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# ON THE ITERATIVE SCHEME GENERATING METHODS USING MEAN-VALUED SEQUENCES

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# ON THE ITERATIVE SCHEME GENERATING METHODS USING MEAN-VALUED SEQUENCES

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Abstract. Using the Mann method and the shrinking projection method, we present generalized forms of iterative scheme generating methods and compared them with prior frameworks. To this end, the properties of mean-valued sequences are leveraged. Subsequently, we establish a convergence theorem similar to that developed by Martinez-Yanes and Xu. This approach highlights the difference between the conventional shrinking projection method and the Martinez-Yanes and Xu variant. The proposed frameworks yield various types of iterative schemes for finding common fixed points, including a three-step iterative scheme. The class of mappings considered incorporate general types, including nonexpansive mappings.

## 1. Introduction

Let $C$ be a nonempty subset of a real Hilbert space $H$ and let $S$ be a mapping from $C$ into $H$. In $H$, an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$ are defined. The notation $F(S) = \{x \in C : Sx = x\}$ is used to represent a set of all fixed points of $S$. A mapping $S : C \to H$ is called *nonexpansive* if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. Due to its broad applicability, the construction of a sequence that converges to a fixed point of a nonexpansive mapping has been a topic of significant research interest. For an overview of fixed point theory and surrounding topics, readers may refer to the monographs by Goebel and Kirk [9], Takahashi [35], and Goebel [8].

Following Baillon [3] and Shimizu and Takahashi [33], Atsushiba and Takahashi [2] introduced the following iterative scheme using a mean-valued sequence:

$$x_{n+1} = a_n x_n + (1 - a_n) \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x_n \tag{1.1}$$

for all $n \in \mathbb{N}$. In (1.1), an initial point $x_1 \in C$ is arbitrarily given and $S, T : C \to C$ are commutative nonexpansive mappings. The sequence $\{a_n\}$ $(\subset [0,1])$ is required to satisfy certain conditions. Atsushiba and Takahashi proved a convergence theorem that weakly approximates a common fixed point of $S$ and $T$ in a framework of a Banach space. Using mean-valued sequences, Kondo [20] proved the following theorem:

**Theorem 1.1** ([20]). *Let $C$ be a nonempty, closed, and convex subset of $H$. Let $S, T : C \to C$ be quasi-nonexpansive and mean-demiclosed mappings such that $F(S) \cap F(T) \neq \emptyset$. Let $P_{F(S)\cap F(T)}$ be the metric projection from $H$ onto $F(S) \cap$*

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*$F(T)$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $a_n+b_n+c_n=1$ for all $n\in\mathbb{N}$, $\underline{\lim}_{n\to\infty}a_nb_n>0$, and $\underline{\lim}_{n\to\infty}a_nc_n>0$. Define a sequence $\{x_n\}$ in $C$ as follows:*

$$x_1 \in C: \text{ given,}$$
$$x_{n+1}=a_nx_n+b_n\frac{1}{n}\sum_{l=0}^{n-1}S^lz_n+c_n\frac{1}{n}\sum_{l=0}^{n-1}T^lw_n \tag{1.2}$$

*for all $n\in\mathbb{N}=\{1,2,\cdots\}$, where $\{z_n\}$ and $\{w_n\}$ are sequences in $C$ that satisfy*

$$\|z_n-q\|\le\|x_n-q\| \quad \text{and} \quad \|w_n-q\|\le\|x_n-q\| \tag{1.3}$$

*for all $q\in F(S)\cap F(T)$ and $n\in\mathbb{N}$. Then, $\{x_n\}$ converges weakly to an element $\widehat{x}$ in $F(S)\cap F(T)$, where $\widehat{x}\equiv\lim_{n\to\infty}P_{F(S)\cap F(T)}x_n$.*

In Theorem 1.1, a "mean-demiclosed mapping" is defined as one where any weak cluster point of a mean-valued sequence (as defined in (1.2)) is a fixed point. This class of mappings includes nonexpansive mappings as special cases, as described in Proposition 2.1. Furthermore, more general types of mappings than nonexpansive mappings also fall within the scope of this theorem, as discussed in the Appendix in the work of Kondo [22].

The required conditions for the sequences $\{z_n\}$ and $\{w_n\}$ in Theorem 1.1 are only the ones specified in (1.3). For example, by setting $z_n=\lambda_nx_n+(1-\lambda_n)Tx_n$ and $w_n=\mu_nx_n+(1-\mu_n)Sx_n$, we obtain the following iterative scheme:

$$\begin{aligned} z_n &= \lambda_nx_n+(1-\lambda_n)Tx_n, \\ w_n &= \mu_nx_n+(1-\mu_n)Sx_n, \\ x_{n+1} &= a_nx_n+b_n\frac{1}{n}\sum_{k=0}^{n-1}S^kz_n+c_n\frac{1}{n}\sum_{l=0}^{n-1}T^lw_n, \end{aligned} \tag{1.4}$$

where an initial point $x_1\in C$ is given. The coefficients of convex combinations $\lambda_n$ and $\mu_n$ are not subject to any restrictive conditions, except for $\lambda_n,\mu_n\in[0,1]$. It can be verified that $z_n$ and $w_n$ in (1.4) satisfy the conditions in (1.3). Note that $z_n$ (resp. $w_n$) in (1.4) depends only on the mapping $T$ (resp. $S$) at least directly. The iterative scheme in (1.4) is a two-step scheme, similar to those presented by Ishikawa [13], Xu [41], Tan and Xu [40], Berinde [4], and Martinez-Yanes and Xu [28]. Furthermore, three-step iterative schemes can be generated from Theorem 1.1. For instance, consider the following formulation:

$$\begin{aligned} w_n &= \mu_nx_n+(1-\mu_n)Tx_n, \\ z_n &= \lambda_nx_n+(1-\lambda_n)Sw_n, \\ x_{n+1} &= a_nx_n+b_n\frac{1}{n}\sum_{k=0}^{n-1}S^kz_n+c_n\frac{1}{n}\sum_{l=0}^{n-1}T^lz_n. \end{aligned} \tag{1.5}$$

