DISCUSSION PAPER SERIES E

Discussion Paper No. E-42

Generalizing the Permanent Income Hypothesis with Homothetic Robust Epstein-Zin Utility

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March 2025

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March 13, 2025

Abstract

This study assumes homothetic robust Epstein-Zin utility and a market model in which the state vector follows a general Markovian diffusion process. The study derives an optimal consumption rule as a generalized permanent income hypothesis (PIH) and demonstrates that the marginal propensity to consume out of the total wealth can be decomposed into five terms, four of which are functions of the state vector. This indicates that the generalized PIH does not ensure the stability of consumption as implied by Friedman's PIH. Additionally, the study derives the conditional expected growth rate of consumption and demonstrates that it can be decomposed into four terms. The first term is the difference between the interest and discount rates. The second and third terms are both positive and interpreted as the "effect of precautionary savings on risk" and "effect of precautionary savings on ambiguity," respectively. The fourth term is interpreted as the "effect of timing of resolution of uncertainty." The stability of the expected growth rate of consumption is not ensured either, because all these terms are functions of the state vector.
Keywords: Homothetic robust utility, Permanent income hypothesis, Growth rate of consumption.

1 Introduction

The permanent income hypothesis (PIH) of Friedman [7] shows that the optimal consumption is proportional to the total wealth that is the sum of nonhuman wealth and "human wealth," the discounted expected value of future labor income. As Wang [20] critiques, this traditional definition of human wealth ignores risk. He proposes the concept of "risk-adjusted human wealth," utilizing risk-adjusted probability to account for risk. This study uses this concept and refers to it as human wealth. Wang [20] then

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derives an optimal consumption rule, which he calls the "generalized PIH." However, Wang [20] assumes CARA utility instead of standard CRRA utility and models the securities market and labor income in a simplified manner. In this sense, this optimal consumption rule is far from a generalized PIH. This study aims to derive the optimal consumption rule as a generalized PIH under a more appropriate utility and more general market model and to elucidate the theoretical structure of the marginal propsensity to consume (MPC) and of the expected growth rate of optimal consumption.

In this study, CRRA utility is generalized in two directions: Epstein-Zin (EZ) utility (Epstein and Zin [6]) and homothetic robust (HR) utility (Maenhout [13]). EZ utility separates the coefficients of the relative risk aversion and elasticity of intertemporal substitution (EIS). Wang, Wang, and Yang [21] assume EZ utility and derive an optimal consumption rule. HR utility, in which Knightian uncertainty is considered, is characterized by the subjective discount rate, relative risk aversion, and relative ambiguity aversion. HR utility is used in robust portfolio studies, such as Skiadas [18], Maenhout [13], [14], Liu [12], Branger, Larsen, and Munk [2], Munk and Rubtsov [15], Yi, Viens, Law, and Li [22], and Kikuchi and Kusuda [9].[1] However, human wealth is not considered in these studies. Homothetic robust Epstein-Zin (HREZ) utility (Maenhout [13]) is a generalization of EZ and HR utility, and is characterized by the subjective discount rate, EIS ψ, relative risk aversion γ, and relative ambiguity aversion θ. Skiadas [18] and Kusuda [11] show that HREZ utility is stochastic differential utility. Schroder and Skiadas [16] show that if $\gamma > 1 > \psi > 0$ or $0 < \gamma < 1 < \psi$ then EZ utility is well defined. Based on the results of many empirical analyses, this study assumes HREZ utility with $\gamma > 1 > \psi > 0$. Note that the assumed HREZ utility includes HR utility (*i.e.*, $\gamma\psi = 1$), EZ utility (*i.e.*, $\theta = 0$), and CRRA utility (*i.e.*, $\gamma\psi = 1, \theta = 0$) as special cases. This study also assumes a market model where the state vector follows a general Markovian diffusion process, and the risk-free rate and volatilities of capital assets and labor income are functions of the state vector. The main results are summarized as follows.

First, I derive an expression of the generalized PIH in which the optimal consumption is the product of the MPC and total wealth and demonstrate that the MPC out of human wealth coincides with that out of nonhuman wealth. Friedman [7] conjectures that the MPC out of human wealth is lower than that out of nonhuman wealth. This conjecture is strongly supported by various studies, including Hall [8]. I show that the traditional definition-based human wealth is greater than our human wealth and that the MPC out of traditional definition-based human wealth is lower than that out of nonhuman wealth.

[1]Kikuchi and Kusuda [10] generalize HR utility so that relative ambiguity aversion depends on age; however, the utility functional is no longer homothetic.

Second, using the partial differential equation (PDE) for the indirect utility function derived from the optimality condition of the agent's consumption/investment problem, I derive an equation that the decomposes MPC into five terms. The first two terms represent a weighted average of the discount rate and risk-free rate, where the weight on the discount rate is the EIS. These terms are presented in previous studies, including Wang *et. al* [21]. The third term represents the effect of the total wealth variance. The fourth and fifth terms represent the intertemporal risk hedging effect and intertemporal ambiguity hedging effect, respectively. Note that the four terms, excluding the first term, are functions of the state vector. Given the possibility of substantial fluctuations in the MPC in response to changes in the state vector, the generalized PIH does not theoretically ensure the stability of consumption implied by Friedman's PIH. Its stability is subject to empirical analysis.

Third, I derive the conditional expected growth rate of consumption from the PDE for the indirect utility function and demonstrate that it can be decomposed into four terms. The first term is proportional to the difference between the interest and discount rates. The second and third terms are proportional to the conditional variance of the growth rate of consumption. They are both positive and interpreted as the "effect of precautionary savings on risk" and "effect of precautionary savings on ambiguity," respectively. The fourth term is proportional to the conditional variance of total wealth return. It is negative (resp., positive) if the agent prefers early (resp., late) resolution. The fourth term is interpreted as the "effect of timing of resolution of uncertainty." The stability of the expected growth rate of consumption is not theoretically ensured either, because all these terms depend on the state vector.

The remainder of this paper is organized as follows. Section 2 introduces HREZ utility and the market model. Section 3 demonstrates the generalized PIH. Section 4 derives the conditional expected growth rate of consumption. Section 5 concludes the study. Appendix presents the proofs of the lemmas and propositions.

2 Basic Setting

I introduce the environment of economy, HREZ utility, and market model. Then, I derive the dynamics of the total wealth.

2.1 Environment

I consider an infinitely lived agent and frictionless markets over the time span $[0, \infty)$. The agent's subjective probability and information structure are modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathrm{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,\infty)}$ is the natural filtration generated by an N-dimensional standard Brownian motion B_t. I indicate the expectation operator under P with E and the conditional expectation operator given $\mathcal{F}_t$ with E_t.

There are markets for consumption goods and financial and nonfinancial capital assets at every date $t \in [0, \infty)$, and the consumer price index p_t is observed. The traded assets are the instantaneously nominal risk-free security called the *money market account*; and $N-1$ types of capital assets. There is also a labor market in which agents' human capital is rented to employers at a wage in each period $[t, t+dt]$. At every date t, P_t and S_t^j denote the prices of the money market account and j-th capital asset, respectively. Let A' and I_N denote the transpose of A and the $N \times N$ identity matrix, respectively.

