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Properties of Homothetic Robust Epstein-Zin Utility

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Abstract This study assumes homothetic robust Epstein-Zin (HREZ) utility and proves that HREZ utility is homothetic stochastic differential utility under certain integrability conditions. The observational indistinguishability among HREZ utilities, ambiguity aversion, risk aversion, and preferences for information are analyzed. Furthermore, by leveraging the convexity or concavity of the normalized aggregator in lieu of the uniform Lipschitz condition, it is proven that HREZ utility is increasing, time consistent, and homothetic.

Keywords Ambiguity aversion · Epstein-Zin utility · Homothetic robust utility · Normalized representation · Stochastic differential utility

JEL C61 · D81

1 Introduction

The global financial crisis reaffirmed the significance of utility, which accounts for Knightian uncertainty. Agents with robust utility, as proposed by Anderson, Hansen, and Sargent [1] and Hansen and Sargent [8], consider the "base probability" to be the most likely probability, while also accounting for other probabilities because the true probability is unknown. Since robust utility lacks homotheticity, Maenhout [14] proposes homothetic robust (HR) utility, which is characterized by a subjective discount rate, relative risk aversion, and relative ambiguity aversion. HR utility is used in robust portfolio studies, including Skiadas [18], Maenhout [15], Liu [13], Branger, Larsen, and Munk [2], Munk

This paper is an independent and substantially developed version of a study (homothetic robust Epstein-Zin utility) from our previously published working paper (Kikuchi and Kusuda [11]), which comprises two studies.

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and Rubtsov [16], Yi, Viens, Law, and Li [19], and Kikuchi and Kusuda [9].[1] HR utility can be interpreted as homothetic robust CRRA utility, because it converges to CRRA utility as ambiguity aversion approaches zero. CRRA utility does not separate the relative risk aversion from the elasticity of intertemporal substitution (EIS). Epstein-Zin (EZ) utility (Epstein and Zin [7]) generalizes CRRA utility by separating these properties while retaining homotheticity. Maenhout [14] introduces homothetic robust Epstein-Zin (HREZ) utility to derive a capital asset pricing model. HREZ utility is characterized by the subjective discount rate, EIS, relative risk aversion, and relative ambiguity aversion. However, Maenhout [14] does not show the properties of HREZ utility.

Skiadas [18] demonstrates that both HR utility and HREZ utility are stochastic differential utilities (SDUs), as proposed by Duffie and Epstein [4]. However, the proof by Skiadas [18] contains errors in the calculation process, as discussed in Section 2. I address the inaccuracies and demonstrate that both HR and HREZ utilities are SDUs. Duffie and Epstein [4] show properties of SDU, including risk aversion, monotonicity for consumption, time consistency, and homotheticity, under the assumption that the felicity function (*i.e.*, the normalized aggregator) satisfies the uniform Lipschitz condition. However, the felicity function of HREZ utility does not meet this condition.

The concepts of ambiguity aversion and comparative ambiguity aversion of utility, which recognizes Knightian uncertainty, are defined by Chen and Epstein [3], who also redefine the concepts of risk aversion and comparative risk aversion under Knightian uncertainty. Skiadas [17] introduces the concept of agents' preferences for information, that is, the concept of whether or not they seek it. However, this concept assumes a unique subjective probability for each agent, making it directly inapplicable to HREZ utility, where agents consider multiple candidate probabilities beyond the base probability. The purpose of this study is to elucidate the properties of HREZ utility, and the main results are summarized as follows.

First, I prove that HREZ utility is SDU under certain integrability conditions by correcting the errors in Skiadas [18]. The class of HREZ utlities satisfying the above condition includes HR, EZ, and CRRA utilities as special cases. Under these conditions, I derive the normalized representation of HREZ utility and show that the felicity function is chracterized by the subjective discount rate, EIS, and *relative uncertainty aversion* which is the sum of relative risk aversion and ambiguity aversion.

Second, I show that HREZ utility is observationally indistinguishable not only from EZ utility but also among HREZ utilities with the same relative uncertainty aversion. This follows naturally, as the felicity function of HREZ utility is characterized by the subjective discount rate, EIS, and relative uncertainty aversion. I also explain the robustness effect of HREZ utility following Maenhout [14].

[1] Kikuchi and Kusuda [10] generalize HR utility so that relative ambiguity aversion depends on age; however, the utility functional is no longer homothetic.

Third, I analyze the ambiguity and risk aversions of HREZ utility using the definitions by Chen and Epstein [3]. I prove the following: (i) if two HREZ utilities share the same subjective discount rate, relative risk aversion, and EIS, then the one with greater relative ambiguity aversion is more ambiguity averse; (ii) if two HREZ utilities share the same subjective discount rate and EIS, then the one with greater relative risk aversion is more risk averse; (iii) HREZ utility is both ambiguity averse and risk averse.

Fourth, I examine preferences for information in HREZ utility based on Skiadas [17]. Since the Skiadas [17]' framework cannot directly apply to HREZ utility, I analyze the preferences in two related utilities: (i) the EZ utility observationally indistinguishable from HREZ utility; and (ii) the restricted HREZ utility, where its domain is restricted to unambiguous consumption plans. I show that the observationally indistinguishable EZ utility is information seeking (resp., averse) if the relative uncertainty aversion is greater (resp., less) than the inverse of EIS, and that the restricted HREZ utility is information seeking (resp., averse) if relative risk aversion is greater (resp., less) than the inverse of EIS.

Fifth, by extending the proof by Duffie and Epstein [4] to cases where the uniform Lipschitz condition does not hold, I demonstrate that HREZ utility is strictly increasing, time consistent, and homothetic. Here, convexity or cocavity in the utility argument of the felicity function substitutes for the uniform Lipschitz condition.

The remainder of this paper is organized as follows: Section 2 introduces HREZ utility and proves that it is SDU. Section 3 explains the observational indistinguishability and robustness effect of HREZ utility. Section 4 analyzes the ambiguity and risk aversions and preference for information of HREZ utility. Section 5 presents other properties of HREZ utility. Section 6 concludes the paper.

2 HREZ Utility and SDU

I introduce HREZ utility and demonstrate that, if relative risk aversion is greater than or equal to EIS, then HREZ utility is interpreted as SDU under certain integrability conditions. Then, I derive the normalized representation of HREZ utility.

2.1 Environment

I consider frictionless markets over the period $[0,T]$. Investors' base probability and information structure are modeled by a complete filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathrm{P})$, where $\mathbb{F}=(\mathcal{F}_t)_{t\in[0,T]}$ is the natural filtration generated by an N-dimensional standard Brownian motion B_t. There is a market for a consumption commodity at every date $t\in[0,T]$. I denote the expectation operator under P by E and the conditional expectation operator given $\mathcal{F}_t$ by E_t. Let A' and I denote the transpose of A and $N\times N$ identity matrix, respectively.

2.2 HREZ Utility

I begin with the continuous-time version (Duffie and Epstein [4]) of Epstein-Zin utility.

$$\tilde{V}_t = \mathrm{E}_t\left[\int_t^T f(c_s, \tilde{V}_s)ds\right], \tag{2.1}$$

where f denotes the normalized aggregator of the form

$$f(c,v) = \beta\frac{c^{1-\psi^{-1}}}{1-\psi^{-1}}\big((1-\gamma)v\big)^{1-\frac{1-\psi^{-1}}{1-\gamma}} - \frac{\beta(1-\gamma)v}{1-\psi^{-1}}, \tag{2.2}$$

where $\beta > 0$ is the subjective discount rate, γ is the relative risk aversion, and ψ is the EIS. In this study, the normalized aggregator is also referred to as the felicity function.

The theoretical range of (γ, ψ) is $(\gamma, \psi) \in (0,1)\times(1,\infty)\cup(1,\infty)\times(0,1)$. Note that if $(\gamma,\psi) \in (0,1)\times(1,\infty)$ then $\tilde{V} > 0$, and if $(\gamma,\psi) \in (1,\infty)\times(0,1)$ then $\tilde{V} < 0$. Many empirical analyses indicate that $\gamma > 1$ and $1 > \psi > 0$. Therefore, I assume that $\gamma > 1 > \psi > 0$.

Remark 1 EZ utility is SDU, and Duffie and Epstein [4] show the properties of SDU satisfying the following uniform Lipschitz condition in its utility argument:

$$|f(c,v) - f(c,\tilde{v})| \le k|v - \tilde{v}|, \quad \forall c, v, \tilde{v}. \tag{2.3}$$

However, EZ utility does not satisfy the uniform Lipschitz condition. This study examines the properties of HREZ utility without assuming the uniform Lipschitz condition. I show that EZ utility is a special case of HREZ utility, and that most of the obtained properties of HREZ utility are also those of EZ utility.