The sequence $\{z_n\}$ in (1.5) fulfills the condition $\|z_n-q\|\le\|x_n-q\|$ in (1.3). For three-step iterative methods, see the work of Noor [31], Dashputre and Diwan [7], Phuengrattana and Suantai [32], and Chugh *et al.* [6]. Four-step and more general types of iterative schemes can also be generated from Theorem 1.1. Thus, this approach can be called an *iterative scheme generating method using mean-valued sequences.*

In 2006, Martinez-Yanes and Xu [28] extended the CQ method by Nakajo and Takahashi [30] and proved strong convergence theorems for finding a fixed point of a nonexpansive mapping. Although Kondo [19, 21] applied the Martinez-Yanes and Xu method with mean-valued sequences, iterative scheme generating methods have not yet been applied to Martinez-Yanes and Xu type iterative schemes. In 2008, Takahashi *et al.* [36] proved a strong convergence theorem using metric projections on shrinking sets. Their method is known as the shrinking projection method. In 2023, Kondo [22] applied iterative scheme generating methods with mean-valued sequences to the CQ method and shrinking projection method and obtained various strong convergence theorems.

In this study, we generalize iterative scheme generating methods using mean-valued sequences. Theorem 1.1 is obtained as a corollary from our result (Theorem 3.1). An iterative scheme generating method with the shrinking projection method addressed in Kondo [22] is also extended (Theorem 4.1). Subsequently, we apply this method to the Martinez-Yanes and Xu iterative scheme with the shrinking projection method (Theorem 5.1). This approach clarifies the difference between the conventional shrinking projection method and that incorporating the Martinez-Yanes and Xu method. By assuming several additional conditions, the proposed iterative scheme generating method can be applied to the Martinez-Yanes and Xu method. Our results yield various types of iterative schemes for finding common fixed points, including two- and three-step iterative schemes. The target mappings are of the general type, which are required to be quasi-nonexpansive with a condition regarding mean-demiclosedness. This class includes nonexpansive mappings and numerous other more general types of mappings.

The remaining article is organized as follows: Section 2 summarizes background information. Section 3 proves a Mann type [27] theorem that generalizes Theorem 1.1. Section 4 provides a generalized version of the iterative scheme generating method with the shrinking projection method. Section 5 elaborates upon the Martinez-Yanes and Xu iterative scheme with the shrinking projection method. Section 6 presents two iterative schemes derived from the result in Section 5 to demonstrate the applicability of the proposed approach. Section 7 concisely concludes this article.

## 2. Preliminaries

This section provides basic information and results. Let $\{x_n\}$ be a sequence in a real Hilbert space $H$ and let $x$ be an element in $H$. We use the notation $x_n \to x$ for strong convergence and $x_n \rightharpoonup x$ for weak convergence. A sequence $\{x_n\}$ converges weakly to $x$ if and only if for every subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \rightharpoonup x$. A closed and convex subset of $H$ is weakly closed.

Let $x, y, z \in H$ and let $a, b, c \in \mathbb{R}$ such that $a+b+c = 1$. According to Maruyama *et al.* [29] and Zegeye and Shahzad [42], the following relation holds:

$$\begin{aligned} (2.1) \qquad & \|ax + by + cz\|^2 \\ & = a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x-y\|^2 - bc\|y-z\|^2 - ca\|z-x\|^2 . \end{aligned}$$

For (2.1), assumptions $a, b, c \in [0,1]$ are not necessary. If $a, b, c \in [0,1]$, then the following expression holds:

$$\|ax + by + cz\|^2 \leq a\|x\|^2 + b\|y\|^2 + c\|z\|^2 . \tag{2.2}$$

Let $F$ be a nonempty, closed, and convex subset of $H$. A *metric projection* from $H$ onto $F$ is denoted by $P_F$, that is, $\|x - P_F x\| \leq \|x - h\|$ for all $x \in H$ and $h \in F$. The metric projection $P_F$ is nonexpansive and satisfies

$$\langle x - P_F x,\ P_F x - h\rangle \geq 0 \quad \text{and} \tag{2.3}$$

$$\|x - P_F x\|^2 + \|P_F x - h\|^2 \leq \|x - h\|^2 \tag{2.4}$$

for all $x \in H$ and $h \in F$. Let $C$ be a nonempty, closed, and convex subset of $H$. Then, a set $D$ defined by

$$D = \{h \in C : 0 \leq \langle x,\ h\rangle + d\} \tag{2.5}$$

is closed and convex, where $x \in H$ and $d \in \mathbb{R}$, as indicated in Lemma 1.3 in the work of Martinez-Yanes and Xu [28].

A mapping $S : C \to H$ with $F(S) \neq \emptyset$ is termed *quasi-nonexpansive* if $\|Sx - q\| \leq \|x - q\|$ for all $x \in C$ and $q \in F(S)$. The set of fixed points of a quasi-nonexpansive mapping is closed and convex, as indicated by Itoh and Takahashi [14]. A nonexpansive mapping with a fixed point is quasi-nonexpansive. Although the following proposition has already been proved in previous studies in more general forms (Lemma 3.1 in Kondo and Takahashi [25] or Lemma 2.3 in Kondo [20]), we present a proof here because the property of a mapping shown in the following proposition is important for this study.

