\} \{1 - (1-\rho)(1-\lambda_{n-1})\} \|x_{n-1} - p\| \\
&\le \cdots \\
&\le \left(\prod_{i=1}^{n} \{1 - (1-\rho)(1-\lambda_i)\} \right) \|x_1 - p\|
\end{aligned}$$

As $\sum_{i=1}^{\infty} (1-\lambda_i) = \infty$ is assumed, we can apply Lemma 2.3 for $\Lambda = 1 - \rho > 0$. Thus, it holds that

$$\prod_{i=1}^{n} (1 - (1-\rho)(1-\lambda_i)) \to 0 \quad \text{as } n \to \infty$$

and we obtain $x_n \to p$. The proof is completed. □

Remark 3.1. *Compare the condition* (3.1) *in Theorem* 3.1 *with* (1.11) *and* (1.12) *in Theorem* 1.3. *In Theorem* 3.1, *conditions such as* (1.12) *are dispensable, whereas the mapping T must be a contraction type that satisfies the condition* (2.2).

According to Lemmas 2.1 and 2.2, Theorem 3.1 applies to Hardy and Rogers type mappings (2.1) and Zamfirescu mappings (1.5):

Theorem 3.2. *Let C be a nonempty, closed, and convex subset of a real Banach space E. Let $T : C \to C$ be a Hardy and Rogers type mapping characterized by the condition* (2.1) *such that $F(T) = \{p\}$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0,1]$ such that $\sum_{n=1}^{\infty} (1-\lambda_n) = \infty$. Define a sequence $\{x_n\}$ in C by the following rule:*

$$x_1 \in C \text{ is given,}$$
$$x_{n+1} = \lambda_n y_n + (1-\lambda_n) T z_n$$

for all $n \in \mathbb{N}$, where $\{y_n\}$ and $\{z_n\}$ are sequences in C that satisfy

$$\|y_n - p\| \le \|x_n - p\| \text{ and } \|z_n - p\| \le \|x_n - p\|$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges in norm to the unique fixed point p.

Theorem 3.3. *Let C be a nonempty, closed, and convex subset of a real Banach space E. Let $T : C \to C$ be a Zamfirescu mapping such that $F(T) = \{p\}$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0,1]$ such that $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$. Define a sequence $\{x_n\}$ in C by the following rule:*

$$
\begin{aligned}
x_1 &\in C \text{ is given,}\\
x_{n+1} &= \lambda_n y_n + (1-\lambda_n) T z_n
\end{aligned}
$$

for all $n \in \mathbb{N}$, where $\{y_n\}$ and $\{z_n\}$ are sequences in C that satisfy

$$\|y_n - p\| \leq \|x_n - p\| \text{ and } \|z_n - p\| \leq \|x_n - p\|$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges in norm to the unique fixed point p.

Remark 3.2. *Set $y_n = z_n = x_n$ for all $n \in \mathbb{N}$ in Theorem 3.3. Then, Theorem 1.2 is derived.*

4. Corollaries

Each theorem from the previous section generates various iterative schemes. In this section, to save space, we will focus exclusively on Theorem 3.3 and its derived variations.

First, set $\lambda_n = 0$ for all $n \in \mathbb{N}$ in Theorem 3.3. The required condition $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$ is then fulfilled, yielding the following result:

Corollary 4.1. *Let C be a nonempty, closed, and convex subset of a real Banach space E. Let $T : C \to C$ be a Zamfirescu mapping such that $F(T) = \{p\}$. Define a sequence $\{x_n\}$ in C by the following rule:*

$$x_{n+1} = T z_n \quad \text{for all } n \in \mathbb{N},$$

where $x_1 \in C$ is given and $\{z_n\}$ is a sequence in C that satisfies

$$\|z_n - p\| \leq \|x_n - p\|$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges in norm to the unique fixed point p.

As in the case of Theorems 3.1–3.3, Corollary 4.1 also generates infinitely many iterative schemes. One variation is as follows:

Corollary 4.2. *Let C be a nonempty, closed, and convex subset of a real Banach space E. Let $T : C \to C$ be a Zamfirescu mapping such that $F(T) = \{p\}$. Let $\{\nu_n\}$ be a sequence of real numbers in the interval $[0,1]$. Define a sequence $\{x_n\}$ in C by the following rule:*

$$
\begin{aligned}
x_1 &\in C \text{ is given,}\\
z_n &= \nu_n x_n + (1-\nu_n) T x_n,\\
x_{n+1} &= T z_n,
\end{aligned}
$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges in norm to the unique fixed point p.