Remark 2 It is clear that if $\psi^{-1} = \gamma$, then the normalized aggregator is simplified as

$$f(c,v) = \beta\frac{c^{1-\gamma}}{1-\gamma} - \beta v \tag{2.4}$$

and EZ utility is reduced to CRRA utility.

Whereas an agent with robust utility regards probability P which is called “base probability” as the most likely probability, they also consider other probabilities because the true probability is unknown. Thus, the agent assumes set $\mathbb{P}$ of all equivalent probability measures[2] as the alternative probabilities. According to Girsanov's theorem, any equivalent probability measure is characterized by a measurable process $(\xi_t)_{t\in[0,T]}$ with Novikov's integrability condition as the following Radon-Nikodym derivative:

$$\mathrm{E}_T\left[\frac{d\mathrm{P}^\xi}{d\mathrm{P}}\right] = \exp\left(\int_0^T \xi_t\, dB_t - \frac{1}{2}\int_0^T |\xi_t|^2 dt\right) \quad \forall T \in (0,\infty). \tag{2.5}$$

[2] A probability measure $\tilde{\mathrm{P}}$ is said to be an *equivalent probability measure of* P if and only if $\mathrm{P}(A) = 0 \Leftrightarrow \tilde{\mathrm{P}}(A) = 0$.

Therefore, the agent chooses the worst-case probability that minimizes the utility among $\mathbb{P}$ for each consumption plan. The agent rationally determines the worst-case probability by considering deviations from P.

Let $\mathcal{C}$ denote the set of adapted consumption processes $(c_t)_{t\in[0,T]}$ satisfying certain requisite integrability condition.

Definition 1 HREZ utility is defined by

$$U(c) = \inf_{\mathrm{P}^\xi \in \mathbb{P}} \mathrm{E}^\xi \left[\int_0^T \left(f(c_t, V_t^\xi) + \frac{(1-\gamma)V_t^\xi}{2\theta} |\xi_t|^2 \right) dt \right], \tag{2.6}$$

where $c \in \mathcal{C}, \mathrm{E}^\xi$ is the expectation under P^ξ, $\theta > 0$ is relative ambiguity aversion, and V_t^ξ is the utility process, defined recursively as follows:

$$V_t^\xi = \mathrm{E}_t^\xi \left[\int_t^T \left(f(c_s, V_s^\xi) + \frac{(1-\gamma)V_s^\xi}{2\theta} |\xi_s|^2 \right) ds \right], \quad V_T^\xi = 0. \tag{2.7}$$

2.3 SDU Representations of HR Utility and HREZ Utility

2.3.1 SDU Representation of HR Utility

First, I prove that HR utility is SDU under certain integrability condition. Assume HR utility; that is, $\gamma = \psi^{-1}$. Then, Eq. (2.7) is rewritten as

$$V_t^\xi = \mathrm{E}_t^\xi \left[\int_t^T \left(\beta \frac{c_s^{1-\gamma}}{1-\gamma} - \beta V_s^\xi + \frac{(1-\gamma)V_s^\xi}{2\theta} |\xi_s|^2 \right) ds \right]. \tag{2.8}$$

Suppose that a progressively measurable pair (V^*, σ^*) satisfies the following stochastic differential equation (SDE):

$$dV_t^* = -\left(\beta \frac{c_t^{1-\gamma}}{1-\gamma} - \beta V_t^* - \frac{\theta}{2(1-\gamma)V_t^*} |\sigma_t^*|^2 \right) dt + (\sigma_t^*)' dB_t, \quad V_T^* = 0. \tag{2.9}$$

By Girsanov theorem, standard Brownian motion under P^ξ is expressed as $B_t^\xi = B_t - \int_0^t \xi_s \, ds$. Thus, Eq. (2.8) is rewritten as

$$dV_t^* = -\left(\beta \frac{c_t^{1-\gamma}}{1-\gamma} - \beta V_t^* - \frac{\theta}{2(1-\gamma)V_t^*} |\sigma_t^*|^2 - (\sigma_t^*)'\xi_t \right) dt + (\sigma_t^*)' dB_t^\xi, \tag{2.10}$$

with $V_T^* = 0$. From Eqs. (2.8) and (2.10), the following equation is derived:[3]

$$V_t^\xi = V_t^* + \mathrm{E}_t^\xi\left[\int_t^T e^{-\beta(s-t)}\Big(h(\sigma_s^*, V_s^\xi) - h(\sigma_s^*, V_s^*) + Q(\xi_s, \sigma_s^*, V_s^\xi)\Big)ds\right], \quad (2.12)$$

where

$$h(\sigma, v) = -\frac{\theta}{2(1-\gamma)v}|\sigma|^2, \quad (2.13)$$

$$Q(\xi_s, \sigma_s^*, V_s^\xi) = \frac{(1-\gamma)V_s^\xi}{2\theta}\left|\xi_s + \frac{\theta}{(1-\gamma)V_s^\xi}\sigma_s^*\right|^2 \geq 0. \quad (2.14)$$

Then, h is concave in its utility argument because the second derivative of h with respect to v is negative, as shown below:

$$h_{vv}(\sigma, v) = -\frac{\theta}{(1-\gamma)v^3}|\sigma|^2 < 0. \quad (2.15)$$

Thus, $V_t^\xi - V_t^*$ is evaluated from below as in the following inequality.

$$V_t^\xi - V_t^* \geq \mathrm{E}_t^\xi\left[\int_t^T e^{-\beta^*(s-t)} h_v(\sigma_s^*, V_s^\xi)(V_s^\xi - V_s^*)ds\right]. \quad (2.16)$$

Under certain integrability condition on h_v, the "stochastic Gronwall-Bellman (SGB) inequality"[4] implies $V_t^\xi \geq V_t^*$ P-$a.s.$ for all $t \in [0, T]$. The "worst-case probability" is then characterized by

$$\xi^* = -\frac{\theta}{(1-\gamma)V^*}\sigma^*, \quad (2.17)$$

and $V^{\xi^*} = V^*$. Therefore, HR utility satisfying the above conditions is SDU of the unnormalized form (2.9).

2.3.2 SDU Representation of HREZ Utility

Next, I prove that HREZ utility is SDU under certain integrability conditions. Assume $\gamma \neq \psi^{-1}$. Let $\beta^* = \dfrac{\beta(1-\gamma)}{1-\psi^{-1}}$. Then, Eq. (2.7) is rewritten as

$$V_t^\xi = \mathrm{E}_t^\xi\left[\int_t^T e^{-\beta^*(s-t)}\left(f^*(c_s, V_s^\xi) + \frac{(1-\gamma)V_s^\xi}{2\theta}|\xi_s|^2\right)ds\right], \quad (2.18)$$

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[3] Skiadas [18] prsents the following equation:

$$V_t^\xi = V_t^* + \frac{(1-\gamma)V_t^\xi}{2\theta}\mathrm{E}_t^\xi\left[\int_t^T e^{-\beta(s-t)}\left|\xi_s + \frac{\theta}{(1-\gamma)V_t^\xi}\sigma_s^*\right|^2 ds\right]. \quad (2.11)$$

In the equation above, $h(\sigma_s, V_s^\xi) - h(\sigma_s, V_s^*)$ in Eq. (2.12) is missing and V_s^ξ in Eq. (2.12) is replaced with V_t^ξ.