**Proposition 2.1** ([25]; see also [20])**.** *Let $S : C \to C$ be a nonexpansive mapping, where $C$ is a nonempty, closed, and convex subset of $H$. For a bounded sequence $\{z_n\}$ in $C$, define $Z_n \equiv \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n\,(\in C)$ for all $n \in \mathbb{N}$. Let $Z_{n_i} \rightharpoonup p$, where $\{Z_{n_i}\}$ is a subsequence of $\{Z_n\}$. Then, $p \in F(S)$ holds.*

*Proof.* As $C$ is closed and convex, it is weakly closed. As $\{Z_{n_i}\}$ is a sequence in $C$ and $Z_{n_i} \rightharpoonup p$, we have that $p \in C$. Hence, $Sp\,(\in C)$ exists. Our aim is to show that $Sp = p$. As $S$ is nonexpansive, it follows that

$$\left\|S^{l+1} z_n - Sp\right\|^2 \leq \left\|S^l z_n - p\right\|^2$$

for all $n \in \mathbb{N}$ and $l \in \mathbb{N} \cup \{0\}$. From this, we have

$$\left\|S^{l+1} z_n - Sp\right\|^2 \leq \left\|S^l z_n - Sp\right\|^2 + 2\left\langle S^l z_n - Sp,\ Sp - p\right\rangle + \|Sp - p\|^2 .$$

Summing these inequalities with respect to $l$ from 0 to $n-1$ and dividing by $n$ yields

$$\frac{1}{n}\|S^n z_n - Sp\|^2 \leq \frac{1}{n}\|z_n - Sp\|^2 + 2\langle Z_n - Sp,\ Sp - p\rangle + \|Sp - p\|^2 .$$

As $\frac{1}{n}\|S^n z_n - Sp\|^2 \geq 0$, we have

$$0 \leq \frac{1}{n}\|z_n - Sp\|^2 + 2\langle Z_n - Sp,\ Sp - p\rangle + \|Sp - p\|^2$$

for all $n \in \mathbb{N}$. Recall that $\{z_n\}$ is bounded and $Z_{n_i} \rightharpoonup p$ is assumed. Replacing $n$ by $n_i$, we obtain

$$0 \leq 2\langle p - Sp,\ Sp - p\rangle + \|Sp - p\|^2$$

by taking the limit as $i \to \infty$. This implies that $0 \leq -\|Sp - p\|^2$. Thus, $Sp = p$. This completes the proof. □

Following the work of Kondo [17], we term a mapping $S : C \to C$ *mean-demiclosed* if

$$Z_{n_j} \rightharpoonup p \text{ (weak convergence) } \Longrightarrow p \in F(S) \tag{2.6}$$

under the setting of Proposition 2.1. According to Proposition 2.1, a nonexpansive mapping is mean-demiclosed.

In the next section, we focus on mappings that are quasi-nonexpansive and mean-demiclosed. Although this class of mappings contains nonexpansive mappings as special cases, it also includes more broad classes of mappings. For example, generalized hybrid mappings [16], normally generalized hybrid mappings [39], 2-generalized hybrid mappings [29], and normally 2-generalized hybrid mappings [24] are quasi-nonexpansive and mean-demiclosed if they have fixed points. Information regarding these types of mappings can be found in the Appendix in the work of Kondo [22].

The following lemma is used in the proof of Theorem 3.1:

**Lemma 2.1** ([37])**.** *Let $P_F$ be the metric projection from $H$ onto $F$, where $F$ is a nonempty, closed, and convex subset of $H$. Let $\{x_n\}$ be a sequence in $H$ such that*

$$\|x_{n+1} - q\| \leq \|x_n - q\| \tag{2.7}$$

*for all $q \in F$ and $n \in \mathbb{N}$. Then, $\{P_F x_n\}$ is convergent in $F$. In other words, there exists $\widehat{x} \in F$ such that $P_F x_n \to \widehat{x}$.*

For the remaining analysis, we assume that there exists a common fixed point of nonlinear mappings. The following is a simplified version of classical results demonstrated in 1965 by Browder [5], Göhde [10], and Kirk [15] in frameworks of Banach spaces:

**Theorem 2.1** ([5, 10, 15])**.** *Let $C$ be a nonempty, closed, convex, and bounded subset of $H$. Let $S, T : C \to C$ be nonexpansive mappings such that $ST = TS$. Then, $S$ and $T$ have a common fixed point.*

For common fixed point theorems for more general types of mappings, see the works of Hojo [11], Kondo [18], and articles cited therein.

## 3. Mann Method

This section presents one of the main theorems of this article, which shows how to approximates common fixed points of two quasi-nonexpansive and mean-demiclosed mappings. Recall that nonexpansive mappings with fixed points are quasi-nonexpansive. Furthermore, from Proposition 2.1, nonexpansive mappings are mean-demiclosed. Hence, the theorem can be applied to nonexpansive mappings under the assumption that the mappings have a common fixed point. The basic elements of the proof draw upon various previous studies, e.g., [16, 23, 26, 29, 39].

**Theorem 3.1.** *Let $C$ be a nonempty, closed, and convex subset of a real Hilbert space $H$. Let $S, T : C \to C$ be quasi-nonexpansive and mean-demiclosed mappings such that $F(S) \cap F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers* *in the interval* $[0,1]$ *such that* $a_n + b_n + c_n = 1$ *for all* $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, *and* $\underline{\lim}_{n\to\infty} a_n c_n > 0$. *Define a sequence* $\{x_n\}$ *in* $C$ *as follows:*

$$x_1 \in C: \text{ given,}$$

$$x_{n+1} = a_n y_n + b_n \frac{1}{n} \sum_{l=0}^{n-1} S^l z_n + c_n \frac{1}{n} \sum_{l=0}^{n-1} T^l w_n \tag{3.1}$$

*for all* $n \in \mathbb{N}$, *where* $\{y_n\}$, $\{z_n\}$, *and* $\{w_n\}$ *are sequences in* $C$ *that satisfy*

$$\|y_n - q\| \le \|x_n - q\|, \quad \|z_n - q\| \le \|x_n - q\|, \quad \|w_n - q\| \le \|x_n - q\| \tag{3.2}$$

*for all* $q \in F(S) \cap F(T)$ *and* $n \in \mathbb{N}$ *and*

$$x_n - y_n \to 0 \tag{3.3}$$

*as* $n \to \infty$. *Then,* $\{x_n\}$ *converges weakly to an element* $\widehat{x}$ *in* $F(S) \cap F(T)$, *where* $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

*Proof.* Define

$$Z_n = \frac{1}{n} \sum_{l=0}^{n-1} S^l z_n \quad \text{and} \quad W_n = \frac{1}{n} \sum_{l=0}^{n-1} T^l w_n$$

for all $n \in \mathbb{N}$. As $C$ is convex, $Z_n$ and $W_n$ are elements in $C$. Now, we can simply state that $x_{n+1} = a_n y_n + b_n Z_n + c_n W_n \, (\in C)$.