Corollary 4.2 presents a two-step iterative method. Its proof is straightforward since the sequence $\{z_n\}$ satisfies the condition $\|z_n - p\| \leq \|x_n - p\|$. For illustration purposes, however, we provide a proof that does not rely on Theorem 3.3:

Proof. We first observe that

$$\|z_n - p\| \leq \|x_n - p\| \quad \text{for all } n \in \mathbb{N}.$$

According to Lemma 2.2, the Zamfirescu mapping T satisfies condition (2.2). Consequently, we can derive the following:

$$\begin{aligned} \|z_n - p\| &= \|\nu_n x_n + (1 - \nu_n) T x_n - p\| \\ &\leq \nu_n \|x_n - p\| + (1 - \nu_n) \|T x_n - p\| \\ &\leq \nu_n \|x_n - p\| + (1 - \nu_n) \rho \|x_n - p\| \\ &\leq \nu_n \|x_n - p\| + (1 - \nu_n) \|x_n - p\| \\ &= \|x_n - p\|, \end{aligned}$$

which confirms our initial assertion. Applying this, we obtain

$$\|x_{n+1} - p\| = \|T z_n - p\| \leq \rho \|z_n - p\| \leq \rho \|x_n - p\|.$$

Iterating this inequality, we find

$$\|x_{n+1} - p\| \leq \rho \|x_n - p\| \leq \rho^2 \|x_{n-1} - p\| \leq \cdots \leq \rho^n \|x_1 - p\| \to 0$$

as $n \to \infty$. Thus, we can conclude that $\{x_n\}$ converges to the unique fixed point p. □

The next corollary is a three-step and split type, which is derived from Theorem 3.3:

Corollary 4.3. *Let C be a nonempty, closed, and convex subset of a real Banach space E and let $T : C \to C$ be a Zamfirescu mapping such that $F(T) = \{p\}$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0, 1]$ such that $\sum_{n=1}^{\infty} (1 - \lambda_n) = \infty$. Let $\{\mu_n\}$, $\{\mu'_n\}$, $\{\nu_n\}$, and $\{\nu'_n\}$ be sequences of real numbers in $[0, 1]$. Define a sequence $\{x_n\}$ in C by the following rule:*

$$\begin{aligned} & x_1 \in C \text{ is given}, \\ & y'_n = \mu'_n x_n + \left(1 - \mu'_n\right) T x_n, \\ & y_n = \mu_n y'_n + (1 - \mu_n) T y'_n, \\ & z'_n = \nu'_n x_n + \left(1 - \nu'_n\right) T x_n, \\ & z_n = \nu_n z'_n + (1 - \nu_n) T z'_n, \\ & x_{n+1} = \lambda_n y_n + (1 - \lambda_n) T z_n \end{aligned} \tag{4.1}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges in norm to the unique fixed point $p \in F(T)$.

Proof. As E is complete and C is closed in E, C is also complete. As $T : C \to C$ is a Zamfirescu mapping, it has a unique fixed point p. According to Theorem 3.3, it is sufficient to show that $\|y_n - p\| \leq \|x_n - p\|$ and $\|z_n - p\| \leq \|x_n - p\|$ for all $n \in \mathbb{N}$. From Lemma 2.2, we can employ the condition (2.2) with $\rho \in (0, 1)$. Consequently, it holds that

$$\begin{aligned} \left\|y_n' - p\right\| &= \left\|\mu_n' x_n + \left(1 - \mu_n'\right) T x_n - p\right\| \\ &\leq \mu_n' \left\|x_n - p\right\| + \left(1 - \mu_n'\right) \left\|T x_n - p\right\| \\ &\leq \mu_n' \left\|x_n - p\right\| + \left(1 - \mu_n'\right) \rho \left\|x_n - p\right\| \\ &\leq \mu_n' \left\|x_n - p\right\| + \left(1 - \mu_n'\right) \left\|x_n - p\right\| \\ &= \left\|x_n - p\right\|. \end{aligned}$$