2.2 HREZ utility

First, I consider the following EZ utility.

$$V_t = \mathrm{E}_t^{\xi} \left[\int_t^{\infty} f(c_s, V_s) ds \right], \tag{2.1}$$

where f is the normalized aggregator shown in Duffie and Epstein [4]) of the form:

$$f(c, v) = \beta \frac{c^{1-\psi^{-1}}}{1-\psi^{-1}} \left((1-\gamma) v \right)^{1 - \frac{1-\psi^{-1}}{1-\gamma}} - \beta \frac{1-\gamma}{1-\psi^{-1}} v, \tag{2.2}$$

where β is the subjective discount rate, γ is the relative risk aversion, and ψ is the EIS. Schroder and Skiadas [16] show that if $\gamma > 1 > \psi > 0$, or $0 < \gamma < 1 < \psi$ then EZ utility is well defined. Numerous empirical analyses indicate that $\gamma > 1$ and $1 > \psi > 0$. Therefore, I assume that $\gamma > 1 > \psi > 0$.

$$U(c) = \inf_{\mathrm{P}^{\xi} \in \mathbb{P}} \mathrm{E}^{\xi} \left[\int_0^{\infty} \left(f(c_t, V_t^{\xi}) + \frac{(1-\gamma) V_t^{\xi}}{2\theta} |\xi_t|^2 \right) dt \right], \tag{2.3}$$

where E^{ξ} is the expectation under P^{ξ}, U_t^{ξ} is the continuation utility process defined recursively as follows:

$$V_t^{\xi} = \mathrm{E}_t^{\xi} \left[\int_t^{\infty} \left(f(c_s, V_s^{\xi}) + \frac{(1-\gamma) V_s^{\xi}}{2\theta} |\xi_s|^2 \right) ds \right], \tag{2.4}$$

Assumption 1. *The agent possesses the HREZ utility of the form (2.3) such that* $\gamma > 1 > \psi > 0$.

2.3 Market Model

I assume the following market model.

Assumption 2. *1. State vector process X_t satisfies the following SDE:*

$$dX_t = \mu(X_t)\, dt + \Sigma(X_t)\, dB_t. \tag{2.5}$$

2. The nominal risk-free rate and market price of risk at t are functions of X_t and expressed as $r(X_t)$ and $\lambda(X_t)$, respectively.

3. The volatility of j-th capital asset and the agent's human capital at t are functions of X_t and expressed as $\sigma_j(X_t)$ and $\sigma_h(X_t)$, respectively.

4. The consumer price index p_t satisfies

$$\frac{dp_t}{p_t} = i(X_t)\, dt + \sigma_p(X_t)' dB_t, \qquad p_0 = 1, \tag{2.6}$$

5. Markets are complete and arbitrage-free.

Let P^* and E^* denote the risk-neutral measure and expectation operator under P^*, respectively. Let L_t denote the representative agent's nominal labor income at time t. Friedman [7] and Hall [8] define human wealth H^{F} by

$$H^{\mathrm{F}}_t = \mathrm{E}_t \left[\int_t^\infty \exp\left(-\int_t^s r(X_u) du \right) L_s ds \right]. \tag{2.7}$$

As Wang [20] points out, this definition of human wealth ignores risk. Following Wang [20], I define nominal human wealth[2] by

$$H_t = \mathrm{E}^*_t \left[\int_t^\infty \exp\left(-\int_t^s r(X_u) du \right) L_s ds \right]. \tag{2.8}$$

The premium on human capital is naturally assumed to be positive. Then, $H^{\mathrm{F}}_t > H_t$, and Friedman's definition overestimates the value of human wealth.

Let $(\vartheta, (\vartheta^j))$ denote a portfolio. Then, nominal nonhuman wealth and nominal total wealth at t are given by

$$A_t = \vartheta_t P_t + \sum_{j=1}^{N-1} \vartheta^j_t S^j_t. \tag{2.9}$$

$$W_t = A_t + H_t. \tag{2.10}$$

Let Y_t denote a nominal value process. Then, the real value process $\bar{Y}_t$ is defind by $\bar{Y}_t = \dfrac{Y_t}{p_t}$. I define the real market price of risk and real risk-free rate by

$$\bar{\lambda}(X_t) = \lambda(X_t) - \sigma_p(X_t), \tag{2.11}$$

$$\bar{r}(X_t) = r(X_t) - i(X_t) + \lambda(X_t)' \sigma_p(X_t), \tag{2.12}$$

[2] Wang [20] refers to as "risk-adjusted human wealth."

respectively.

Let $\Phi_t^j = \dfrac{\vartheta_t^j S_t^j}{A_t}$ for every $j \in \{1, \cdots, N-1\}$.

Lemma 1. *Under Assumption 2, given a control* (c, ς) *and* W_0 ¿ *0, the agent's real total wealth process satisfies*

$$\frac{d\bar{W}_t}{\bar{W}_t} = \left(\bar{r}(X_t) + \varsigma_t' \bar{\lambda}(X_t) - \frac{c_t}{\bar{W}_t}\right) dt + \varsigma_t' \, dB_t. \tag{2.13}$$

where

$$\varsigma_t = \frac{\bar{A}_t}{\bar{W}_t} \sum_{j=1}^{N-1} \Phi_t^j \sigma_j(X_t) + \frac{\bar{H}_t}{\bar{W}_t} \sigma_h(X_t) - \sigma_p(X_t). \tag{2.14}$$

Proof. See Appendix A.1. □

Let $\mathbf{X}_t' = (\bar{W}_t, X_t')$. (c, ς) denotes a control. The control that satisfies Eq. (2.13) with the initial state $\mathbf{X}_0 = (W_0, X_0')'$ is an admissible control. Let $\mathcal{B}(\mathbf{X}_0)$ denote the set of admissible controls. I refer to ς as *investment control.*

Remark 1. *Eq. (2.13) represents the instantaneous real rate of return on the real total wealth at time t. The conditional expectation and variance of the real total wealth return are*

$$\mathrm{E}_t \left[\frac{d\bar{W}_t}{\bar{W}_t}\right] = \left(\bar{r}(X_t) + \varsigma_t' \bar{\lambda}(X_t) - \frac{c_t}{\bar{W}_t}\right) dt, \tag{2.15}$$

$$\mathrm{Var}_t \left[\frac{d\bar{W}_t}{\bar{W}_t}\right] = |\varsigma_t|^2 dt, \tag{2.16}$$

respectively. Increasing investment control can increase the expectation of total wealth return; however, it also increases its variance.

The agent's indirect utility function and consumption–investment problem are recursively defined by

$$J^\xi(\mathbf{X}_t) = \sup_{(c,\varsigma) \in \mathcal{B}(\mathbf{X}_t)} \mathrm{E}_t^\xi \left[\int_t^\infty \left(f(c_s, J_s^\xi) + \frac{(1-\gamma) J_s^\xi}{2\theta} |\xi_s|^2\right) ds\right]. \tag{2.17}$$

3 Generalized PIH

I derive the optimal consumption rule as a generalized PIH and demonstrate that it can be decomposed into five terms.