[4] See Appendix B in Duffie and Epstein [4].

where

$$f^*(c,v) = \frac{\beta}{1-\psi^{-1}} c^{1-\psi^{-1}} \big((1-\gamma)v\big)^{1-\frac{1-\psi^{-1}}{1-\gamma}}. \tag{2.19}$$

Suppose that a progressively measurable pair (V^*, σ^*) satisfies

$$dV_t^* = -\left(f^*(c_t, V_t^*) - \beta^* V_t^* - \frac{\theta}{2(1-\gamma)V_t^*}|\sigma_t^*|^2\right) dt + (\sigma_t^*)' dB_t, \quad V_T^* = 0. \tag{2.20}$$

Then, the following equation holds:[5]

$$\begin{aligned} V_t^\xi = V_t^* + \mathrm{E}_t^\xi \bigg[\int_t^T e^{-\beta^*(s-t)} \Big(& f^*(c_s, V_s^\xi) - f^*(c_s, V_s^*) \\ & + h(\sigma_s^*, V_s^\xi) - h(\sigma_s^*, V_s^*) + Q(\xi_s, \sigma_s^*, V_s^\xi)\Big) ds\bigg], \end{aligned} \tag{2.22}$$

where h and Q are given by Eqs. (2.13) and (2.14), respectively. The second derivative of f^* with respect to v is given by

$$f_{vv}^*(c,v) = \beta(\gamma - \psi^{-1}) c^{1-\psi^{-1}} \big((1-\gamma)v\big)^{-\frac{1-\psi^{-1}}{1-\gamma}-1} \begin{cases} > 0, & \text{if } \gamma > \psi^{-1}, \\ < 0, & \text{if } \gamma < \psi^{-1}. \end{cases} \tag{2.23}$$

Hence, $V_t^\xi - V_t^*$ is evaluted from below as in the following:

$$V_t^\xi - V_t^* \geq \begin{cases} \mathrm{E}_t^\xi \left[\int_t^T e^{-\beta^*(s-t)} \big(f_v^*(c_s, V_s^*) + h_v(\sigma_s^*, V_s^\xi)\big)(V_s^\xi - V_s^*) ds\right], & \text{if } \gamma > \psi^{-1}, \\ \mathrm{E}_t^\xi \left[\int_t^T e^{-\beta^*(s-t)} \big(f_v^*(c_s, V_s^\xi) + h_v(\sigma_s^*, V_s^\xi)\big)(V_s^\xi - V_s^*) ds\right], & \text{if } \gamma < \psi^{-1}. \end{cases} \tag{2.24}$$

Under certain integrability conditions on f_v^* and h_v, the SGB inequality implies $V_t^\xi \geq V_t^*$ P-*a.s.* for all $t \in [0,T]$. Then, the worst-case probability is characterized by Eq. (2.17), and $V^{\xi^*} = V^*$. Therefore, HREZ utility satisfying the above conditions is SDU of the unnormalized form (2.20).

Remark 3 In Eq. (2.20), as $\theta \searrow 0$, V^* converges to EZ utility, and HREZ utility is a generalization of EZ utility. Thus, the class of HREZ utlities satisfying this assumption includes HR utlity (*i.e.*, $\gamma = \psi^{-1}$), EZ utility (*i.e.*, $\theta = 0+$), and CRRA utility (*i.e.*, $\gamma = \psi^{-1}, \theta = 0+$) as special cases.

It is posited hereafter that the requisite integrability conditions are satisfied.

[5] Skiadas [18] shows the following equation:

$$V_t^\xi = V_t^* + \mathrm{E}_t^\xi \left[\int_t^T e^{-\beta^*(s-t)} \Big(f^*(c_s, V_s^\xi) - f^*(c_s, V_s^*) + Q(\xi_s, \sigma_s^*, V_s^\xi)\Big) ds\right], \tag{2.21}$$

where $Q(\xi_s, \sigma_s^*, V_s^*)$ is given by Eq. (2.14). In the above equation, $h(\sigma_s, V_s^\xi) - h(\sigma_s, V_s^*)$ in Eq.(2.22) is missing.

2.4 Normalized Representation of HREZ Utility

Using an ordinally equivalent utility[6] (OEU) introduced by Duffie and Epstein [4], I show the normalized form of V^*. Let

$$\mathcal{U} = \gamma + \theta. \tag{2.25}$$

Define an OEU process $\bar{V}_t$ of V_t^* by $\bar{V}_t = \bar{\varphi}(V_t^*)$ where

$$\bar{\varphi}(V_t^*) = \frac{1}{1-\mathcal{U}}\big((1-\gamma)V_t^*\big)^{1-\frac{\theta}{1-\gamma}}. \tag{2.26}$$

Proposition 1 *Let U be HREZ utility with $(\beta, \gamma, \psi, \theta)$. Then, U has the following normalized representation:*

$$\bar{V}_t = \mathrm{E}_t\left[\int_t^T \Big(\bar{f}(c_s, \bar{V}_s) - \bar{\beta}\bar{V}_s\Big) ds\right], \tag{2.27}$$

or equivalently,

$$\bar{V}_t = \mathrm{E}_t\left[\int_t^T e^{-\bar{\beta}(s-t)} \bar{f}(c_s, \bar{V}_s) ds\right], \tag{2.28}$$

where $\bar{\beta} = \dfrac{\beta(1-\mathcal{U})}{1-\psi^{-1}}$ *and*

$$\bar{f}(c_s, \bar{V}_s) = \beta \frac{c^{1-\psi^{-1}}}{1-\psi^{-1}}\big((1-\mathcal{U})v\big)^{1-\frac{1-\psi^{-1}}{1-\mathcal{U}}}. \tag{2.29}$$

The felicity function $\bar{f}$ is strictly increasing and concave in its consumption argument. In its utility argument, $\bar{f}$ is convex (resp., concave) if $\gamma + \theta > \psi^{-1}$ (resp., $\gamma + \theta < \psi^{-1}$), and linear if $\gamma + \theta = \psi^{-1}$.

Proof See Appendix A.1.

Kikuchi and Kusuda [9] refers to the coefficient $\mathcal{U} = \gamma + \theta$ as the relative uncertainty aversion. Note that relative risk aversion in the normalized aggregator of EZ utility in Eq. (2.2) is replaced by relative uncertainty aversion in that of HREZ utility in Eq. (2.29).

3 Observational Indistinguishability, Robustness Effects, and Identification Method

I explain the observational indistinguishability and robustness effects of HREZ utility and introduce an identification method proposed by Kikuchi and Kusuda [12].

[6] Two utility functionals U and $\bar{U}$ are ordinally equivalent if there is a strictly increasing and twice continuously differentiable function φ with $\varphi(0) = 0$ such that $\bar{U} = \varphi \circ U$.

3.1 HR Utility and EZ Utility

First, I analyze the observational indistinguishability and robustness effects of HR utility. To consider HR utility with (β, γ, θ), I substitute $\psi^{-1} = \gamma$ into Eq. (2.29), and obtain the normalized aggregator $\tilde{f}$ of HR utility:

$$\tilde{f}(c,v) = \beta \frac{c^{1-\gamma}}{1-\gamma}\Big(\big(1-(\gamma+\theta)\big)v\Big)^{-\frac{1-\gamma}{1-(\gamma+\theta)}} - \frac{\beta}{1-\gamma}\big(1-(\gamma+\theta)\big)v. \quad (3.1)$$

I compare the above HR utility with the EZ utility, such that $(\beta, \hat{\gamma}, \psi)$ satisfies $\psi = \gamma^{-1}$ and $\hat{\gamma} = \gamma + \theta$. Then, from Eq. (2.2), the normalized aggregator of EZ utility agrees with that in Eq. (3.1). Thus, these utilities are observationally indistinguishable. Maenhout [14] suggests that these utilities are observationally indistinguishable by showing that optimal portfolios based on these utilities coincide under one particular model: that is, the Black-Scholes model. Maenhout [14] also explains that although HREZ utility with (β, γ, θ) is observationally indistinguishable from EZ utility with $(\beta, \hat{\gamma}, \psi)$, HREZ utility has robustness effects that EZ utility does not have, as follows: Given that nonrobust agents have CRRA utility with $(\beta, \hat{\gamma})$, they are equally willing to substitute over time as across states, because $\hat{\gamma}$ is the inverse of the EIS. Robustness makes the agent less willing to substitute across states as the relative risk aversion becomes $\hat{\gamma} + \theta > \hat{\gamma}$, without altering the willingness to substitute intertemporally, as the EIS remains $\hat{\gamma}^{-1}$.

3.2 HREZ Utility and EZ Utility

Next, I analyze the observational indistinguishability and robustness effects of HREZ utility. Eqs. (2.2) and (2.29) show that the HREZ utility with $(\beta, \gamma, \psi, \theta)$ is observationally indistinguishable from the EZ utility with $(\beta, \hat{\gamma}, \psi)$ if $\hat{\gamma} = \gamma + \theta$. Following Maenhout [14], I nterpret the robustness effects of HREZ utlity as follows. Suppose that the nonrobust agent has EZ utility with $(\beta, \hat{\gamma}, \psi)$. What robustness does is to make the agent less willing to substitute across states as the relative uncertainty aversion becomes $\gamma + \theta > \gamma$, without altering the willingness to substitute intertemporally as the EIS remains ψ.