Using this yields $\|y_n - p\| \leq \|x_n - p\|$. Indeed,

$$\begin{aligned} \left\|y_n - p\right\| &= \left\|\mu_n y_n' + (1 - \mu_n) T y_n' - p\right\| \\ &\leq \mu_n \left\|y_n' - p\right\| + (1 - \mu_n) \left\|T y_n' - p\right\| \\ &\leq \mu_n \left\|y_n' - p\right\| + (1 - \mu_n) \left\|y_n' - p\right\| \\ &\leq \mu_n \left\|x_n - p\right\| + (1 - \mu_n) \left\|x_n - p\right\| \\ &= \left\|x_n - p\right\|. \end{aligned}$$

Similarly, we can verify that $\|z_n - p\| \leq \|x_n - p\|$ for all $n \in \mathbb{N}$. Thus, we obtain the desired result. □

The iterative scheme in Corollary 4.3 is a three-step and split version. The construction of x_{n+1} is split into "y-part" and "z-part." Setting $\mu_n' = \nu_n' = 1$ in (4.1), we obtain $y_n' = z_n' = x_n$. Then, the following two-step and split type iterative scheme is deduced:

$$\begin{aligned} y_n &= \mu_n x_n + (1 - \mu_n) T x_n, \\ z_n &= \nu_n x_n + (1 - \nu_n) T x_n, \\ x_{n+1} &= \lambda_n y_n + (1 - \lambda_n) T z_n \end{aligned} \tag{4.2}$$

for all $n \in \mathbb{N}$, where $x_1 \in C$ is given arbitrarily. Furthermore, substituting $\mu_n = 1$ in (4.2), we have the Ishikawa iterative method (1.8).

We can also derive the following corollary, which is also a three-step type:

Corollary 4.4. *Let C be a nonempty, closed, and convex subset of a real Banach space E and let $T : C \to C$ be a Zamfirescu mapping such that $F(T) = \{p\}$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0, 1]$ such that $\sum_{n=1}^{\infty} (1 - \lambda_n) = \infty$. Let $\{\mu_n\}$, $\{\nu_n\}$, $\{\eta_n\}$, $\{\theta_n\}$, and $\{\iota_n\}$ be sequences of real numbers in $[0, 1]$. Define a sequence $\{x_n\}$ in C by the* *following rule:*

$$\begin{aligned} x_1 &\in C \text{ is given,} \\ z_n &= \eta_n x_n + \theta_n T x_n + \iota_n T^2 x_n + (1 - \eta_n - \theta_n - \iota_n) T^3 x_n, \\ y_n &= \mu_n z_n + \nu_n T z_n + (1 - \mu_n - \nu_n) T^2 z_n, \\ x_{n+1} &= \lambda_n y_n + (1 - \lambda_n) T y_n \end{aligned}$$

for all $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *converges in norm to the unique fixed point* $p \in F(T)$.

Proof. As C is complete, the Zamfirescu mapping $T : C \to C$ has a unique fixed point p. First, observe that $\|z_n - p\| \leq \|x_n - p\|$ for all $n \in \mathbb{N}$. From Lemma 2.2, T satisfies (2.2). Thus, we have

$$\begin{aligned} & \|z_n - p\| \\ = \ & \left\|\eta_n x_n + \theta_n T x_n + \iota_n T^2 x_n + (1 - \eta_n - \theta_n - \iota_n) T^3 x_n - p\right\| \\ \leq \ & \eta_n \|x_n - p\| + \theta_n \|T x_n - p\| + \iota_n \left\|T^2 x_n - p\right\| \\ & + (1 - \eta_n - \theta_n - \iota_n) \left\|T^3 x_n - p\right\| \\ \leq \ & \|x_n - p\|. \end{aligned}$$

Using this, we can obtain $\|y_n - p\| \leq \|x_n - p\|$ for all $n \in \mathbb{N}$. Therefore, from Theorem 3.3, the desired result holds. □

Approximation methods involving terms such as $T^2 z_n$ have been used by Maruyama *et al.* [25] to deal with a general class of mappings; see also Kondo [15] and papers cited therein. The following corollary also follows from Theorem 3.3, where other mappings appear in the statements. Recall the condition of quasi-nonexpansive mappings (1.9).