3.1 Worst-case Probability

As the standard Brownian motion under P^{ξ} is given by $B_t^{\xi} = B_t - \int_0^t \xi_s \, ds$, the SDE for the state vector under P^{ξ} is expressed as

$$d\mathbf{X}_t = \left(\begin{pmatrix} \bar{W}_t(\bar{r}(X_t) + \varsigma_t' \bar{\lambda}(X_t)) - c_t \\ \mu(X_t) \end{pmatrix} + \begin{pmatrix} \bar{W}_t \varsigma_t' \\ \Sigma(X_t) \end{pmatrix} \xi_t \right) dt + \begin{pmatrix} \bar{W}_t \varsigma_t' \\ \Sigma(X_t) \end{pmatrix} dB_t^{\xi}. \tag{3.1}$$

Thus, the HJB equation for problem (2.17) is given by

$$\begin{aligned} 0 = \sup_{(\hat{c},\hat{\varsigma}) \in \mathbb{R}_+ \times \mathbb{R}^N} \inf_{\hat{\xi} \in \mathbb{R}^N} \Bigg\{ & \begin{pmatrix} \bar{w}\Big(\bar{r}(x) + \hat{\varsigma}'\bar{\lambda}(x)\Big) - \hat{c} \\ \mu(x) \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix} + \hat{\xi}' \begin{pmatrix} \bar{w}\hat{\varsigma}' \\ \Sigma(x) \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix} \\ & + \frac{1}{2} \operatorname{tr} \left[\begin{pmatrix} \bar{w}\hat{\varsigma}' \\ \Sigma(x) \end{pmatrix} \begin{pmatrix} \bar{w}\hat{\varsigma}' \\ \Sigma(x) \end{pmatrix}' \begin{pmatrix} J_{ww} & J_{wx} \\ J_{xw} & J_{xx} \end{pmatrix} \right] + f(\hat{c}, J) + \frac{(1-\gamma)J}{2\theta} |\hat{\xi}|^2 \Bigg\}. \end{aligned} \tag{3.2}$$

It is easy to see that the worst-case probability satisfies

$$\hat{\xi}^* = -\frac{\theta}{(1-\gamma)J} \begin{pmatrix} \bar{w}\hat{\varsigma}' \\ \Sigma(x) \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix}. \tag{3.3}$$

Substituting $\hat{\xi}^*$ into HJB Eq. (3.2) yields

$$\begin{aligned} \sup_{(\hat{c},\hat{\varsigma}) \in \mathbb{R}_+ \times \mathbb{R}^N} \Bigg[& \begin{pmatrix} \bar{w}\left(\bar{r}(x) + \hat{\varsigma}'\bar{\lambda}(x)\right) - \hat{c} \\ \mu(x) \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix} + \frac{1}{2} \operatorname{tr} \left[\begin{pmatrix} \bar{w}\hat{\varsigma}' \\ \Sigma(x) \end{pmatrix} \begin{pmatrix} w\hat{\varsigma}' \\ \Sigma(x) \end{pmatrix}' \begin{pmatrix} J_{ww} & J_{wx} \\ J_{xw} & J_{xx} \end{pmatrix} \right] \\ & + f(\hat{c}, J) - \frac{\theta}{2(1-\gamma)J} \begin{pmatrix} J_w \\ J_x \end{pmatrix}' \begin{pmatrix} \bar{w}\hat{\varsigma}' \\ \Sigma(x) \end{pmatrix} \begin{pmatrix} \bar{w}\hat{\varsigma}' \\ \Sigma(x) \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix} \Bigg] = 0. \end{aligned} \tag{3.4}$$

3.2 A First Expression of Generalized PIH

Lemma 2. *Under Assumptions 1 and 2, optimal control and optimal total wealth satisfy Eqs. (3.5)–(3.6) and (3.17), and the indirect utility satisfies PDE (3.8).*

$$\hat{c}^* = \beta^{\psi} \big((1-\gamma)J\big)^{\frac{1-\gamma\psi}{1-\gamma}} J_w^{-\psi}, \tag{3.5}$$

$$\hat{\varsigma}^* = \frac{1}{\mathcal{U}_t} \left(\bar{\lambda}(x) + \Sigma(x)' \frac{J_{xw}}{J_w} + \frac{\theta}{\gamma - 1} \Sigma(x)' \frac{J_x}{J} \right), \tag{3.6}$$

where $\mathcal{U}_t$ is given by

$$\mathcal{U}_t = -\frac{\bar{W}_t^* J_{ww}}{J_w} + \theta \frac{\bar{W}_t^* J_w}{(1-\gamma)J}, \tag{3.7}$$

$$\frac{1}{2}\mathrm{tr}\left[\Sigma(x)\Sigma(x)'J_{xx}\right] - \frac{\theta}{2(1-\gamma)J}\left|\Sigma(x)'J_x\right|^2 - \frac{|\pi_t|^2}{2\bar{w}^{*2}\left(J_{ww} - \dfrac{\theta J_w^2}{(1-\gamma)J}\right)}$$
$$+ \bar{r}(x)\bar{w}^* J_w + \mu(x)'J_x - \frac{1}{1-\psi}\hat{c}^* J_w - \frac{\beta(1-\gamma)}{1-\psi^{-1}}J = 0, \quad (3.8)$$

where π_t *is given by*

$$\pi_t = -\mathcal{U}_t \bar{W}_t^* J_w \hat{\varsigma}^* = -\bar{W}_t^* J_w \left(\bar{\lambda}(X_t) + \Sigma(X_t)'\frac{J_{xw}}{J_w} - \frac{\theta}{1-\gamma}\Sigma(X_t)'\frac{J_x}{J}\right). \quad (3.9)$$

Proof. See Appendix A.2. □

From PDE (3.8), I conjecture that the indirect utility function takes the form

$$J(\mathbf{X}_t) = \frac{\bar{W}_t^{1-\gamma}}{1-\gamma}\big(G(X_t)\big)^{-\frac{1-\gamma}{1-\psi}}. \quad (3.10)$$

Then, $\mathcal{U}_t$ in Eq. (3.7) is expressed as

$$\mathcal{U}_t = \gamma + \theta. \quad (3.11)$$

The sum of the relative risk aversion and the relative ambiguity aversion is refered to as the *relative uncertainty aversion*. Substituting $J_w = (1-\gamma)\dfrac{J}{\bar{w}^*}$ and $\mathcal{U}_t = \gamma + \theta$. into PDE (3.8) leads to the following equation.[3]

$$\frac{1}{2}\mathrm{tr}\left[\Sigma(x)\Sigma(x)'\frac{J_{xx}}{J}\right] + \frac{1-\gamma}{2}(\gamma+\theta)|\hat{\varsigma}^*|^2 - \frac{\theta}{2(1-\gamma)}\left|\Sigma(x)'\frac{J_x}{J}\right|^2$$
$$+ (1-\gamma)\bar{r}(x) + \mu(x)'\frac{J_x}{J} - \frac{1-\gamma}{1-\psi}\frac{\hat{c}^*}{\bar{w}^*} - \frac{\beta(1-\gamma)}{1-\psi^{-1}} = 0. \quad (3.14)$$

Then, optimal consumption is expressed as

$$\hat{c}^* = m(X_t)\bar{W}_t^* = m(X_t)(\bar{A}_t^* + \bar{H}_t), \quad (3.15)$$

where

$$m(x) = \psi\beta + (1-\psi)\,\bar{r}(x) + m_1(x) + m_2(x) + m_3(x), \quad (3.16)$$
$$\bar{W}_t^* = \exp\left(\int_0^t \left(\bar{r}(X_s) + (\hat{\varsigma}^*)'\bar{\lambda}(X_s) - m(X_s) - \frac{1}{2}|\hat{\varsigma}^*|^2\right)dt + (\varsigma_s^*)'\,dB_s\right), \quad (3.17)$$

[3] Here, I use the following results.

$$|\pi_t|^2 = (\gamma-1)^2(\gamma+\theta)^2|\hat{\varsigma}^*|^2, \quad (3.12)$$
$$\bar{w}^{*2}\left(J_{ww} - \frac{\theta J_w^2}{(1-\gamma)J}\right) = (\gamma-1)(\gamma+\theta). \quad (3.13)$$

with

$$m_1(x) = \frac{1}{2}(1-\psi)(\gamma+\theta)|\hat{\varsigma}^*|^2, \tag{3.18}$$

$$m_2(x) = -\frac{1-\psi}{\gamma-1}\left(\mu(x)'\frac{J_x}{J} + \frac{1}{2}\mathrm{tr}\left[\Sigma(x)\Sigma(x)'\frac{J_{xx}}{J}\right]\right), \tag{3.19}$$

$$m_3(x) = -\frac{(1-\psi)\theta}{2(\gamma-1)^2}\left|\Sigma(x)'\frac{J_x}{J}\right|^2. \tag{3.20}$$