3.3 HREZ Utilities with the Common Relative Uncertainty Aversions

HREZ utilities are not only observationally indistinguishable from EZ utilities, but also observationally indistinguishable among HREZ utilities. Eq. (2.29) in Proposition 1 shows that the felicity function of HREZ utility with $(\beta, \gamma, \theta, \psi)$ is characterized by $(\beta, \mathcal{U}, \psi)$ where $\mathcal{U} = \gamma + \theta$. Thus, HREZ utilities with common relative uncertainty aversion are observationally indistinguishable.

3.4 Identification Method

To solve this observational indistinguishability problem, Kikuchi and Kusuda [12] focus on the fact that the worst-case probability characterized by Eq. (2.17) depends on relative ambiguity aversion. They assume a quadratic security market model where state X_t is a stationary process. Let $R(X_t)$ denote the instantaneous return on the S&P500 at time t. Let E^* denote the expectation under worst-case probability. Kikuchi and Kusuda [12] introduce the notion of the *premium* $\bar{\nu}(X_t)$ and *worst-case premium* $\hat{\nu}(X_t)$, defined by

$$\bar{\nu}(X_t) = \mathrm{E}_t\left[R(X_t)\right] - r(X_t), \qquad \hat{\nu}(X_t) = \mathrm{E}_t^*\left[R(X_t)\right] - r(X_t). \tag{3.2}$$

They then introduce the notion of the *long-term premium* $\bar{\nu}$ and *worst-case long-term premium* $\hat{\nu}$, defined by

$$\bar{\nu} = \mathrm{E}_t\left[\lim_{t\to\infty} \bar{\nu}(X_t)\right], \qquad \hat{\nu} = \mathrm{E}^*[\lim_{t\to\infty} \hat{\nu}(X_t)]. \tag{3.3}$$

Kikuchi and Kusuda [12] derive analytical expressions for the long-term and worst-case long-term premiums.

Suppose that an agent has HREZ utility with $(\beta, \gamma, \theta, \psi)$ and that we observe their consumption, wealth, and portfolio. We can then recognize $(\beta, \mathcal{U}, \psi)$ where $\mathcal{U} = \gamma + \theta$. However, we cannot identify γ or θ. Furthermore, we cannot exclude the possibility that $\theta = 0$, that is, EZ utility, or $\gamma = \psi^{-1}$, that is, HR utility. Note that the agent does not recognize their own utility, and therefore cannot inform us of γ or θ. Kikuchi and Kusuda [12] show that if they inform us of their long-term expected rate of return and their worst-case long-term expected rate of return, then we can calculate γ and θ using the analytical expressions for the long-term and worst-case long-term premiums.

4 Ambiguity and Risk Aversions and Preferences for Information

I analyze the ambiguity and risk aversions of HREZ utility based on the notions defined by Chen and Epstein [3]. I then examine preferences for information using the definition introduced by Skiadas [17].

4.1 Comparative Ambiguity Aversion

Chen and Epstein [3] define the notion of *comparative ambiguity aversion* as follows (for formal arguments, see Epstein [5] and Epstein and Zhang [6]): Event $A \in \mathcal{F}_T$ is said to be *unambiguous* if $\tilde{\mathrm{P}}(A) = \mathrm{P}(A)$ for every $\tilde{P} \in \mathbb{P}$. Let $\mathcal{R}$ denote the class of unambiguous events. Let $\mathcal{R}_t = \mathcal{R} \cap \mathcal{F}_t$ for every $t \in [0, T]$. The consumption process c is said to be *unambiguous* if c_t is $\mathcal{R}_t$-measurable for every $t \in [0, T]$. Let $\mathcal{C}_{\mathcal{R}}$ denote the set of all unambiguous consumption processes. Comparative ambiguity aversion is defined as

Definition 2 Let U and $\tilde{U}$ be HREZ utilities with the corresponding unambiguous classes $\mathcal{R}$ and $\tilde{\mathcal{R}}$ of unambiguous events. $\tilde{U}$ is said to be more ambiguity averse than U if $\mathcal{R} \supset \tilde{\mathcal{R}}$ and if for every $c \in \mathcal{C}$ and every $\tilde{\mathcal{R}}$-unambiguous consumption plan $c^{\mathcal{R}} \in \mathcal{C}_{\tilde{\mathcal{R}}}$, the following holds:

$$U(c) \le U(c^{\mathcal{R}}) \quad \Longrightarrow \quad \tilde{U}(c) \le \tilde{U}(c^{\mathcal{R}}). \tag{4.1}$$

The condition $\mathcal{R} \supset \tilde{\mathcal{R}}$ in the above definiton means that a more ambiguous averse agent views more events as ambiguous.

Lemma 1 *Let U be HREZ utility with $(\beta, \gamma, \psi, \theta)$. Let $c^{\mathcal{R}} \in \mathcal{C}_{\mathcal{R}}$ and V_t denote the utility process of $U(c^{\mathcal{R}})$. Then, V_t satisfies*

$$V_t = \mathrm{E}_t \left[\int_t^T e^{-\beta^*(s-t)} f^*(c_s^{\mathcal{R}}, V_s)\, ds \right], \tag{4.2}$$

where f^* is given by Eq. (2.19). Function f^* is concave in its consumption argument. In its utility argument, f^* is convex (resp., concave) if $\gamma > \psi^{-1}$ (resp., $\gamma < \psi^{-1}$), and linear if $\gamma = \psi^{-1}$.

Proof See Appendix A.2.

Remark 4 Note that f^* depends only on (β, γ, ψ), but not on θ. Thus, Eq. (4.2) shows that $U(c^{\mathcal{R}})$ dedends only on (β, γ, ψ), but not on θ. Given that $c^{\mathcal{R}}$ is an unambiguous consumption plan, this is a natural consequence.

Proposition 2 *Let U and $\tilde{U}$ be HREZ utilities with $(\beta, \gamma, \psi, \theta)$ and $(\beta, \gamma, \psi, \tilde{\theta})$. If $\tilde{\theta} > \theta$, then $\tilde{U}$ is more ambiguity averse than U.*

Proof See Appendix A.3.

4.2 Comparative Risk Aversion

Let $\bar{c}_t = \mathrm{E}[c_t]$ for every $t \in [0, T]$. The following notion of *comparative risk aversion* is introduced by Chen and Epstein [3]:

Definition 3 Let U and $\tilde{U}$ be HREZ utilities with the corresponding classes $\mathcal{R}$ and $\tilde{\mathcal{R}}$ of unambiguous events. $\tilde{U}$ is said to be more risk averse than U if $\mathcal{R} \subset \tilde{\mathcal{R}}$ and if for every $c^{\mathcal{R}} \in \mathcal{C}_{\mathcal{R}}$, the following holds:

$$U(c^{\mathcal{R}}) \leq U(\bar{c}) \quad \Longrightarrow \quad \tilde{U}(c^{\mathcal{R}}) \leq \tilde{U}(\bar{c}). \tag{4.3}$$

Lemma 2 *Let U be HREZ utility with $(\beta, \gamma, \psi, \theta)$. Let $c^{\mathcal{R}} \in \mathcal{C}_{\mathcal{R}}$ and let V_t denote the utility process of $U(c^{\mathcal{R}})$. Define $\hat{V}_t = \hat{\varphi}(V_t; \gamma)$ where*

$$\hat{\varphi}(v; \gamma) = \big((1-\gamma)v\big)^{\frac{1}{1-\gamma}}. \tag{4.4}$$

Then, there exists an adapted process $\hat{\sigma}$ such that

$$d\hat{V}_t = -\left(\hat{f}(c_s^{\mathcal{R}}, \hat{V}_s) - \hat{\beta}\hat{V}_t - \frac{\gamma}{2\hat{V}_t}|\hat{\sigma}_t|^2 \right) dt + \hat{\sigma}_t' dB_t, \tag{4.5}$$

where $\hat{\beta} = \dfrac{\beta}{1-\psi^{-1}}$ and

$$\hat{f}(c, v) = \frac{\beta}{1-\psi^{-1}} c^{1-\psi^{-1}} v^{\psi^{-1}}. \tag{4.6}$$

Proof See Appendix A.4.

Remark 5 Let $\bar{V}_t$ denote the utility process of $U(\bar{c})$. Then, the SDE (4.5) is simplified as the following ODE.

$$d\bar{V}_t = -\left(\hat{f}(\bar{c}_s, \hat{V}_s) - \hat{\beta}\bar{V}_t\right) dt. \tag{4.7}$$

Therefore, $\bar{V}_t$ has the representation of the form.

$$\bar{V}_t = \int_t^T e^{-\hat{\beta}(s-t)} \hat{f}(\bar{c}_s, \bar{V}_s)\, ds. \tag{4.8}$$

Eq. (4.8) shows that $U(\bar{c})$ depends only on (β, ψ) but not on (γ, θ). Given that $\bar{c}$ is a deterministic consumption plan, this is a natural result.