Observe that

$$\|Z_n - q\| \le \|z_n - q\| \quad \text{and} \quad \|W_n - q\| \le \|w_n - q\| \tag{3.4}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Indeed, as $S$ is quasi-nonexpansive and $q \in F(S) \cap F(T) \subset F(S)$, it follows that

$$\begin{aligned} \|Z_n - q\| &= \left\| \frac{1}{n} \sum_{l=0}^{n-1} S^l z_n - q \right\| = \frac{1}{n} \left\| \sum_{l=0}^{n-1} S^l z_n - nq \right\| \\ &= \frac{1}{n} \left\| \sum_{l=0}^{n-1} \left( S^l z_n - q \right) \right\| \le \frac{1}{n} \sum_{l=0}^{n-1} \left\| S^l z_n - q \right\| \\ &\le \frac{1}{n} \sum_{l=0}^{n-1} \|z_n - q\| = \|z_n - q\|. \end{aligned} \tag{3.5}$$

Similarly, the second part of (3.4) also holds true as $T$ is quasi-nonexpansive and $q \in F(S) \cap F(T) \subset F(T)$.

We verify that

$$\|x_{n+1} - q\| \le \|x_n - q\| \tag{3.6}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Indeed, from (3.4) and (3.2), it follows that

$$\begin{aligned} \|x_{n+1} - q\| &= \|a_n y_n + b_n Z_n + c_n W_n - q\| \\ &= \|a_n (y_n - q) + b_n (Z_n - q) + c_n (W_n - q)\| \\ &\le a_n \|y_n - q\| + b_n \|Z_n - q\| + c_n \|W_n - q\| \\ &\le a_n \|y_n - q\| + b_n \|z_n - q\| + c_n \|w_n - q\| \\ &\le a_n \|x_n - q\| + b_n \|x_n - q\| + c_n \|x_n - q\| \\ &= \|x_n - q\|. \end{aligned}$$

Thus, (3.6) holds as claimed. According to (3.6), the sequence $\{\|x_n - q\|\}$ is convergent for all $q \in F(S) \cap F(T)$, and $\{x_n\}$ is bounded. Furthermore, from Lemma 2.1, we have that $\{P_{F(S)\cap F(T)} x_n\}$ is convergent in $F(S) \cap F(T)$. Thus, $\widehat{x} = \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$ exists in $F(S) \cap F(T)$.

Next, we aim to demonstrate that

$$y_n - Z_n \to 0 \text{ and } y_n - W_n \to 0 \tag{3.7}$$

as $n \to \infty$. Here, $q \in F(S) \cap F(T)$ is arbitrarily selected. Using (2.1), (3.4), and (3.2), we obtain the following expressions:

$$\begin{aligned}
& \|x_{n+1} - q\|^2 \\
= \ & \|a_n (y_n - q) + b_n (Z_n - q) + c_n (W_n - q)\|^2 \\
= \ & a_n \|y_n - q\|^2 + b_n \|Z_n - q\|^2 + c_n \|W_n - q\|^2 \\
& -a_n b_n \|y_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - y_n\|^2 \\
\\
\leq \ & a_n \|y_n - q\|^2 + b_n \|z_n - q\|^2 + c_n \|w_n - q\|^2 \\
& -a_n b_n \|y_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - y_n\|^2 \\
\leq \ & a_n \|x_n - q\|^2 + b_n \|x_n - q\|^2 + c_n \|x_n - q\|^2 \\
& -a_n b_n \|y_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - y_n\|^2 \\
= \ & \|x_n - q\|^2 \\
& -a_n b_n \|y_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - y_n\|^2 .
\end{aligned}$$

As $b_n c_n \|Z_n - W_n\|^2 \geq 0$, we obtain

$$a_n b_n \|y_n - Z_n\|^2 + a_n c_n \|y_n - W_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 .$$

As $\{\|x_n - q\|\}$ is convergent, we have from the assumptions $\underline{\lim}_{n\to\infty} a_n b_n > 0$ and $\underline{\lim}_{n\to\infty} a_n c_n > 0$ that (3.7) holds true as claimed.

Observe that

$$x_n - Z_n \to 0 \text{ and } x_n - W_n \to 0 \tag{3.8}$$

as $n \to \infty$. Indeed, from (3.3) and (3.7), it follows that

$$\|x_n - Z_n\| \leq \|x_n - y_n\| + \|y_n - Z_n\| \to 0$$

as $n \to \infty$. Similarly, we can show that $x_n - W_n \to 0$.

Our goal is to prove that $x_n \rightharpoonup \widehat{x} \left(\equiv \lim_{k\to\infty} P_{F(S)\cap F(T)} x_k\right)$. To this end, it is sufficient to show that for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \rightharpoonup \widehat{x}$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. As $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \rightharpoonup p$ for some $p \in H$. From (3.8), we have that $Z_{n_j} \rightharpoonup p$ and $W_{n_j} \rightharpoonup p$. As $S$ and $T$ are mean-demiclosed (2.6), we obtain $p \in F(S) \cap F(T)$. From (2.3), it follows that

$$\left\langle x_{n_j} - P_{F(S)\cap F(T)} x_{n_j},\ P_{F(S)\cap F(T)} x_{n_j} - p \right\rangle \geq 0$$

for all $j \in \mathbb{N}$. As $x_{n_j} \rightharpoonup p$ and $P_{F(S)\cap F(T)} x_n \to \widehat{x}$, it holds in the limit as $j \to \infty$ that $\langle p - \widehat{x},\ \widehat{x} - p\rangle \geq 0$. Thus, $p = \widehat{x}$. This indicates that $x_n \rightharpoonup \widehat{x}$. The proof is thus complete. □

Setting $y_n = x_n$ for all $n \in \mathbb{N}$ in Theorem 3.1, we obtain Theorem 1.1 as a corollary. Therefore, the iterative schemes (1.4) and (1.5) presented in the Introduction are generated from Theorem 3.1. In (3.1), the idea using a sequence $\{y_n\}$ that satisfies $\|y_n - q\| \leq \|x_n - q\|$ and $x_n - y_n \to 0$ instead of $\{x_n\}$ is derived from the recent work of Kondo [23].