Remark 2. *Eq. (3.15) shows a generalized PIH under HREZ utility. Following Wang et. al [21], I refer to $m(x)$ as the MPC. Eq. (3.15) shows that the MPC out of human wealth coincides with that out of nonhuman wealth. Friedman [7] conjectures that the MPC out of human wealth is lower than that out of nonhuman wealth. This conjecture is strongly supported by various studies, including Hall [8]. However, Eq. (3.15) shows that the MPC out of human wealth is equivalent to that out of nonhuman wealth. This is interpreted as follows. First, Friedman's definition of human wealth overestimates its value, i.e., $\bar{H}_t^{\mathrm{F}} > \bar{H}_t$. Let $\alpha_t = \frac{\bar{H}_t}{\bar{H}_t^F}$. Then, Eq. (3.15) is rewritten as*

$$\hat{c}^* = m(X_t)(\bar{A}_t^* + \bar{H}_t) = m(X_t)\bar{A}_t^* + m(X_t)\alpha_t\bar{H}_t^F. \tag{3.21}$$

As $\alpha_t < 1$, the "MPC $m(X_t)\alpha_t$ out of human wealth" is lower than the MPC $m(X_t)$ out of nonhuman wealth. The empirical support for Friedman's misguided conjecture stems from an overvaluation of human wealth. Eq. (3.15) is a generalization of (25) in Wang [20] and of (17) in Wang et al. [21].

Remark 3. *The MPC $m(X_t)$ is decomposed into five terms. The first two terms represent a weighted average of the discount rate and the risk-free rate, where the weight on the discount rate is the EIS. These terms are presented in previous studies, including Wang et. al [21]. The third term represents the effect of total wealth variance. The fourth and fifth terms are interpreted to represent the intertemporal risk hedging effect and intertemporal ambiguity hedging effect, respectively. Note that the four terms, excluding the first term, are functions of the state vector. Consequently, the MPC may exhibit significant fluctuations in response to changes in the state vector. This means that the generalized PIH does not guarantee the stability of consumption implied by Friedman's PIH, leaving its validation to empirical analysis.*

Remark 4. *Note that it is assumed that $\gamma > 1 > \psi > 0$. The third term regarding the total wealth variance effect is positive and decreasing in the EIS and increasing in the relative uncertainty aversion. The fifth term regarding intertemporal ambiguity hedging effect is negative and decreasing in the relative ambiguity aversion. The sign of the fourth term with respect to the intertemporal risk hedging effect is indeterminate.*

3.3 A Second Expression of Generalized PIH

By inserting Eq. (3.6) and the partial derivatives of J into the PDE (3.8), we obtain the following proposition.

Proposition 1. *Under Assumptions 1 and 2, the indirect utility function, optimal consumption, and optimal investment for problem (2.17) satisfy Eqs. (3.10), (3.22) and (3.23), respectively. Function G constituting the indirect utility function is a solution to PDE (3.25).*

$$\hat{c}^* = \frac{\beta^\psi}{G(X_t)}\bar{W}_t^*, \tag{3.22}$$

$$\hat{\varsigma}^* = \frac{1}{\gamma+\theta}\bar{\lambda}(X_t) + \left(1 - \frac{1}{\gamma+\theta}\right)\frac{1}{1-\psi}\Sigma(X_t)'\frac{G_x(X_t)}{G(X_t)}, \tag{3.23}$$

where $\bar{W}_t^*$ is given by

$$\bar{W}_t^* = \exp\left(\int_0^t \left(\bar{r}(X_s) + (\hat{\varsigma}^*)'\bar{\lambda}(X_s) - \frac{\beta^\psi}{G(X_t)} - \frac{1}{2}|\hat{\varsigma}^*|^2\right)dt + (\varsigma_s^*)'\,dB_s\right), \tag{3.24}$$

$$\frac{1}{2}\operatorname{tr}\left[\Sigma(x)\Sigma(x)'\frac{G_{xx}}{G}\right] + \frac{(\gamma+\theta)\psi - 1}{2(\gamma+\theta)(1-\psi)}\left|\Sigma(x)'\frac{G_x}{G}\right|^2 + \left(\mu(x) - \frac{\gamma+\theta-1}{\gamma+\theta}\Sigma(x)\bar{\lambda}(x)\right)'\frac{G_x}{G}$$
$$+ \frac{\beta^\psi}{G} - \frac{1-\psi}{2(\gamma+\theta)}|\bar{\lambda}(x)|^2 - (1-\psi)\bar{r}(x) - \beta\psi = 0. \tag{3.25}$$

Proof. See Appendix A.3. □

It follows from Eq. (3.22) that the optimal consumption is expressed as

$$\hat{c}^* = \beta^\psi G^{-1}(x)\bar{W}_t^* = m(x)(\bar{A}_t^* + \bar{H}_t), \tag{3.26}$$

where the term $m_1(x)$ is given by Eq. (3.18) and the terms $m_2(x)$ and $m_3(x)$ are rewritten as

$$m_2(x) = -\frac{(1-\psi)\gamma}{\gamma-1}\left\{\mu(x)'\frac{G_x}{G}\right.$$
$$\left.-\frac{1}{2}\frac{\gamma-1}{1-\psi}\left(\frac{\gamma-\psi}{1-\psi}\left|\Sigma(x)'\frac{G_x}{G}\right|^2 + \operatorname{tr}\left[\Sigma(x)\Sigma(x)'\frac{G_{xx}}{G}\right]\right)\right\}, \tag{3.27}$$
$$m_3(x) = -\frac{(1-\psi)\gamma^2\theta}{2(\gamma-1)^2}\left|\Sigma(x)'\frac{G_x}{G}\right|^2.$$

Remark 5. *By specifying the market model as either an affine model (Duffie and Kan [5]) or a quadratic model (Batbold, Kikuchi, and Kusuda [1]), the model can be estimated if time series data on human capital is available. Then, a loglinear approximate solution (Campbell and Viceira [3]) of the PDE (3.25) is obtained, and $m_1(X_t), m_2(X_t)$, and $m_3(X_t)$ are computed. However, because time series data on human capital are unavailable, they must be estimated.*

4 Conditional Expected Growth Rate of Consumption

I derive the conditional expected growth rate of consumption and demonstrate that it can be decomposed into four terms.