Proposition 3 *Let U and U^* be HREZ utilities with $(\beta, \gamma, \psi, \theta)$ and $(\beta, \gamma^*, \psi, \theta^*)$. If $\gamma^* > \gamma$, then U^* is more risk averse than U.*

Proof See Appendix A.5.

4.3 Ambiguity Aversion

To define ambiguity aversion of utility, Chen and Epstein [3] introduce the notion of *probabilistically sophisticated utility for timeless prospects*[7]. The following definition is interpreted as the notion of probabilistically sophisticated utility for the case of HREZ utility in continuous-time settings.

Definition 4 For HREZ utility with $(\beta, \gamma, \psi, \theta)$, the corresponding probabilistically sophisticated utility is EZ utility with (β, γ, ψ).

Then, ambiguity aversion is defined as follows.

Definition 5 Let U be HREZ utility. Let $\tilde{U}$ be the corresponding probabilistically sophisticated utility. Then, U is ambiguity averse if for every unambiguous consumption plan $c^{\mathcal{R}} \in \mathcal{C}_{\mathcal{R}}$ and every consumption plan $c \in \mathcal{C}$, the following holds:

$$\tilde{U}(c) \le \tilde{U}(c^{\mathcal{R}}) \quad \Longrightarrow \quad U(c) \le U(c^{\mathcal{R}}). \tag{4.9}$$

Proposition 4 *HREZ utility is ambiguity averse.*

Proof Let U be HREZ utility with $(\beta, \gamma, \psi, \theta)$. Let $\tilde{U}$ be the corresponding EZ utility with (β, γ, ψ). Assume that $c^{\mathcal{R}} \in \mathcal{C}_{\mathcal{R}}$ and $c \in \mathcal{C}$ are such that $\tilde{U}(c) \le \tilde{U}(c^{\mathcal{R}})$. First, $\tilde{U}(c^{\mathcal{R}}) = U(c^{\mathcal{R}})$ holds from Lemma 1. As $\tilde{U}$ is interpreted as HREZ utility with $(\beta, \gamma, \psi, 0+)$, from Propositon 2, the following holds:

$$U(c) \le \tilde{U}(c). \tag{4.10}$$

Thus, it follows from Eq. (4.10) and $\tilde{U}(c^{\mathcal{R}}) = U(c^{\mathcal{R}})$ that $U(c) - U(c^{\mathcal{R}}) = U(c) - \tilde{U}(c) + \tilde{U}(c) - \tilde{U}(c^{\mathcal{R}}) + \tilde{U}(c^{\mathcal{R}}) - U(c^{\mathcal{R}}) \le 0$. Therefore, U is ambiguity averse.

[7] for definition, see Chen and Epstein [3].

4.4 Risk Aversion

Chen and Epstein [3] provide the following definition of *risk aversion*:

Definition 6 Utility U is risk averse if for every $c^{\mathcal{R}} \in \mathcal{C}_{\mathcal{R}}$,

$$U(c^{\mathcal{R}}) \leq U(\bar{c}^{\mathcal{R}}). \tag{4.11}$$

Extending the proof by Duffie and Epstein [4] to the case of HREZ utility, I show that HREZ utility is risk averse.

Proposition 5 *HREZ utility is risk averse.*

Proof See Appendix A.6.

4.5 Preferences for Information

I analyze the preferences for information of HREZ utility based on Skiadas [17], who introduce the concept of preferences for information. He demonstrates that SDU is *information seeking* (resp., *averse*) (for definitions, see Skiadas [17]) if the normalized aggregator is convex (resp., concave) in its utility argument (Proposition A in Skiadas [17]). However, this concept assumes that each agent's subjective probability is unique, rendering it directly inapplicable to HREZ utility. Therefore, I analyze preferences for information of the following two utilities related to HREZ utility, rather than HREZ utility itself. The first utility is the observationally indistinguishable EZ utility from HREZ utility. The second is a restricted HREZ utility, where the domain is restricted to the set $\mathcal{C}_{\mathcal{R}}$ of unambiguous consumption plans. For any event on unambiguous consumption plans, evaluations by all candidate probability measures are identical. Therefore, the subjective probability measure of an agent with HREZ utility coincides with the base probability.

Proposition 6 *Let U denote HREZ utility with $(\beta, \gamma, \psi, \theta)$. Let $U_{\mathcal{R}}$ be HREZ utility with $(\beta, \gamma, \psi, \theta)$, in which the domain of HREZ utility is restricted to unambiguous consumption plans $\mathcal{C}_{\mathcal{R}}$, and let $\tilde{U}$ be EZ utility with $(\beta, \hat{\gamma}, \psi)$ where $\hat{\gamma} = \gamma + \theta$. Then, the following 1 and 2 hold.*

1. *$\tilde{U}$ is information seeking (resp., averse) if $\gamma + \theta > \psi^{-1}$ (resp., $\gamma + \theta < \psi^{-1}$).*
2. *$U_{\mathcal{R}}$ is information seeking (resp., averse) if $\gamma > \psi^{-1}$ (resp., $\gamma < \psi^{-1}$).*

Proof First, consider $U_{\mathcal{R}}$. Then, from Lemma 1, $U_{\mathcal{R}}$ has the normalized aggregator f^* given by Eq. (2.19), and f^* is convex (resp., concave) in its utility argument, if $\gamma > \psi^{-1}$ (resp., $\gamma < \psi^{-1}$). Thus, $U_{\mathcal{R}}$ is information seeking (resp., concave), if $\gamma > \psi^{-1}$ (resp., $\gamma < \psi^{-1}$). Next, consider $\tilde{U}$. Then, $\tilde{U}$ has a normalized aggregator $\bar{f}$ given by Eq. (2.29), and $\bar{f}$ is convex (tesp., concave) in its utility argument, if $\gamma + \theta > \psi^{-1}$ (resp., $\gamma + \theta < \psi^{-1}$). Therefore, $U_{\mathcal{R}}$ is information seeking (resp., averse) if $\gamma + \theta > \psi^{-1}$ (resp., $\gamma + \theta < \psi^{-1}$).

Remark 6 Note that rhe inverse ψ^{-1} of EIS can be interpreted as temporal variation aversion. According to Proposition 6.2, if the domain of HREZ utility is restricted to the set of unambiguous consumption plans, the restricted HREZ utility is information seeking (resp., averse) if the relative risk aversion is greater (resp., less) than the inverse of EIS, *i.e.*, temporal variation aversion. This is interpreted to mean that an agent whose relative risk aversion is greater than temporal variation aversion seeks information to avoid risk because the agent prioritizes avoiding risk rather than temporal variation.

5 Other Properties

I demonstrate that HREZ utility is strictly increasing and time consistent by extending the proof by Duffie and Epstein [4] to cases where the uniform Lipschitz condition is not satisfied. I exploit convexity or concavity in the utility argument of the felicity function in lieu of the uniform Lipschitz condition. Finally, I show that HREZ utility is homothetic.

5.1 Monotonicity for Consumption

First, I show that HREZ utility is strictly increasing.

Proposition 7 *HREZ utility is strictly increasing.*

Proof See Appendix A.7.

5.2 Time Consistency

Next, I present that HREZ utility is *time consistent* (for definition, see Duffie and Epstein [4]). As demonstrated in the proof of Proposition 4 by Duffie and Epstein [4], any SDU that satisfies monotonicity for terminal value is time consistent. To show that HREZ utility satisfies monotonicity for terminal value, I tentatively extend the definition of HREZ utility so that there is a terminal reward at some $[0, T]$-valued stopping time τ. The terminal reward is defined by some $\mathcal{F}_\tau$-measurable integrable random variable Y. Then, Proposition A1 in Duffie and Epstein [4] implies that there is a unique integrable semimartingale $V^{c,Y}$ that solves the following equation:

$$V_t^{c,Y} = \mathrm{E}_t \left[\int_t^\tau e^{-\bar{\beta}(s-t)} \bar{f}(c_s, V_s^{c,Y}) ds + e^{-\bar{\beta}(\tau-t)} Y \right], \quad t \in [0, T), \tag{5.1}$$

where $\bar{f}$ is given by Eq. (2.29).