## 4. Takahashi–Takeuchi–Kubota Method

This section presents a strong convergence theorem for finding a common fixed point of two nonlinear mappings. We use the shrinking projection method proposed by Takahashi *et al.* [36] together with mean-valued sequences. The basic element of the proof has been developed in many prior studies, for instance, [12, 17, 22, 38].

For proving theorems in the following sections, we relax a condition pertaining to mappings, compared with that in Theorem 3.1. Consider the following setting: Let $C$ be a nonempty, closed, and convex subset of a real Hilbert space $H$. Moreover, let $S : C \to C$ with $F(S) \neq \emptyset$ and let $\{z_n\}$ be a bounded sequence in $C$. Define $Z_n = \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n \, (\in C)$. Following Kondo [17], consider the following condition:

$$Z_{n_j} \to p \text{ (strong convergence) } \Longrightarrow p \in F(S), \tag{4.1}$$

where $\{Z_{n_j}\}$ is a subsequence of $\{Z_n\}$. A mean-demiclosed mapping (2.6) satisfies the condition (4.1), and thus, broad classes of mappings, including nonexpansive mappings, satisfy this condition (4.1). In the following analysis, quasi-nonexpansive mappings with the condition (4.1) are considered.

**Theorem 4.1.** *Let $C$ be a nonempty, closed, and convex subset of a real Hilbert space $H$. Let $S, T : C \to C$ be quasi-nonexpansive mappings that satisfy the condition (4.1). Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$. Let $\{u_n\}$ be a sequence in $H$ such that $u_n \to u \, (\in H)$. Define a sequence $\{x_n\}$ in $C$ as follows:*

$$\begin{aligned} x_1 &= x \in C : \text{ given,} \\ C_1 &= C, \\ X_n &= a_n y_n + b_n \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n + c_n \frac{1}{n}\sum_{l=0}^{n-1} T^l w_n, \\ C_{n+1} &= \{h \in C_n : \|X_n - h\| \leq \|x_n - h\|\}, \\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned}$$

*for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in $C$ that satisfy*

$$\|y_n - q\| \leq \|x_n - q\|, \quad \|z_n - q\| \leq \|x_n - q\|, \quad \|w_n - q\| \leq \|x_n - q\| \tag{4.2}$$

*for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ and*

$$x_n - y_n \to 0 \tag{4.3}$$

*as $n \to \infty$. Then, $\{x_n\}$ converges strongly to an element $\widehat{u}$ in $F(S) \cap F(T)$, where $\widehat{u} = P_{F(S)\cap F(T)} u$.*

*Proof.* In this proof, we use again the notation

$$Z_n = \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n \quad \text{and} \quad W_n = \frac{1}{n}\sum_{l=0}^{n-1} T^l w_n$$

for simplicity, where $\{z_n\}$ and $\{w_n\}$ are given. As $C$ is convex, $\{Z_n\}$ and $\{W_n\}$ are sequences in $C$. In this case, $X_n = a_n y_n + b_n Z_n + c_n W_n \, (\in C)$.

We show that (a) $C_n$ is closed and convex, (b) $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$, and (c) the sequences $\{x_n\}, \{y_n\}, \{z_n\}, \{w_n\}, \{X_n\}$ in $C$ and $\{C_n\}$ are properly defined. First, we consider the case in which $n = 1$.

(i) Given $x_1 \in C_1 \, (= C)$, we can choose $y_1$, $z_1$, and $w_1 \in C$ such that (4.2) and (4.3) are satisfied for $n = 1$. For instance, if we set $y_1 = z_1 = w_1 = x_1$, then the condition (4.2) is fulfilled. With similar settings for all $n \in \mathbb{N}$, the condition (4.3) will be satisfied. With $x_1, y_1, z_1, w_1 \in C$, $X_1$ and $C_2$ are defined as follows:

$$\begin{aligned} &X_1 = a_1 y_1 + b_1 Z_1 + c_1 W_1 \in C \quad \text{and} \\ &C_2 = \{h \in C_1 : \|X_1 - h\| \leq \|x_1 - h\|\}. \end{aligned}$$

As $C_1$ is closed and convex, $C_2$ is also closed and convex. We verify that $F(S) \cap F(T) \subset C_2$. Let $q \in F(S) \cap F(T) \, (\subset C_1)$. It follows from (4.2) that

$$\begin{aligned} \|X_1 - q\| &= \|a_1 y_1 + b_1 Z_1 + c_1 W_1 - q\| \\ &= \|a_1 y_1 + b_1 z_1 + c_1 w_1 - q\| \\ &\leq a_1 \|y_1 - q\| + b_1 \|z_1 - q\| + c_1 \|w_1 - q\| \\ &\leq a_1 \|x_1 - q\| + b_1 \|x_1 - q\| + c_1 \|x_1 - q\| = \|x_1 - q\|, \end{aligned}$$

which means that $q \in C_2$. Therefore, $F(S) \cap F(T) \subset C_2$ as claimed. As $F(S) \cap F(T) \neq \emptyset$ is assumed, we have $C_2 \neq \emptyset$. Consequently, the metric projection $P_{C_2}$ exists and $x_2 = P_{C_2} u_2$ is defined.

(ii) Given $x_2 \in C_2 \, (\subset C_1 = C)$, we can choose $y_2$, $z_2$, and $w_2 \in C$ such that (4.2) and (4.3) are satisfied for $n = 2$. Furthermore, $X_2$ and $C_3$ are defined as follows:

$$\begin{aligned} &X_2 = a_2 y_2 + b_2 Z_2 + c_2 W_2 \in C \quad \text{and} \\ &C_3 = \{h \in C_2 : \|X_2 - h\| \leq \|x_2 - h\|\}. \end{aligned}$$

Using the same reasoning as that in the case of (i), we can verify that $C_3$ is closed and convex and $F(S) \cap F(T) \subset C_3$. As $F(S) \cap F(T) \neq \emptyset$ is assumed, it holds that $C_3 \neq \emptyset$. Thus, the metric projection $P_{C_3}$ exists and $x_3 = P_{C_3} u_3$ is defined.