4.1 Expression of Consumption Growth Rate

The optimal consumption process is governed by the following SDE.

$$\begin{aligned}\frac{d\hat{c}^*}{\hat{c}^*} &= \frac{d\bar{W}_t^*}{\bar{W}_t^*} - \frac{dG(X_t)}{G(X_t)} + \left|\frac{dG(X_t)}{G(X_t)}\right|^2 - \frac{d\bar{W}_t^*}{\bar{W}_t^*}\frac{dG(X_t)}{G(X_t)} \\ &= \Bigg(\bar{r}(X_t) + (\hat{\varsigma}^*)'\bar{\lambda}(X_t) - \frac{\beta^\psi}{G} - \mu(X_t)'\frac{G_x}{G} - \frac{1}{2}\mathrm{tr}\left[\Sigma(X_t)\Sigma(X_t)'\frac{G_{xx}}{G}\right] \\ &\qquad + \left|\Sigma(X_t)'\frac{G_x}{G}\right|^2 - (\hat{\varsigma}^*)'\Sigma(X_t)'\frac{G_x}{G}\Bigg)dt + (\sigma_t^c)'dB_t. \quad (4.1)\end{aligned}$$

where

$$\sigma_t^c = \hat{\varsigma}^* - \Sigma(X_t)'\frac{G_x}{G}. \quad (4.2)$$

It follows from Eq. (3.23) that the volatility of optimal consumption is rewritten as

$$\sigma_t^c = \hat{\varsigma}^* - \Sigma(X_t)'\frac{G_x}{G} = \frac{1}{\gamma+\theta}\bar{\lambda}(X_t) + \frac{(\gamma+\theta)\psi - 1}{(\gamma+\theta)(1-\psi)}\Sigma(X_t)'\frac{G_x}{G}. \quad (4.3)$$

Thus, the real market price of risk is expressed as

$$\begin{aligned}\bar{\lambda}(X_t) &= (\gamma+\theta)\sigma_t^c - \frac{(\gamma+\theta)\psi-1}{1-\psi}\Sigma(X_t)'\frac{G_x}{G} = (\gamma+\theta)\sigma_t^c - \frac{(\gamma+\theta)\psi-1}{1-\psi}\big(\hat{\varsigma}^* - \sigma_t^c\big) \\ &= \frac{1}{1-\psi}\Big((\gamma+\theta-1)\sigma_t^c - \big((\gamma+\theta)\psi - 1\big)\hat{\varsigma}^*\Big).\end{aligned} \quad (4.4)$$

Proposition 2. *Under Assumptions 1 and 2, the conditional expected growth rate of consumption is expressed as*

$$\begin{aligned}\mathrm{E}_t\left[\frac{d\hat{c}^*}{\hat{c}^*}\right] &= \psi(\bar{r}(X_t) - \beta)\,dt + \frac{\gamma+\theta-\psi}{2(1-\psi)}\mathrm{Var}_t\left[\frac{d\hat{c}^*}{\hat{c}^*}\right] \\ &\qquad - \frac{\psi\big((\gamma+\theta)\psi-1\big)}{2(1-\psi)}\mathrm{Var}_t\left[\frac{d\bar{W}_t^*}{\bar{W}_t^*}\right]. \quad (4.5)\end{aligned}$$

Proof. See Appendix A.4. □

Remark 6. *From Eqs. (4.2) and (2.13), the following equations are obtained.*

$$\mathrm{Var}_t\left[\frac{d\hat{c}^*}{\hat{c}^*}\right] = \left(\hat{\varsigma}^* - \Sigma(X_t)'\frac{G_x(X_t)}{G(X_t)}\right)dt, \tag{4.6}$$

$$\mathrm{Var}_t\left[\frac{d\bar{W}_t^*}{\bar{W}_t^*}\right] = \hat{\varsigma}^* dt. \tag{4.7}$$

Thus, all three terms in Eq. (4.5) are functions of the state vector. Therefore, the stability of the expected growth rate of consumption is not guaranteed.

4.2 Decomposition of Consumption Growth Rate

Eq. (4.5) shows that the conditional expected growth rate of consumption is decomposed into three terms. The first term is the difference between the interest and discount rates. The second and third terms are the conditional variances of the growth rates of consumption and total wealth, respectively. In the following, I demonstrate that the conditional expected growth rate of consumption can be decomposed into four terms and analyze the effect of each term through the analysis in the case of CRRA utility and EZ utility.

In the case of CRRA utility ($\gamma\psi = 1, \theta = 0$), Eq. (4.5) is simplified as the following familiar expression.

$$\mathrm{E}_t\left[\frac{d\hat{c}^*}{\hat{c}^*}\right] = \frac{1}{\gamma}(\bar{r}_t - \beta)\,dt + \frac{1}{2}(\gamma+1)\mathrm{Var}_t\left[\frac{d\hat{c}^*}{\hat{c}^*}\right]. \tag{4.8}$$

The conditional expected growth rate of consumption is increasing in the conditional variance of the consumption growth rate and is proportional to the relative prudence $\gamma + 1$. This term is interpreted as the effect of precautionary savings. Eq. (4.8) shows that the consumption growth rate is independent of the variance of the wealth growth rate. However, this interpretation is misleading and is only valid in the case where utility is CRRA utility.

In the case of EZ utility ($\gamma\psi \neq 1, \theta = 0$), Eq. (4.5) is simplified as

$$\mathrm{E}_t\left[\frac{d\hat{c}^*}{\hat{c}^*}\right] = \psi(\bar{r}_t - \beta)\,dt + \frac{\gamma-\psi}{2(1-\psi)}\mathrm{Var}_t\left[\frac{d\hat{c}^*}{\hat{c}^*}\right] - \frac{\psi(\gamma\psi-1)}{2(1-\psi)}\mathrm{Var}_t\left[\frac{d\bar{W}_t^*}{\bar{W}_t^*}\right]. \tag{4.9}$$

The second term is positive and interpreted as the "effect of precautionary savings on risk." Skiadas [17] shows that EZ utility is information seeking (resp., averse), if $\gamma > \psi^{-1}$ (resp., $\gamma < \psi^{-1}$). Thus, the coefficient of the third term is negative (positive) when the agent is information seeking (resp., averse), as demonstrated in the following equation.

$$-\frac{\psi(\gamma\psi-1)}{2(1-\psi)}\begin{cases} < 0, & \text{if } \gamma > \psi^{-1}, \\ > 0, & \text{if } \gamma < \psi^{-1}. \end{cases} \tag{4.10}$$

Remark 7. *Eq. (2.13) shows that the agent's decision to increase (resp., decrease) consumption and decrease (resp., increase) savings/investment leads to a decrease (resp., an increase) in the conditional expectation of total wealth, but to a decrease (resp., an increase) in its conditional variance. Thus, the agent's decision to increase (resp., decrease) consumption and decrease (resp., increase) savings/investment reduces (resp., increaes) risk. Therefore, if an agent is information seeking and prefers early resolution then the agent prefers consumption to saving/investment. Conversely, if an agent is information averse and prefers late resolution then the agent prefers saving/investment to consumption. The third term can thus be interpreted as the "effect of timing of resolution of risk."*

In the case of HREZ utility, Eq. (4.5) is rewritten as

$$\mathrm{E}_t\left[\frac{d\hat{c}^*}{\hat{c}^*}\right] = \psi(\bar{r}_t - \beta)\,dt + \frac{\gamma-\psi}{2(1-\psi)}\mathrm{Var}_t\left[\frac{d\hat{c}^*}{\hat{c}^*}\right] + \frac{\theta}{2(1-\psi)}\mathrm{Var}_t\left[\frac{d\hat{c}^*}{\hat{c}^*}\right] - \frac{\psi\big((\gamma+\theta)\psi-1\big)}{2(1-\psi)}\mathrm{Var}_t\left[\frac{d\bar{W}_t^*}{\bar{W}_t^*}\right]. \tag{4.11}$$

Remark 8. *The third term is positive and interpreted as the "effect of precautionary savings on ambiguity." The sign of the coefficient of the fourth term is evaluated as follows.*

$$-\frac{\psi\big((\gamma+\theta)\psi-1\big)}{2(1-\psi)}\begin{cases} <0, & \text{if } \gamma+\theta>\psi^{-1}, \\ >0, & \text{if } \gamma+\theta<\psi^{-1}. \end{cases} \tag{4.12}$$