Proposition 8 *Let U be HREZ utility with $(\beta, \gamma, \psi, \theta)$. Let τ be a $[0, T]$-valued stopping time. Suppose $Y > \tilde{Y}$, for $\mathcal{F}_\tau$-measurable integrable random variables Y and $\tilde{Y}$. For any given $c \in \mathcal{C}$, let $V = V^{c,Y}$ and $\tilde{V} = V^{c,\tilde{Y}}$ be defined by Eq. (5.1). Then, $V \geq \tilde{V}$ and U is time consistent.*

Proof See Appendix A.8.

5.3 Homotheticity

Finally, I demonstrate the homotheticty of HREZ utility. A utility functional U is *homothetic* if for any consumption plan c and $\tilde{c}$, and any scalar $\alpha > 0$, $U(\alpha\tilde{c}) \geq U(\alpha c) \Leftrightarrow U(\tilde{c}) \geq U(c)$.

Proposition 9 *HREZ utility is homothetic.*

Proof See Appendix A.9.

Remark 7 Kikuchi and Kusuda [10] demonstrate that both the optimal consumption-wealth ratio and optimal portfolio based on HR utility depend only on the state, but not on time or wealth. This result is attributed to the homotheticity of HR utility.

6 Conclusion

The propseties of HREZ utility were studied. First, I proved that HREZ utility is SDU under certain integrability conditions by correcting the erroneous proof in Skiadas [18]. It was assumed hereafter that the requisite integrability conditions are satisfied. I then derived the normalized representation of HREZ utility and that the felicity function is chracterized by the subjective discount rate, EIS, and relative uncertainty aversion. Second, I showed that HREZ utility is observationally indistinguishable not only from EZ utility but also among HREZ utilities with common relative uncertainty aversion. This is a natural consequence of the fact that the felicity function in the normalized HREZ utility is characterized by the subjective discount rate, EIS, and relative uncertainty aversion. I explained the robustness effects of HREZ utility, and introduced an identification method, as proposed by Kikuchi and Kusuda [12].

Third, I analyzed ambiguity and risk aversions of HREZ utility based on the definitions provided by Chen and Epstein [3]. Specifically, I proved the following: (i) if the subjective discount rate, relative risk aversion, and EIS are the same for two HREZ utilities, then the utility with greater relative ambiguity aversion is more ambiguity averse; (ii) if the subjective discount rate and EIS are the same for two HREZ utilities, then the utility with greater relative risk aversion is more risk averse; (iii) HREZ utility exhibits both ambiguity and risk aversions. Fourth, I examined preferences for information of HREZ utility referencing concepts in Skiadas [17]. Since Skiadas [17]' framework for preferences for information cannot be directly applied to HREZ utility, I instead analyzed preferences for information of two related utilities: (i) the observationally indistinguishable EZ utility from HREZ utility, and (ii) a restricted version of HREZ utility, where the domain is restricted to unambiguous consumption plans. I demonstrated that the observationally indistinguishable EZ utility is information seeking (resp., averse) if the sum of relative risk aversion and relative ambiguity aversion is greater (resp., less) than the inverse of EIS, and that the restricted HREZ utility is information seeking (resp., averse) if relative risk aversion is greater (resp., less) than the inverse of EIS.

Fifth, I demonstrated that HREZ utility is strictly increasing and time consistent by extending the proof by Duffie and Epstein [4] to cases where the uniform Lipschitz condition is not satisfied. I exploit convexity or concavity in the utility argument of the felicity function in lieu of the uniform Lipschitz condition. Finally, I showd that HREZ utility is homothetic.

A Proofs

A.1 Proof of Proposition 1

From Ito's lemma, the SDE for $\bar{V}_t$ is calculated as

$$\begin{aligned}
d\bar{V}_t =& \frac{d\bar{\varphi}}{dv}(V_t^*)dV_t^* + \frac{1}{2}\frac{d\bar{\varphi}^2}{dv^2}(V_t^*)(dV_t^*)^2 \\
=&\big((1-\gamma)V_t^*\big)^{-\frac{\theta}{1-\gamma}}\left\{-\left(f^*(c_t,V_t^*) - \beta^* V_t^* - \frac{\theta}{2(1-\gamma)V_t^*}|\sigma_t^*|^2\right)dt + (\sigma_t^*)' dB_t\right\} \\
&+ \frac{1}{2}\big((1-\gamma)V_t^*\big)^{-\frac{\theta}{1-\gamma}}\left(-\frac{\theta}{(1-\gamma)V_t^*}\right)|\sigma_t^*|^2 dt \\
=&-\left\{\big((1-\gamma)V_t^*\big)^{-\frac{\theta}{1-\gamma}} f^*(c_t,V_t^*) - \frac{\beta^*}{1-\gamma}\big((1-\gamma)V_t^*\big)^{1-\frac{\theta}{1-\gamma}}\right\}dt + \bar{\sigma}_t' dB_t \\
=&-\bar{f}(c_t,\bar{V}_t)\,dt + \bar{\sigma}_t' dB_t,
\end{aligned} \tag{A.1}$$

where $\bar{\sigma}_t = \big((1-\gamma)V_t^*\big)^{-\frac{\theta}{1-\gamma}}\sigma_t^*$ and Eq. (2.29). Therefore, $\bar{V}_t$ has the normalized representations (2.27) and (2.28). The derivatives of $\bar{f}$ are calculated as

$$\bar{f}_c(c,v) = \beta c^{-\psi^{-1}}\Big(\big(1-\mathcal{U}\big)v\Big)^{1-\frac{1-\psi^{-1}}{1-\mathcal{U}}} > 0, \tag{A.2}$$

$$\bar{f}_{cc}(c,v) = -\beta\psi^{-1}\big((1-\mathcal{U})v\big)^{1-\frac{1-\psi^{-1}}{1-\mathcal{U}}} < 0, \tag{A.3}$$

$$\bar{f}_{vv}(c,v) = \beta(\gamma+\theta-\psi^{-1})\big((1-\mathcal{U})v\big)^{-\frac{1-\psi^{-1}}{1-\mathcal{U}}-1}. \tag{A.4}$$

Thus, $\bar{f}$ is strictly increasing and concave in its consumption argument. In its utility argument, $\bar{f}$ is convex (resp., concave) if $\gamma+\theta > \psi^{-1}$ (resp., $\gamma+\theta < \psi^{-1}$), and linear if $\gamma+\theta = \psi^{-1}$.

A.2 Proof of Lemma 1

In the case of $c^{\mathcal{R}} \in \mathcal{C}_{\mathcal{R}}$, Eq. (2.18) is rewritten as

$$V_t^{\xi} = \mathrm{E}_t\left[\int_t^T e^{-\beta^*(s-t)}\left(f^*(c_s^{\mathcal{R}},V_s^{\xi}) + \frac{(1-\gamma)V_s^{\xi}}{2\theta}|\xi_s|^2\right)ds\right]. \tag{A.5}$$

Subtracting Eq. (4.2) from Eq. (A.5) yields

$$V_t^{\xi} - V_t = \mathrm{E}_t\left[\int_t^T e^{-\beta^*(s-t)}\left(f^*(c_s^{\mathcal{R}},V_s^{\xi}) - f^*(c_s^{\mathcal{R}},V_s) + \frac{(1-\gamma)V_s^{\xi}}{2\theta}|\xi_s|^2\right)ds\right]. \tag{A.6}$$

Then, the following holds:

$$f_{vv}^*(c,v) = \beta(\gamma-\psi^{-1})c^{1-\psi^{-1}}\big((1-\gamma)v\big)^{-\frac{1-\psi^{-1}}{1-\gamma}-1}. \tag{A.7}$$

Thus, in its utility argument, f^* is convex (resp., concave) if $\gamma > \psi^{-1}$ (resp., $\gamma < \psi^{-1}$), and linear if $\gamma = \psi^{-1}$. Hence, $V_t^\xi - V_t$ is evaulated from below as in the following:

$$V_t^\xi - V_t \geq \begin{cases} \mathrm{E}_t\left[\int_t^T e^{-\beta^*(s-t)} f_v^*(c_s^{\mathcal{R}}, V_s^*)(V_s^\xi - V_s)\, ds\right], & \text{if } \gamma \geq \psi^{-1}, \\ \mathrm{E}_t^\xi\left[\int_t^T e^{-\beta^*(s-t)} f_v^*(c_s, V_s^\xi)(V_s^\xi - V_s^*)\, ds\right], & \text{if } \gamma < \psi^{-1}. \end{cases} \tag{A.8}$$