Repeating the same analysis, we can prove (a), (b), and (c) as claimed.

Define $\overline{u}_n = P_{C_n} u \, (\in C_n)$. As $C_n \subset C_{n-1} \subset \cdots \subset C_1 = C$, $\{\overline{u}_n\}$ is a sequence in $C$. As $\overline{u}_n = P_{C_n} u$ and $F(S) \cap F(T) \subset C_n$, it follows that

$$\|u - \overline{u}_n\| \leq \|u - q\| \tag{4.4}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. This outcome shows that $\{\overline{u}_n\}$ is bounded. Furthermore, as $\overline{u}_n = P_{C_n} u$ and $\overline{u}_{n+1} = P_{C_{n+1}} u \in C_{n+1} \subset C_n$, we obtain that

$$\|u - \overline{u}_n\| \leq \|u - \overline{u}_{n+1}\|$$

for all $n \in \mathbb{N}$. This shows that the sequence $\{\|u - \overline{u}_n\|\}$ of real numbers is monotone increasing. As $\{\overline{u}_n\}$ is bounded, $\{\|u - \overline{u}_n\|\}$ is also bounded. Thus, $\{\|u - \overline{u}_n\|\}$ is convergent.

Subsequently, we demonstrate that $\{\overline{u}_n\}$ is convergent in $C$. In other words, there exists $\overline{u} \in C$ such that

$$\overline{u}_n \to \overline{u}. \tag{4.5}$$

Let $m, n \in \mathbb{N}$ such that $m \geq n$. As $\overline{u}_n = P_{C_n} u$ and $\overline{u}_m = P_{C_m} u \in C_m \subset C_n$, we have from (2.4) that

$$\|u - \overline{u}_n\|^2 + \|\overline{u}_n - \overline{u}_m\|^2 \leq \|u - \overline{u}_m\|^2 .$$

As $\{\|u - \overline{u}_n\|\}$ is convergent, it can be stated that $\overline{u}_n - \overline{u}_m \to 0$ as $m, n \to \infty$. This indicates that $\{\overline{u}_n\}$ is a Cauchy sequence in $C$. As $C$ is closed in a real Hilbert space $H$, it is complete. Hence, there exists $\overline{u} \in C$ such that $\overline{u}_n \to \overline{u}$ as claimed.

Next, observe that $\{x_n\}$ has the same limit point, that is,

$$x_n \to \overline{u}. \tag{4.6}$$

Indeed, as the metric projection $P_{C_n}$ is nonexpansive and $u_n \to u$ is assumed, it follows from (4.5) that

$$\begin{aligned}
\|x_n - \overline{u}\| &\leq \|x_n - \overline{u}_n\| + \|\overline{u}_n - \overline{u}\| \\
&= \|P_{C_n} u_n - P_{C_n} u\| + \|\overline{u}_n - \overline{u}\| \\
&\leq \|u_n - u\| + \|\overline{u}_n - \overline{u}\| \to 0
\end{aligned}$$

as claimed. As $\{x_n\}$ is convergent, it is bounded.

We prove that

$$x_n - X_n \to 0. \tag{4.7}$$

Indeed, as $\{x_n\}$ is convergent, it holds that $x_n - x_{n+1} \to 0$. From $x_{n+1} = P_{C_{n+1}} u_{n+1} \in C_{n+1}$, it follows that $\|X_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| \to 0$. Therefore, we have

$$\|x_n - X_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - X_n\| \to 0$$

as claimed. As $\{x_n\}$ is bounded, $\{X_n\}$ is also bounded according to (4.7).

Now, note that

$$\|Z_n - q\| \leq \|z_n - q\| \quad \text{and} \quad \|W_n - q\| \leq \|w_n - q\| \tag{4.8}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. These inequalities in (4.8) can be proved in the same manner as (3.5), given that $S$ and $T$ are quasi-nonexpansive. Using (4.8), we demonstrate that

$$y_n - Z_n \to 0 \quad \text{and} \quad y_n - W_n \to 0. \tag{4.9}$$

Here, we arbitrarily select $q \in F(S) \cap F(T)$. From (2.1), (4.8), and (4.2), it follows that

$$\begin{aligned}
& \|X_n - q\|^2 \\
= \;& \|a_n (y_n - q) + b_n (Z_n - q) + c_n (W_n - q)\|^2 \\
= \;& a_n \|y_n - q\|^2 + b_n \|Z_n - q\|^2 + c_n \|W_n - q\|^2 \\
& - a_n b_n \|y_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - y_n\|^2 \\
\leq \;& a_n \|y_n - q\|^2 + b_n \|z_n - q\|^2 + c_n \|w_n - q\|^2 \\
& - a_n b_n \|y_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - y_n\|^2 \\
\leq \;& a_n \|x_n - q\|^2 + b_n \|x_n - q\|^2 + c_n \|x_n - q\|^2 \\
& - a_n b_n \|y_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - y_n\|^2 \\
= \;& \|x_n - q\|^2 \\
& - a_n b_n \|y_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - y_n\|^2 .
\end{aligned}$$

As $b_n c_n \|Z_n - W_n\|^2 \geq 0$, it follows that

$$\begin{aligned} & a_n b_n \|y_n - Z_n\|^2 + a_n c_n \|y_n - W_n\|^2 \\ & \leq \|x_n - q\|^2 - \|X_n - q\|^2 \\ & \leq (\|x_n - q\| + \|X_n - q\|) \left| \|x_n - q\| - \|X_n - q\| \right| \\ & \leq (\|x_n - q\| + \|X_n - q\|) \|x_n - X_n\| . \end{aligned}$$

As $\{x_n\}$ and $\{X_n\}$ are bounded, we obtain (4.9) from (4.7) and the assumptions $\underline{\lim}_{n\to\infty} a_n b_n > 0$ and $\underline{\lim}_{n\to\infty} a_n c_n > 0$.

Next, we show that

$$x_n - Z_n \to 0 \quad \text{and} \quad x_n - W_n \to 0. \tag{4.10}$$

Indeed, from (4.3) and (4.9), it holds that

$$\|x_n - Z_n\| \leq \|x_n - y_n\| + \|y_n - Z_n\| \to 0.$$

The second part in (4.10) can be similarly verified.