Kusuda [11] shows that HREZ utility is interpreted as information seeking (resp., averse) and prefers early (resp., late) resolution of uncertainty if $\gamma+\theta>\psi^{-1}$ (resp., $\gamma+\theta<\psi^{-1}$).[4] *Thus, the fourth term can be interpreted as the "effect of timing of resolution of uncertainty."*

Furthermore, $\mathrm{Var}_t\left[\frac{d\bar{W}_t^*}{\bar{W}_t^*}\right]$ in Eq. (4.11) is decomposed into the following three terms.

$$\mathrm{Var}_t\left[\frac{d\bar{W}_t^*}{\bar{W}_t^*}\right] = \left(\frac{\bar{A}_t^*}{\bar{W}_t^*}\right)^2\mathrm{Var}_t\left[\frac{d\bar{A}_t^*}{\bar{A}_t^*}\right] + \left(\frac{\bar{H}_t}{\bar{W}_t^*}\right)^2\mathrm{Var}_t\left[\frac{d\bar{H}_t}{\bar{H}_t}\right] + 2\frac{\bar{A}_t^*}{\bar{W}_t^*}\frac{\bar{H}_t}{\bar{W}_t^*}\mathrm{Cov}_t\left(\frac{d\bar{A}_t^*}{\bar{A}_t^*}, \frac{d\bar{H}_t}{\bar{H}_t}\right). \tag{4.13}$$

5 Conclusion

I assumed HREZ utility with $\gamma > 1 > \psi > 0$, and a market model in which the state vector process is a general Markovian diffusion process,

[4]For details, see Kusuda [11].

and the risk-free rate and volatilities of capital assets and labor income are functions of the state vector. First, I derived an expression of generalized PIH in which the optimal consumption is the product of the MPC and total wealth and demonstrated that the MPC out of human wealth coincides with that out of nonhuman wealth. I showed that traditional definition-based human wealth is greater than Wang's definition of human wealth and that the MPC out of traditional definition-based human wealth is lower than that out of nonhuman wealth, as conjectured by Friedman [7]. Second, I showed that the MPC can be decomposed into five terms. The first two terms represent the weighted average of the discount rate and the risk-free rate, where the weight on the discount rate is the EIS. These terms were presented in previous studies, including Wang *et. al* [21]. The third term represents the effect of the total wealth variance. The fourth and fifth terms represent the intertemporal risk hedging effect and intertemporal ambiguity hedging effect, respectively. Given that the four terms, excluding the first term, are functions of the state vector, the generalized PIH does not theoretically guarantee the stability of consumption implied by Friedman's PIH. Rather, its stability is subject to empirical analysis.

Finally, I derived the conditional expected growth rate of consumption and demonstrated that it can be decomposed into four terms. The first term is the difference between the interest and discount rates. The second and third terms are both positive and interpreted as the "effect of precautionary savings on risk" and "effect of precautionary savings on ambiguity," respectively. The fourth term is negative (resp., positive) if the agent prefers early (resp., late) resolution and is interpreted as the "effect of timing of resolution of uncertainty." Given that all of these terms are functions of the state vector, the stability of the expected growth rate of consumption is not theoretically guaranteed.

As explained in Remark 5, by specifying the market model as an affine model or a quadratic model, the model can be estimated if time series data on human capital is available and numerical analysis of the stability of the MPC and expected growth rate of consumption can be performed. However, because time series data on human capital are unavailable, they must be estimated. Estimating human capital and conducting numerical analysis of the stability of the MPC and expected growth rate of consumption is a topic for future research.

A Proofs

A.1 Proof of Lemma 1

Let D^j denote the income process arising from the holdings of the j-th capital asset. Then, the agent's nominal total wealth process satisfies

$$\begin{aligned}
\frac{dW_t}{W_t} &= \frac{1}{W_t}\left\{\vartheta_t dP_t + \sum_{j=1}^{N-1}\vartheta_t^j\left(dS_t^j + D_t^j dt\right) + dH_t + L_t dt - p_t c_t dt\right\} \\
&= \frac{A_t}{W_t}\left\{\frac{\vartheta_t P_t}{A_t}\frac{dP_t}{P_t} + \sum_{j=1}^{N-1}\frac{\vartheta_t^j S_t^j}{A_t}\frac{dS_t^j + D_t^j dt}{S_t^j}\right\} + \frac{H_t}{W_t}\frac{dH_t + L_t dt}{H_t} - \frac{c_t}{\bar{W}_t}dt \\
&= \frac{\bar{A}_t}{\bar{W}_t}\left(1 - \sum_{j=1}^{N-1}\Phi_t^j\right)\frac{dP_t}{P_t} + \sum_{j=1}^{N-1}\Phi_t^j\frac{dS_t^j + D_t^j dt}{S_t^j}\Bigg\} + \frac{\bar{H}_t}{\bar{W}_t}\frac{dH_t + L_t dt}{H_t} - \frac{c_t}{\bar{W}_t}dt.
\end{aligned} \tag{A.1}$$

From the no arbitrage condition, the dynamics of P_t, S_t^j, and H_t are given by

$$\frac{dP_t}{P_t} = r(X_t)dt, \tag{A.2}$$

$$\frac{dS_t^j + D_t^j dt}{S_t^j} = \big(r(X_t) + \sigma_j(X_t)'\lambda(X_t)\big)dt + \sigma_j(X_t)'dB_t, \tag{A.3}$$

$$\frac{dH_t + L_t dt}{H_t} = \big(r(X_t) + \sigma_h(X_t)'\lambda(X_t)\big)dt + \sigma_h(X_t)'dB_t. \tag{A.4}$$

Substituting Eqs. (A.2)-(A.4) into the above equation and using the investment control ς_t yields

$$\frac{dW_t}{W_t} = \left(r(X_t) + \lambda(X_t)'(\varsigma_t + \sigma_p(X_t)) - \frac{c_t}{\bar{W}_t}\right)dt + (\varsigma_t + \sigma_p(X_t))'\, dB_t. \tag{A.5}$$

Noting that

$$\frac{dW_t}{W_t} = \frac{d(p_t\bar{W}_t)}{p_t\bar{W}_t} = \frac{d\bar{W}_t}{\bar{W}_t} + \frac{dp_t}{p_t} + \frac{d\bar{W}_t}{\bar{W}_t}\frac{dp_t}{p_t},$$

and the volatility of $\bar{W}_t$ is equal to ς_t, we have

$$\frac{d\bar{W}_t}{\bar{W}_t} = \frac{dW_t}{W_t} - i(X_t)dt - \sigma_p(X_t)'dB_t - \varsigma_t'\sigma_p(X_t)\,dt.$$

By inserting Eq. (A.5) into the equation above and using Eqs. (2.11) and (2.12), I obtain Eq. (2.13).