Then, the SGB inequality implies $V_t^\xi \geq V_t$ P-$a.s.$ for all $t \in [0,T]$. Therefore, the minimizer of ξ is given by $\xi_t = 0$ for every $t \in [0,T]$, and $U(c^{\mathcal{R}})$ satisfies Eq. (A.5). Finally, f^* is concave in its consumption argument, as shown in the following:

$$f_{cc}^*(c,v) = -\beta\psi^{-1}c^{-\psi^{-1}}\Big((1-\gamma)v\Big)^{1-\frac{1-\psi^{-1}}{1-\gamma}} < 0. \tag{A.9}$$

A.3 Proof of Proposition 2

It is obvious that $\mathcal{R} = \tilde{\mathcal{R}}$. It follows by Lemma 1 that $U(c^{\mathcal{R}}) = \tilde{U}(c^{\mathcal{R}})$ for every $c^{\mathcal{R}} \in \mathcal{C}_{\mathcal{R}}$. Let $c \in \mathcal{C}$ such that $U(c) \leq U(c^{\mathcal{R}})$. Let OEUs $\bar{U} = \bar{\varphi}(U)$ and $\check{U} = \bar{\varphi}(\tilde{U})$ where $\bar{\varphi}$ is given by Eq. (2.26). Then, I obtain Eq. (2.27) and

$$\check{V}_t = \mathrm{E}_t\left[\int_t^T \big(\bar{f}(c_s, \check{V}_s) - q(\check{\sigma}_t, \check{V}_t)\big)\, ds\right]. \tag{A.10}$$

where

$$q(\sigma, v) = \frac{1}{2}(\tilde{\theta} - \theta)\big((1-\gamma)v\big)^{-\frac{\theta}{1-\gamma}-1}|\sigma|^2 \geq 0. \tag{A.11}$$

Let $\bar{\beta} = \dfrac{\beta(1-\mathcal{U})}{1-\psi^{-1}}$. Then, Eqs. (2.27) and (A.10) are rewritten as

$$\bar{V}_t = \mathrm{E}_t\left[\int_t^T e^{-\bar{\beta}(s-t)}\hat{f}(c_s, \bar{V}_s) ds\right], \tag{A.12}$$

$$\check{V}_t = \mathrm{E}_t\left[\int_t^T e^{-\bar{\beta}(s-t)}\Big(\hat{f}(c_s, \check{V}_s) - q(\check{\sigma}_t, \check{V}_t)\Big)\, ds\right], \tag{A.13}$$

where

$$\hat{f}(c,v) = \frac{\beta}{1-\psi^{-1}}(1-\mathcal{U})v\Big(c\big((1-\mathcal{U})v\big)^{-\frac{1}{1-\mathcal{U}}}\Big)^{1-\psi^{-1}}. \tag{A.14}$$

Note that $\mathcal{U} = \gamma + \theta > \psi^{-1}$ because $\gamma \geq \psi^{-1}$ and $\theta > 0$. Then, in its utility argument, $\hat{f}$ is convex (resp., concave) if $\gamma + \theta > \psi^{-1}$ (resp., $\gamma + \theta < \psi^{-1}$), and linear if $\gamma + \theta = \psi^{-1}$ as shown in the following:

$$\hat{f}_{vv}(c,v) = \beta\big(\mathcal{U} - \psi^{-1}\big)c^{1-\psi^{-1}}\Big((1-\mathcal{U})v\Big)^{-\frac{1-\psi^{-1}}{1-(\gamma+\theta)}-1}. \tag{A.15}$$

Subrtracting Eq. (A.13) from Eq. (A.12) yields

$$\begin{aligned} \bar{V}_t - \check{V}_t &= \mathrm{E}_t\left[\int_t^T e^{-\bar{\beta}(s-t)}\Big(\hat{f}(c_s, \bar{V}_s) - \hat{f}(c_s, \check{V}_s) + q(\check{\sigma}_t, \check{V}_t)\Big)\, ds\right] \\ &\geq \begin{cases} \mathrm{E}_t\left[\int_t^T e^{-\bar{\beta}(s-t)}\hat{f}_v(c_s, \check{V}_s)(\bar{V}_s - \check{V}_s)\, ds\right], & \text{if } \gamma \geq \psi^{-1}, \\ \mathrm{E}_t\left[\int_t^T e^{-\bar{\beta}(s-t)}\hat{f}_v(c_s, \bar{V}_s)(\bar{V}_s - \check{V}_s)\, ds\right], & \text{if } \gamma < \psi^{-1}. \end{cases} \end{aligned} \tag{A.16}$$

The SGB inequality implies $\bar{V}_t \geq \check{V}_t$ P-$a.s.$ for all $t \in [0,T]$. Therefore, $\tilde{U}(c) \leq U(c) \leq U(c^{\mathcal{R}}) = \tilde{U}(c^{\mathcal{R}})$.

A.4 Proof of Lemma 2

By Lemma 1, Eq. (4.2) and thus the following holds.

$$V_t = \mathrm{E}_t \left[\int_t^T \Big(f^*(c_s^{\mathcal{R}}, V_s) - \beta^* V_s \Big)\, ds \right]. \tag{A.17}$$

Then, by martingale representation theorem, there exists an adapted process σ such that the SDE for V_t satisfies

$$dV_t = -\Big(f^*(c_s^{\mathcal{R}}, V_s) - \beta^* V_s \Big)\, dt + \sigma_t' dB_t. \tag{A.18}$$

From Ito's lemma, the ODE process $\hat{V}_t = \hat{\varphi}(V_t;\gamma)$ satisfies

$$\begin{aligned} d\hat{V}_t &= \frac{d\hat{\varphi}}{dv}(V_t;\gamma) dV_t + \frac{1}{2}\frac{d\bar{\varphi}^2}{dv^2}(V_t;\gamma)(dV_t)^2 \\ &= \big((1-\gamma)V_t\big)^{\frac{1}{1-\gamma}-1} \left\{ - \Big(f^*(c_t^{\mathcal{R}}, V_t) - \beta^* V_t \Big)\, dt + \sigma_t' dB_t + \frac{1}{2}\frac{\gamma}{V_t}|\sigma_t|^2 dt \right\} \\ &= -\left(\hat{f}(c_t^{\mathcal{R}}, \hat{V}_t) - \hat{\beta}\hat{V}_t - \frac{\gamma}{2\hat{V}_t}|\hat{\sigma}_t|^2 \right) dt + \hat{\sigma}_t' dB_t, \end{aligned} \tag{A.19}$$

where $\hat{\sigma}_t = \big((1-\gamma)V_t\big)^{\frac{1}{1-\gamma}-1}\sigma_t$.

A.5 Proof of Proposition 3

It is obvious that $\mathcal{R} = \tilde{\mathcal{R}}$. Let $c^{\mathcal{R}} \in \mathcal{C}_{\mathcal{R}}$ such that $U(c^{\mathcal{R}}) \leq U(\bar{c})$. Let V and V^* denote the utility process of $U(c^{\mathcal{R}})$ and $U^*(c^{\mathcal{R}})$, respectively. First, by Lemma 2.2, $U(\bar{c}) = U^*(\bar{c})$. Thus, it suffices to show $U^*(c^{\mathcal{R}}) \leq U(c^{\mathcal{R}})$. By Lemma 1, V_t satisfies Eq. (4.2). Define $\hat{V}_t^* = \hat{\varphi}(V_t^*;\gamma)$ where $\hat{\varphi}$ is given by Eq. (4.4). Then, by Lemma 2.1,

$$d\hat{V}_t = -\left(\hat{f}(c_s^{\mathcal{R}}, \hat{V}_s) - \hat{\beta}\hat{V}_t - \frac{\gamma}{2\hat{V}_t}|\hat{\sigma}_t|^2 \right) dt + \hat{\sigma}_t' dB_t. \tag{A.20}$$

Let $\tilde{V}_t^* = \tilde{\varphi}(\hat{V}_t^*)$ where

$$\tilde{\varphi}(v) = \frac{1}{1-\gamma} v^{1-\gamma}. \tag{A.21}$$

By Ito's lemma, $\tilde{V}_t$ satisfies

$$\tilde{V}_t^* = \mathrm{E}_t \left[\int_t^T \left(f^*(c_s^{\mathcal{R}}, \tilde{V}_s^*) - \beta^* \tilde{V}_s^* - \frac{\gamma^* - \gamma}{2\hat{V}_s^*}|\hat{\sigma}_s^*|^2 \right) ds \right], \tag{A.22}$$

where f^* is given by Eq. (2.19) and $\beta^* = \dfrac{\beta(1-\gamma)}{1-\psi^{-1}}$. Eq. (A.22) is rewritten as

$$\tilde{V}_t^* = \mathrm{E}_t \left[\int_t^T e^{-\beta^*(s-t)} \left(f^*(c_s^{\mathcal{R}}, \tilde{V}_s^*) - \frac{\gamma^* - \gamma}{2\hat{V}_s^*}|\hat{\sigma}_s^*|^2 \right) ds \right]. \tag{A.23}$$