From (4.6) and (4.10), we have $Z_n \to \overline{u}$ and $W_n \to \overline{u}$. Therefore, from (4.1), we obtain $\overline{u} \in F(S) \cap F(T)$.

Finally, we demonstrate that

$$\overline{u} \left( = \lim_{n\to\infty} \overline{u}_n = \lim_{n\to\infty} x_n \right) = \widehat{u} \left( = P_{F(S)\cap F(T)} u \right).$$

As $\overline{u} \in F(S) \cap F(T)$ and $\widehat{u} = P_{F(S)\cap F(T)} u$, it is sufficient to show that $\|u - \overline{u}\| \leq \|u - \widehat{u}\|$. As $\widehat{u} \in F(S) \cap F(T)$, from (4.4), it holds that $\|u - \overline{u}_n\| \leq \|u - \widehat{u}\|$. From (4.5), we obtain $\|u - \overline{u}\| \leq \|u - \widehat{u}\|$. Thus, we have that $\overline{u} = \widehat{u}$. Given (4.6), it can be stated that $x_n \to \widehat{u} (= \overline{u})$. This completes the proof. □

Setting $y_n = x_n$ in Theorem 4.1 yields the following corollary, corresponding to Theorem 4 in the work of Kondo [22]:

**Corollary 4.1** ([22]). *Let $C$ be a nonempty, closed, and convex subset of $H$. Let $S, T : C \to C$ be quasi-nonexpansive mappings that satisfy the condition* (4.1). *Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$. Let $\{u_n\}$ be a sequence in $H$ such that $u_n \to u (\in H)$. Define a sequence $\{x_n\}$ in $C$ as follows:*

$$\begin{aligned} & x_1 = x \in C : \text{ given}, \\ & C_1 = C, \\ & X_n = a_n x_n + b_n \frac{1}{n} \sum_{l=0}^{n-1} S^l z_n + c_n \frac{1}{n} \sum_{l=0}^{n-1} T^l w_n, \\ & C_{n+1} = \{ h \in C_n : \|X_n - h\| \leq \|x_n - h\| \}, \\ & x_{n+1} = P_{C_{n+1}} u_{n+1} \end{aligned}$$

*for all $n \in \mathbb{N}$, where $\{z_n\}$ and $\{w_n\}$ are sequences in $C$ that satisfy*

$$\|z_n - q\| \leq \|x_n - q\| \quad \text{and} \quad \|w_n - q\| \leq \|x_n - q\| \tag{4.11}$$

*for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to an element $\widehat{u}$ in $F(S) \cap F(T)$, where $\widehat{u} = P_{F(S)\cap F(T)} u$.*

From this corollary, various types of iterative schemes can be generated, as discussed in Section 5 in Kondo [22].

## 5. Martinez-Yanes and Xu Method

This section presents a strong convergence theorem for finding a common fixed point of nonlinear mappings. We use the Martinez-Yanes and Xu iterative method (see Theorem 2.1 in [28]) alongside the shrinking projection method [36] and mean-valued sequences. To the authors' best knowledge, this is the first attempt to apply the iterative scheme generating method to the Martinez-Yanes and Xu method. The fundamentals of the following proof have been improved in many studies; see, for instance, [1, 19, 21].

**Theorem 5.1.** *Let $C$ be a nonempty, closed, and convex subset of a real Hilbert space $H$. Let $S, T : C \to C$ be quasi-nonexpansive mappings that satisfy the condition* (4.1)*. Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$. Let $\{u_n\}$ be a sequence in $H$ such that $u_n \to u\,(\in H)$. Define a sequence $\{x_n\}$ in $C$ as follows:*

$$
\begin{aligned}
x_1 &= x \in C : \text{ given}, \\
C_1 &= C, \\
X_n &= a_n y_n + b_n \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n + c_n \frac{1}{n}\sum_{l=0}^{n-1} T^l w_n, \\
C_{n+1} &= \left\{ h \in C_n : \|X_n - h\|^2 \leq a_n \|y_n - h\|^2 + b_n \|z_n - h\|^2 + c_n \|w_n - h\|^2 \right\}, \\
x_{n+1} &= P_{C_{n+1}} u_{n+1}
\end{aligned}
$$

*for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in $C$ that satisfy*

$$
\|y_n - q\| \leq \|x_n - q\|, \quad \|z_n - q\| \leq \|x_n - q\|, \quad \|w_n - q\| \leq \|x_n - q\| \tag{5.1}
$$

*for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ and*

$$
x_n - y_n \to 0, \quad x_n - z_n \to 0, \quad x_n - w_n \to 0 \tag{5.2}
$$

*as $n \to \infty$. Then, $\{x_n\}$ converges strongly to an element $\widehat{u}$ in $F(S) \cap F(T)$, where $\widehat{u} = P_{F(S)\cap F(T)} u$.*

**Remark 5.1.** *In the definition of $C_{n+1}$,*

$$
\|X_n - h\|^2 \leq a_n \|y_n - h\|^2 + b_n \|z_n - h\|^2 + c_n \|w_n - h\|^2
$$

$$
\iff \quad 0 \leq a_n \|y_n\|^2 + b_n \|z_n\|^2 + c_n \|w_n\|^2 - \|X_n\|^2 - 2\langle a y_n + b z_n + c w_n - X_n,\ h\rangle \tag{5.3}
$$

$$
\iff \quad \|X_n - h\|^2 \leq \|y_n - h\|^2 + b_n \left( \|z_n\|^2 - \|y_n\|^2 + 2\langle z_n - y_n,\ h\rangle \right) + c_n \left( \|w_n\|^2 - \|y_n\|^2 + 2\langle w_n - y_n,\ h\rangle \right). \tag{5.4}
$$

*From* (5.4)*, we can see that Theorem* 5.1 *corresponds to the Martinez-Yanes and Xu type. According to* (2.5) *and* (5.3)*, the set $C_{n+1}$ is closed and convex if $C_n$ is closed and convex, given $X_n, y_n, z_n, w_n \in C$ and $a_n, b_n, c_n \in \mathbb{R}$.*

*Proof.* We again use the notation

$$
Z_n = \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n \quad \text{and} \quad W_n = \frac{1}{n}\sum_{l=0}^{n-1} T^l w_n,
$$

where $\{z_n\}$ and $\{w_n\}$ are given. The mean-valued sequences $\{Z_n\}$ and $\{W_n\}$ lie in $C$ as $C$ is convex. Then, we have $X_n = a_n y_n + b_n Z_n + c_n W_n \, (\in C)$.