A.2 Proof of Lemma 2

It is clear that the optimal control $\hat{u}^* = (\hat{c}^*, \hat{\varsigma}^*)$ in HJB Eq. (3.4) satisfies Eqs. (3.5) and (3.6) . Substituting the optimal control $(\hat{c}^*, \hat{\varsigma}^*)$ into HJB Eq. (3.4), I have

$$\Big(\bar{w}^*(\bar{r}(x)+\bar{\lambda}(x)'\varsigma^*)-\hat{c}^*\Big)J_w+\mu(x)'J_x+\frac{1}{2}\operatorname{tr}\left[\begin{pmatrix}\bar{w}^*(\hat{\varsigma}^*)'\\ \Sigma(x)\end{pmatrix}\begin{pmatrix}\bar{w}^*(\hat{\varsigma}^*)'\\ \Sigma(x)\end{pmatrix}'\begin{pmatrix}J_{ww} & J_{wx}\\ J_{xw} & J_{xx}\end{pmatrix}\right]$$
$$+f(\hat{c}^*,J)-\frac{\theta}{2(1-\gamma)J}\begin{pmatrix}J_w\\ J_x\end{pmatrix}'\begin{pmatrix}\bar{w}^*(\hat{\varsigma}^*)'\\ \Sigma(x)\end{pmatrix}\begin{pmatrix}\bar{w}^*(\hat{\varsigma}^*)'\\ \Sigma(x)\end{pmatrix}'\begin{pmatrix}J_w\\ J_x\end{pmatrix}=0. \quad (A.6)$$

The investment-related terms in HJB Eq. (A.6) are organized as:

$$\begin{aligned}&\bar{w}^*J_w\bar{\lambda}(x)'\hat{\varsigma}^*+\frac{1}{2}\operatorname{tr}\left[\begin{pmatrix}\bar{W}_t^*(\hat{\varsigma}^*)'\\ \Sigma(x)\end{pmatrix}\begin{pmatrix}\bar{w}^*(\hat{\varsigma}^*)'\\ \Sigma(x)\end{pmatrix}'\begin{pmatrix}J_{ww} & J_{wx}\\ J_{xw} & J_{xx}\end{pmatrix}\right]\\ &\qquad-\frac{\theta}{2(1-\gamma)J}\begin{pmatrix}J_w\\ J_x\end{pmatrix}'\begin{pmatrix}\bar{w}^*(\hat{\varsigma}^*)'\\ \Sigma(x)\end{pmatrix}\begin{pmatrix}\bar{w}^*(\hat{\varsigma}^*)'\\ \Sigma(x)\end{pmatrix}'\begin{pmatrix}J_w\\ J_x\end{pmatrix}\\ &=\frac{1}{2}\operatorname{tr}\left[\Sigma(x)\Sigma(x)'J_{xx}\right]-\frac{\theta}{2(1-\gamma)J}\left|\Sigma(x)'J_x\right|^2-\frac{|\pi_t|^2}{2\bar{w}^{*2}\left(J_{ww}-\dfrac{\theta J_w^2}{(1-\gamma)J}\right)}.\end{aligned} \quad (A.7)$$

The consumption-related terms in HJB Eq. (A.6) are computed as

$$\begin{aligned}-\hat{c}^*J_w+f(\hat{c}^*,J)&=-\hat{c}^*J_w+\frac{1}{1-\psi^{-1}}\hat{c}^*J_w-\frac{\beta(1-\gamma)}{1-\psi^{-1}}J\\ &=-\frac{1}{1-\psi}\hat{c}^*J_w-\frac{\beta(1-\gamma)}{1-\psi^{-1}}J.\end{aligned} \quad (A.8)$$

By substituting Eqs. (A.7) and (A.8) into HJB Eq. (A.6), the PDE (3.8) for J is obtained:

A.3 Proof of Proposition 1

First, the optimal consumption control (3.22) is obtained as follows:

$$\hat{c}^*=\beta^{\psi}\big((1-\gamma)J\big)^{\left(1-\frac{1-\psi^{-1}}{1-\gamma}\right)\psi}\left(\frac{(1-\gamma)J}{\bar{w}^*}\right)^{-\psi}=\beta^{\psi}\bar{w}^{*\psi}\left(\bar{w}^{*1-\gamma}G^{-\frac{1-\gamma}{1-\psi}}\right)^{\frac{1-\psi}{1-\gamma}}=\beta^{\psi}\frac{\bar{w}^*}{G}. \quad (A.9)$$

Second, the derivatives of J are given by

$$\begin{aligned}&\bar{w}^*J_w=(1-\gamma)J, \qquad J_x=\frac{\gamma-1}{1-\psi}J\frac{G_x}{G}, \qquad \bar{w}^{*2}J_{ww}=\gamma(\gamma-1)J,\\ &\bar{w}^*J_{xw}=-\frac{(\gamma-1)^2}{1-\psi}J\frac{G_x}{G}, \qquad J_{xx}=\frac{\gamma-1}{1-\psi}J\left\{\left(\frac{\gamma-1}{1-\psi}-1\right)\frac{G_x}{G}\frac{G_x'}{G}+\frac{G_{xx}}{G}\right\}.\end{aligned} \quad (A.10)$$

Then, $\mathcal{U}_t$ in Eq. (3.7) is expressed as Eq. (3.11), and π_t in Eq. (3.9) is calculated as

$$\pi_t = (\gamma - 1)J\left(\bar{\lambda}(x) + \frac{\gamma + \theta - 1}{1 - \psi}\Sigma(x)'\frac{G_x}{G}\right), \tag{A.11}$$

I also obtain

$$\bar{w}^2 J_{ww} - \frac{\theta \bar{w}^2 J_w^2}{(1-\gamma)J} = \gamma(\gamma - 1)J - (1-\gamma)\theta J = (\gamma - 1)(\gamma + \theta)J. \tag{A.12}$$

Therefore, by inserting Eq. (3.11) into Eq. (3.6), I obtain the optimal investment control (3.23). From Eq. (A.11), the first to third terms in PDE (3.8) are calculated as

$$\begin{aligned}
&\frac{1}{2}\operatorname{tr}\left[\Sigma(x)\Sigma(x)'J_{xx}\right] - \frac{\theta}{2(1-\gamma)J}\left|\Sigma(x)'J_x\right|^2 - \frac{|\pi_t|^2}{2\bar{w}^{*2}\left(J_{ww} - \frac{\theta J_w^2}{(1-\gamma)J}\right)} \\
=&\frac{\gamma - 1}{2(1-\psi)}J \operatorname{tr}\left[\Sigma(x)\Sigma(x)'\left(\frac{\gamma + \psi - 2}{1-\psi}\frac{G_x}{G}\frac{G_x'}{G} + \frac{G_{xx}}{G}\right)\right] + \frac{(\gamma - 1)\theta}{2(1-\psi)^2}\left|\Sigma(x)'\frac{G_x}{G}\right|^2 \\
&- \frac{\gamma - 1}{2(\gamma + \theta)}J\left|\bar{\lambda}(x) + \frac{\gamma + \theta - 1}{1-\psi}\Sigma(x)'\frac{G_x}{G}\right|^2 \\
=&(\gamma - 1)J\left\{\frac{1}{2(1-\psi)}\operatorname{tr}\left[\Sigma(x)\Sigma(x)'\frac{G_{xx}}{G}\right] - \frac{1}{2(\gamma+\theta)}|\bar{\lambda}(x)|^2 - \frac{\gamma + \theta - 1}{(\gamma+\theta)(1-\psi)}\bar{\lambda}(x)'\Sigma(x)'\frac{G_x}{G}\right. \\
&\left. + \frac{1}{2(\gamma+\theta)(1-\psi)^2}\left\{(\gamma+\theta)(\gamma - 2 + \psi) + (\gamma + \theta)\theta - (\gamma + \theta - 1)^2\right\}\left|\Sigma(x)'\frac{G_x}{G}\right|^2\right\} \\
=&(\gamma - 1)J\left\{\frac{1}{2(1-\psi)}\operatorname{tr}\left[\Sigma(x)\Sigma(x)'\frac{G_{xx}}{G}\right] - \frac{1}{2(\gamma+\theta)}|\bar{\lambda}(x)|^2\right. \\
&\left. - \frac{\gamma + \theta - 1}{(\gamma+\theta)(1-\psi)}\bar{\lambda}(x)'\Sigma(x)'\frac{G_x}{G} + \frac{(\gamma+\theta)\psi - 1}{2(\gamma+\theta)(1-\psi)^2}\left|\Sigma(x)'\frac{G_x}{G}\right|^2\right\}.
\end{aligned} \tag{A.13}$$

From Eq. (A.9), the sixth term in PDE (3.8) is calculated as

$$-\frac{1}{1-\psi}\hat{c}^* J_w = \frac{\beta^\psi(\gamma - 1)}{1-\psi}\frac{J}{G}, \tag{A.14}$$

Substituting Eqs. (A.13) and (A.14) into Eq. (3.8) and dividing by $\frac{\gamma - 1}{1-\psi}J$ yields PDE (3.25).