Subtracting $\tilde{V}_t^*$ from V_t yields

$$V_t - \tilde{V}_t^* = \mathrm{E}_t \left[\int_t^T e^{-\beta^*(s-t)} \left(f^*(c_s^{\mathcal{R}}, V_s) - f^*(c_s^{\mathcal{R}}, \tilde{V}_s^*) + \frac{\gamma^* - \gamma}{2\hat{V}_s^*}|\hat{\sigma}_s^*|^2 \right) ds \right], \tag{A.24}$$

As $\gamma^* - \gamma > 0$, $V_t - \tilde{V}_t^*$ is evaluted as

$$V_t - \tilde{V}_t^* \geq \begin{cases} \mathrm{E}_t \left[\int_t^T e^{-\beta^*(s-t)} f_v^*(c_s^{\mathcal{R}}, \tilde{V}_s^*)\big(V_s - \tilde{V}_s^*\big) ds \right], & \text{if } \gamma \geq \psi^{-1}, \\ \mathrm{E}_t \left[\int_t^T e^{-\beta^*(s-t)} f_v^*(c_s^{\mathcal{R}}, V_s)\big(V_s - \tilde{V}_s^*\big) ds \right], & \text{if } \gamma < \psi^{-1}. \end{cases} \tag{A.25}$$

Therefore, $U^*(c^{\mathcal{R}}) \leq U(c^{\mathcal{R}})$.

A.6 Proof of Proposition 5

Let U denote HREZ utility with $(\beta, \gamma, \psi, \theta)$. Let $c^{\mathcal{R}} \in \mathcal{C}_{\mathcal{R}}$. Let $V_0 = U(c)$ and $\bar{V}_0 = U(\bar{c})$. From Lemma 1, we have

$$\bar{V}_t - V_t = \mathrm{E}_t\left[\int_t^T e^{-\beta^*(s-t)}\Big(f^*(\bar{c}_s, \bar{V}_s) - f^*(c_s^{\mathcal{R}}, V_s)\Big)\,ds\right]. \tag{A.26}$$

where f^* is given by Eq. (2.19). Using Fubini's Theorem for conditional expectations, we obtain

$$\begin{aligned}\bar{V}_t - V_t = \mathrm{E}_t\Big[\int_t^T \Big\{& e^{-\beta^*(s-t)}\mathrm{E}_t\Big[f^*(\bar{c}_s, \bar{V}_s) - f^*(c_s^{\mathcal{R}}, \bar{V}_s)\Big] \\ &+ e^{-\beta^*(s-t)}\Big(f^*(c_s^{\mathcal{R}}, \bar{V}_s) - f^*(c_s^{\mathcal{R}}, V_s)\Big)\Big\}ds\Big].\end{aligned} \tag{A.27}$$

As f^* is concave in its consmuption argument (Lemma 1), by Jensen's Inequality for conditional expectations,

$$\mathrm{E}_t\Big[f^*(\bar{c}_s, \bar{V}_s) - f^*(c_s^{\mathcal{R}}, \bar{V}_s)\Big] \geq 0. \tag{A.28}$$

Then, we have

$$f^*(c_s^{\mathcal{R}}, \bar{V}_s) - f^*(c_s^{\mathcal{R}}, V_s) \geq \begin{cases} f_v^*(\bar{c}_s, V_s)(\bar{V}_s - V_s), & \text{if } \gamma \geq \psi^{-1}, \\ f_v^*(\bar{c}_s, \bar{V}_s)(\bar{V}_s - V_s), & \text{if } \gamma < \psi^{-1}. \end{cases} \tag{A.29}$$

Hence, SGB inequality implies $\bar{V}_t \geq V_t$ P-$a.s.$ for all $t \in [0, T]$. Therefore, $U(\bar{c}) \geq U(c)$.

A.7 Proof of Proposition 7

Let U denote HREZ utility with $(\beta, \gamma, \psi, \theta)$. Let $c, \tilde{c} \in \mathcal{C}$ with $c \geq \tilde{c}$. Let $V_0 = U(c)$ and $\tilde{V}_0 = U(\tilde{c})$. We have

$$V_t - \tilde{V}_t = \mathrm{E}_t\left[\int_t^\tau e^{-\bar{\beta}(s-t)}\Big(\bar{f}(c_s, V_s) - \bar{f}(\tilde{c}_s, \tilde{V}_s)\Big)ds\right], \tag{A.30}$$

and

$$\begin{aligned}\bar{f}(c_s, V_s) - \bar{f}(\tilde{c}_s, \tilde{V}_s) &= \bar{f}(c_s, V_s) - \bar{f}(\tilde{c}_s, V_s) + \bar{f}(\tilde{c}_s, V_s) - \bar{f}(\tilde{c}_s, \tilde{V}_s) \\ &\geq \bar{f}(c_s, V_s) - \bar{f}(\tilde{c}_s, V_s) + \begin{cases} \bar{f}_v(\tilde{c}_s, \tilde{V}_s)(V_s - \tilde{V}_s), & \text{if } \gamma \geq \psi^{-1}, \\ \bar{f}_v(\tilde{c}_s, V_s)(V_s - \tilde{V}_s), & \text{if } \gamma < \psi^{-1}. \end{cases}\end{aligned} \tag{A.31}$$

Given that $\bar{f}$ is strictly increasing in its consumption argument (Proposition 1), the result follows by the SGB inequality.

A.8 Proof of Proposition 8

First, suppose $\tau = T$. We have

$$V_t^{c,Y} - V_t^{c,\tilde{Y}} = \mathrm{E}_t\left[\int_t^\tau e^{-\bar{\beta}(s-t)}\Big(\bar{f}(c_s, V_s^{c,Y}) - \bar{f}(c_s, V_s^{c,\tilde{Y}})\Big)ds + Y - \tilde{Y}\right]. \tag{A.32}$$

We have

$$V_t^{c,Y} - V_t^{c,\tilde{Y}} \geq \begin{cases} \mathrm{E}_t\left[\int_t^\tau e^{-\bar{\beta}(s-t)}\bar{f}_v(c_s, V_s^{c,\tilde{Y}})\big(V_s^{c,Y} - V_s^{c,\tilde{Y}}\big)\,ds + Y - \tilde{Y}\right], & \text{if } \gamma \geq \psi^{-1}, \\ \mathrm{E}_t\left[\int_t^\tau e^{-\bar{\beta}(s-t)}\bar{f}_v(c_s, V_s^{c,Y})\big(V_s^{c,Y} - V_s^{c,\tilde{Y}}\big)\,ds + Y - \tilde{Y}\right], & \text{if } \gamma < \psi^{-1}. \end{cases} \tag{A.33}$$

Thus, the SGB inequality implies $V_t^{c,Y} \geq V_t^{c,\tilde{Y}}$ P-$a.s.$ for all $t \in [0, T]$. For general τ, we replace $\bar{f}(c_s, V_s)$ by $1_{s \leq \tau}\bar{f}(c_s, V_s)$ throughout the above and obtain the same answer.

A.9 Proof of Proposition 9

The utility process V^* is given by Eq. (2.20). Define the OEU process $\hat{V}_t$ of V_t^* as $\hat{V}_t = \hat{\varphi}(V_t^*)$ where $\hat{\varphi}$ is given by Eq. (4.4). Then, from Ito's lemma, $\hat{V}_t$ satisfies

$$d\hat{V}_t = -\left(\hat{f}(c_s, \hat{V}_s) - \hat{\beta}\hat{V}_t - \frac{\gamma + \theta}{2\hat{V}_t}|\hat{\sigma}_t|^2\right) dt + \hat{\sigma}_t' dB_t, \quad \hat{V}_T = 0, \tag{A.34}$$

where $\hat{\beta} = \dfrac{\beta}{1-\psi}$ and $\hat{f}$ is given by Eq. (4.6). From Proposition 8 in Duffie and Epstein [4], SDE (A.34) and Eq. (4.6) show that U is homothetic.

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Declaration of Interests

I have nothing to declare.

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