Corollary 4.5. *Let* C *be a nonempty, closed, and convex subset of a real Banach space* E. *Let* $T : C \to C$ *be a Zamfirescu mapping such that* $F(T) = \{p\}$. *Let* $U, V : C \to C$ *be qusi-nonexpansive mappings such that* $p \in F(U) \cap F(V)$. *Let* $\{\lambda_n\}$ *be a sequence of real numbers in the interval* $[0, 1]$ *such that* $\sum_{n=1}^{\infty} (1 - \lambda_n) = \infty$. *Let* $\{\mu_n\}$ *and* $\{\nu_n\}$ *be sequences of real numbers in* $[0, 1]$. *Define a sequence* $\{x_n\}$ *in* C *by the following rule:*

$$\begin{aligned} x_1 &\in C \text{ is given,} \\ y_n &= \mu_n x_n + \nu_n U x_n + (1 - \mu_n - \nu_n) V x_n, \\ x_{n+1} &= \lambda_n x_n + (1 - \lambda_n) T y_n \end{aligned}$$

for all $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *converges in norm to the unique fixed point* p *of* T.

Proof. As $p \in F(U) \cap F(V)$ and U and $V : C \to C$ are qusi-nonexpansive (1.9), it follows that

$$\begin{aligned}
&\|y_n - p\| \\
&= \|\mu_n x_n + \nu_n U x_n + (1 - \mu_n - \nu_n) V x_n - p\| \\
&\leq \mu_n \|x_n - p\| + \nu_n \|U x_n - p\| + (1 - \mu_n - \nu_n) \|V x_n - p\| \\
&\leq \mu_n \|x_n - p\| + \nu_n \|x_n - p\| + (1 - \mu_n - \nu_n) \|x_n - p\| \\
&= \|x_n - p\| .
\end{aligned}$$

Consequently, from Theorem 3.3, we obtain the desired result. □

It is not yet known what kind of quasi-nonexpansive mappings U and V should be used to increase the convergence speed.

5. Application

In this section, we consider a variational inequality problem (VIP) and show that the outcomes of this study can be applied to optimization techniques. For VIPs, see Noor [27], Yamada [37], Xu and Kim [36], and Truong *et al.* [35].

Let H be a real Hilbert space with an inner product $\langle \cdot, \ \cdot \rangle$ and the induced norm $\|\cdot\|$ and let C be a nonempty subset of H. For a mapping $A : C \to H$, we denote by

$$VI(C, A) = \{x \in C : \langle y - x, \ Ax \rangle \geq 0 \text{ for all } y \in C\}$$

the set of solutions of the VIP. The following types of mappings are frequently used in the literature:

(i) A mapping $A : C \to H$ is called *K-Lipschitz continuous* if there exists $K > 0$ such that

$$\|Ax - Ay\| \leq K \|x - y\| \quad \text{for all } x, y \in C. \tag{5.1}$$

If $K < 1$, A is a contraction mapping. If $K = 1$, A is a nonexpansive mapping. Let $A, B : C \to C$. If A is K-Lipschitz continuous and B is L-Lipschitz continuous, then the composite mapping $A \circ B$ is KL-Lipschitz continuous. The composite mapping $A \circ B$ is simply written as AB when no ambiguity arises.

(ii) A mapping $A : C \to H$ is termed *η-strongly monotone* if there exists $\eta > 0$ such that

$$\eta \|x - y\|^2 \leq \langle x - y, \ Ax - Ay \rangle \quad \text{for all } x, y \in C. \tag{5.2}$$

(iii) Let C be a nonempty, closed, and convex subset of H. For $x \in H$, there exists a unique element $u \in C$ such that

$$\|x - u\| \leq \|x - y\| \quad \text{for all } y \in C.$$

This correspondence from x to u is called the *metric projection* from H onto C and denoted by P_C. It is known that a metric projection P_C is nonexpansive and satisfies

$$\langle x - P_C x,\ P_C x - y\rangle \geq 0 \ \text{ for all } y \in C.$$

Conversely, if $u \in C$ and $\langle x - u,\ u - y\rangle \geq 0$ for all $y \in C$, then $u = P_C x$. For these points, see Lemma 2.1 in Kondo [21].

To solve VIPs via the fixed point theory, the following two lemmas are crucial. Although they are known in the literature (see, for instance, Lemmas 4.1 and 4.2 in Kondo [17]), we provide proofs for self-completeness. Denote by I the identity mapping defined on C.