A.4 Proof of Proposition 2

Let $\bar{r}_t = \bar{r}(X_t)$ and $\bar{\lambda}_t = \bar{\lambda}(X_t)$. Substituting Eq. (3.23) into the conditional expectation of Eq. (4.1) yields

$$
\begin{aligned}
\mathrm{E}_t\left[\frac{d\hat{c}^*}{\hat{c}^*}\right] &= \Bigg(\bar{r}(X_t) + (\hat{\varsigma}^*)'\bar{\lambda}(X_t) - \frac{\beta^\psi}{G} - \mu(X_t)'\frac{G_x}{G} - \frac{1}{2}\mathrm{tr}\left[\Sigma(X_t)\Sigma(X_t)'\frac{G_{xx}}{G}\right] \\
&+ \left|\Sigma(X_t)'\frac{G_x}{G}\right|^2 - (\hat{\varsigma}^*)'\Sigma(X_t)'\frac{G_x}{G}\Bigg)dt \\
&= \Bigg\{\bar{r}_t + \frac{1}{\gamma+\theta}\left(\bar{\lambda}_t + \frac{\gamma+\theta-1}{1-\psi}\Sigma(X_t)'\frac{G_x}{G}\right)'\left(\bar{\lambda}_t - \Sigma(X_t)'\frac{G_x}{G}\right) + \left|\Sigma(X_t)'\frac{G_x}{G}\right|^2 \\
&- \frac{1}{2}\mathrm{tr}\left[\Sigma(X_t)\Sigma(X_t)'\frac{G_{xx}}{G}\right] - \mu(x)'\frac{G_x}{G} - \frac{\beta^\psi}{G}\Bigg\}dt \\
&= \Bigg\{\bar{r}_t + \frac{1}{\gamma+\theta}|\bar{\lambda}_t|^2 - \frac{(\gamma+\theta)\psi-1}{(\gamma+\theta)(1-\psi)}\left|\Sigma(X_t)'\frac{G_x}{G}\right|^2 + \frac{1}{\gamma+\theta}\left(\frac{\gamma+\theta-1}{1-\psi}-1\right)\bar{\lambda}_t'\Sigma(X_t)'\frac{G_x}{G} \\
&- \left(\frac{1}{2}\mathrm{tr}\left[\Sigma(X_t)\Sigma(X_t)'\frac{G_{xx}}{G}\right] + \mu(x)'\frac{G_x}{G} + \frac{\beta^\psi}{G}\right)\Bigg\}dt.
\end{aligned} \tag{A.15}
$$

Expanding the above equation using PDE (3.25) results in the following equation.

$$
\begin{aligned}
\mathrm{E}_t\left[\frac{d\hat{c}^*}{\hat{c}^*}\right] &= \Bigg\{\bar{r}_t + \frac{1}{\gamma+\theta}|\bar{\lambda}_t|^2 - \frac{(\gamma+\theta)\psi-1}{(\gamma+\theta)(1-\psi)}\left|\Sigma(X_t)'\frac{G_x}{G}\right|^2 + \frac{1}{\gamma+\theta}\left(\frac{\gamma+\theta-1}{1-\psi}-1\right)\bar{\lambda}_t'\Sigma(X_t)'\frac{G_x}{G} \\
&+ \frac{(\gamma+\theta)\psi-1}{2(\gamma+\theta)(1-\psi)}\left|\Sigma(X_t)'\frac{G_x}{G}\right|^2 - \left(\frac{\gamma+\theta-1}{\gamma+\theta}\Sigma(X_t)\bar{\lambda}_t\right)'\frac{G_x}{G} - \frac{1-\psi}{2(\gamma+\theta)}|\bar{\lambda}_t|^2 - (1-\psi)\bar{r}_t - \beta\psi \\
&= \Bigg\{\psi(\bar{r}_t - \beta) + \frac{1+\psi}{2(\gamma+\theta)}|\bar{\lambda}_t|^2 - \frac{(\gamma+\theta)\psi-1}{2(\gamma+\theta)(1-\psi)}\left|\Sigma(X_t)'\frac{G_x}{G}\right|^2 + \frac{(\gamma+\theta)\psi-1}{(\gamma+\theta)(1-\psi)}\bar{\lambda}_t'\Sigma(X_t)'\frac{G_x}{G}\Bigg\}dt \\
&= \Bigg\{\psi(\bar{r}_t - \beta) + \frac{1}{2(\gamma+\theta)}\Bigg\{(1+\psi)|\bar{\lambda}_t|^2 - \frac{(\gamma+\theta)\psi-1}{1-\psi}\left(\left|\Sigma(X_t)'\frac{G_x}{G}\right|^2 - 2\bar{\lambda}_t'\Sigma(X_t)'\frac{G_x}{G}\right)\Bigg\}\Bigg\}dt
\end{aligned} \tag{A.16}
$$

Substituting Eqs. (4.2) and (4.4) into the above equation, I have

$$\begin{aligned}
\mathrm{E}_t\left[\frac{d\hat{c}^*}{\hat{c}^*}\right] &= \Bigg\{\psi(\bar{r}_t-\beta)+\frac{1}{2(\gamma+\theta)}\bigg\{\frac{1+\psi}{(1-\psi)^2}\big|(\gamma+\theta-1)\sigma_t^c-\big((\gamma+\theta)\psi-1\big)\hat{\varsigma}^*\big|^2 \\
&\quad -\frac{(\gamma+\theta)\psi-1}{1-\psi}\bigg(|\sigma_t^c-\hat{\varsigma}^*|^2+\frac{2}{1-\psi}\Big((\gamma+\theta-1)\sigma_t^c-\big((\gamma+\theta)\psi-1\big)\hat{\varsigma}^*\Big)'(\sigma_t^c-\hat{\varsigma}^*)\bigg)\bigg\}\Bigg\}dt \\
&= \Bigg\{\psi(\bar{r}_t-\beta)+\frac{1}{2(\gamma+\theta)}\bigg\{(\gamma+\theta)\frac{\gamma+\theta-\psi}{1-\psi}|\sigma_t^c|^2-\frac{\psi\big((\gamma+\theta)-\psi\big)(\gamma+\theta)}{1-\psi}|\hat{\varsigma}^*|^2\bigg\}\Bigg\}dt \\
&= \bigg\{\psi(\bar{r}_t-\beta)-\frac{\psi\big((\gamma+\theta)\psi-1\big)}{2(1-\psi)}|\hat{\varsigma}^*|^2+\frac{\gamma+\theta-\psi}{2(1-\psi)}|\sigma_t^c|^2\bigg\}dt.
\end{aligned} \tag{A.17}$$

Therefore, I obtain Eq. (4.5).

Acknowledgments

I would like to express my profound gratitude to Prof. Kentaro KIKUCHI for his invaluable advice. I would also like to thank Editage (www.editage.com) for assisting with English language editing.

Declaration of Interests

I have nothing to declare.

Funding Sources

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

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