Lemma 5.1. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H, let P_C be the metric projection from H onto C, and let A be a mapping from C into H. Then, it holds that $VI(C, A) = F(P_C(I - \sigma A))$ for all $\sigma > 0$.*

Proof. The desired result is verified as follows:

$$\begin{aligned}
& x \in F(P_C(I - \sigma A)) \\
& \quad \Longleftrightarrow x = P_C(x - \sigma Ax) \\
& \quad \Longleftrightarrow \langle (x - \sigma Ax) - x,\ x - y\rangle \geq 0 \text{ for all } y \in C \\
& \quad \Longleftrightarrow \langle Ax,\ x - y\rangle \leq 0 \text{ for all } y \in C \\
& \quad \Longleftrightarrow x \in VI(C, A).
\end{aligned}$$

This concludes the proof. □

Lemma 5.2. *Let $A : C \to H$ be an η-strongly monotone and K-Lipschitz continuous mapping, where C is a nonempty subset of H and $0 < \eta < K$. Then, for $\sigma \in \left(0,\ \frac{2\eta}{K^2}\right)$, $I - \sigma A$ is a contraction mapping from C into H.*

Proof. Let $x, y \in C$. As A is η-strongly monotone and K-Lipschitz continuous, from (5.2) and (5.1), it holds that

$$\begin{aligned}
& \|(I - \sigma A)x - (I - \sigma A)y\|^2 \\
& = \|x - y - \sigma(Ax - Ay)\|^2 \\
& = \|x - y\|^2 - 2\sigma\langle x - y,\ Ax - Ay\rangle + \sigma^2\|Ax - Ay\|^2 \\
& \leq \|x - y\|^2 - 2\sigma\eta\|x - y\|^2 + \sigma^2K^2\|x - y\|^2 \\
& = \left\{1 - \sigma\left(2\eta - \sigma K^2\right)\right\}\|x - y\|^2.
\end{aligned}$$

Using the conditions $0 < \sigma < 2\eta/K^2$ and $0 < \eta < K$, we can prove that

$$0 < 1 - \sigma\left(2\eta - \sigma k^2\right) < 1.$$

This indicates that $I - \sigma A$ is a contraction mapping. □

As a metric projection is nonexpansive, the self-mapping $P_C(I - \sigma A)$ on C is a contraction mapping in the situation of Lemma 5.2. Furthermore, according to Lemma 5.1, it holds that $VI(C, A) = F(P_C(I - \sigma A))$. Therefore, from the Banach contraction principle (Theorem 1.1), the set

$VI(C,A)\,(= F(P_C(I-\sigma A)))$ consists of only one element and Picard iterative method is effective to approximate the unique element of $VI(C,A)$. More general types of iterative schemes introduced in this study are also available. Here, we present a two-step and split version as an application of Corollary 4.3 with (4.2):

Theorem 5.1. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Denote by P_C the metric projection from H onto C. Let $A : C \to H$ be an η-strongly monotone and K-Lipschitz continuous mapping, where $0 < \eta < K$. Let $\sigma \in \left(0, \frac{2\eta}{K^2}\right)$ and let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0,1]$ such that $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$. Let $\{\mu_n\}$ and $\{\nu_n\}$ be sequences of real numbers in $[0,1]$. Define a sequence $\{x_n\}$ in C by the following rule:*

$$\begin{aligned} x_1 &\in C \text{ is given,} \\ y_n &= \mu_n x_n + (1-\mu_n) P_C(I-\sigma A) x_n, \\ z_n &= \nu_n x_n + (1-\nu_n) P_C(I-\sigma A) x_n, \\ x_{n+1} &= \lambda_n y_n + (1-\lambda_n) P_C(I-\sigma A) z_n \end{aligned} \tag{5.3}$$

for all $n \in \mathbb{N}$, where I is the identity mapping defined on C. Then, $\{x_n\}$ converges in norm to a unique element of $VI(C,A)$.

Proof. From Lemma 5.2, $I - \sigma A$ is a contraction mapping from C into H. As the metric projection P_C is nonexpansive, the composite mapping $P_C(I-\sigma A)$ is also a contraction mapping from C into itself and consequently, it has a unique fixed point $p \in F(P_C(I-\sigma A))$. From Lemma 5.1, $p \in F(P_C(I-\sigma A)) = VI(C,A)$. As $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$ is assumed, from Corollary 4.3 with (4.2), the sequence $\{x_n\}$ converges in norm to $p \in F(P_C(I-\sigma A)) = VI(C,A)$. This completes the proof. ☐

Setting $\mu_n = \nu_n = 1$ and $\lambda_n = 0$ in (5.3), the Picard iterative method is deduced. Therefore, the iterative scheme (5.3) is more general than the Picard method.

Acknowledgements. The author would like to thank Ryosui Academic Foundation of Shiga university for its financial support.

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