We prove that (a) $C_n$ is closed and convex, (b) $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$, and (c) the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, and $\{X_n\}$ in $C$ and $\{C_n\}$ are properly defined. We start with the case of $n = 1$.

(i) Given $x_1 \in C_1 \, (= C)$, we can choose $y_1$, $z_1$, and $w_1 \in C$ such that (5.1) and (5.2) are satisfied for $n = 1$. For example, setting $y_1 = z_1 = w_1 = x_1$, the condition (5.1) is satisfied. Furthermore, by choosing $y_n, z_n, w_n$ in a similar manner for all $n \in \mathbb{N}$, the condition (5.2) will be satisfied. With $x_1, y_1, z_1, w_1 \in C$, $X_1$ and $C_2$ are defined as follows:

$$\begin{aligned} X_1 &= a_1 y_1 + b_1 Z_1 + c_1 W_1 \\ &= a_1 y_1 + b_1 z_1 + c_1 w_1 \in C \quad \text{and} \\ C_2 &= \left\{ h \in C_1 : \|X_1 - h\|^2 \le a_1 \|y_1 - h\|^2 + b_1 \|z_1 - h\|^2 + c_1 \|w_1 - h\|^2 \right\}. \end{aligned}$$

From (2.5) and (5.3), we see that $C_2$ is closed and convex as $C_1$ is closed and convex. Observe that $F(S) \cap F(T) \subset C_2$. Choose $q \in F(S) \cap F(T) \, (\subset C_1)$ arbitrarily. Using (2.2), we have

$$\begin{aligned} \|X_1 - q\|^2 &= \|a_1 y_1 + b_1 z_1 + c_1 w_1 - q\|^2 \\ &= \|a_1 (y_1 - q) + b_1 (z_1 - q) + c_1 (w_1 - q)\|^2 \\ &\le a_1 \|y_1 - q\|^2 + b_1 \|z_1 - q\|^2 + c_1 \|w_1 - q\|^2 . \end{aligned}$$

This indicates that $q \in C_2$. Thus, $F(S) \cap F(T) \subset C_2$ as claimed. As $F(S) \cap F(T) \neq \emptyset$ is assumed, $C_2$ is also nonempty. Consequently, the metric projection $P_{C_2}$ exists and $x_2 = P_{C_2} u_2$ is defined.

(ii) Given $x_2 \in C_2 \, (\subset C_1 = C)$, we can choose $y_2$, $z_2$, and $w_2 \in C$ such that (5.1) and (5.2) are satisfied for $n = 2$. With these elements, $X_2$ and $C_3$ are defined as follows:

$$\begin{aligned} X_2 &= a_2 y_2 + b_2 Z_2 + c_2 W_2 \\ &= a_2 y_2 + b_2 \frac{1}{2} (z_2 + S z_2) + c_2 \frac{1}{2} (w_2 + T w_2) \in C \quad \text{and} \\ C_3 &= \left\{ h \in C_2 : \|X_2 - h\|^2 \le a_2 \|y_2 - h\|^2 + b_2 \|z_2 - h\|^2 + c_2 \|w_2 - h\|^2 \right\}. \end{aligned}$$

Using the same reasoning as in case (i), we can verify that $C_3$ is closed and convex and $F(S) \cap F(T) \subset C_3$. As $F(S) \cap F(T) \neq \emptyset$ is assumed, $C_3 \neq \emptyset$. Thus, the metric projection $P_{C_3}$ exists and $x_3 = P_{C_3} u_3$ is defined.

Repeating the same analysis, we can prove (a), (b), and (c).

Define $\overline{u}_n = P_{C_n} u \in C_n$. As the sequence $\{C_n\}$ of sets is shrinking, that is, $C_n \subset C_{n-1} \subset \cdots \subset C_1 = C$, $\{\overline{u}_n\}$ is a sequence in $C$. Observe that

$$\|u - \overline{u}_n\| \le \|u - q\| \tag{5.5}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. This follows from the definition $\overline{u}_n = P_{C_n} u$ and the fact that $q \in F(S) \cap F(T) \subset C_n$. Then, from (5.5), $\{\overline{u}_n\}$ is bounded.

Next, we show that

$$\|u - \overline{u}_n\| \le \|u - \overline{u}_{n+1}\| \tag{5.6}$$

for all $n \in \mathbb{N}$. As $\overline{u}_n = P_{C_n} u$ and $\overline{u}_{n+1} = P_{C_{n+1}} u \in C_{n+1} \subset C_n$, the inequality (5.6) follows, which means that $\{\|u - \overline{u}_n\|\}$ is monotone increasing. As $\{\overline{u}_n\}$ is bounded, $\{\|u - \overline{u}_n\|\}$ is a convergent sequence in $\mathbb{R}$.

We claim that the sequence $\{\overline{u}_n\}$ is convergent in $C$, that is, there exists $\overline{u} \in C$ such that

$$\overline{u}_n \to \overline{u}. \tag{5.7}$$

To prove this, we verify that $\{\overline{u}_n\}$ is a Cauchy sequence in $C$. Let $m, n \in \mathbb{N}$ such that $m \geq n$. As $\overline{u}_n = P_{C_n} u$ and $\overline{u}_m = P_{C_m} u \in C_m \subset C_n$, using (2.4), we have

$$\|u - \overline{u}_n\|^2 + \|\overline{u}_n - \overline{u}_m\|^2 \leq \|u - \overline{u}_m\|^2 .$$

Given that $\{\|u - \overline{u}_n\|\}$ is convergent, it follows that $\overline{u}_n - \overline{u}_m \to 0$ as $m, n \to \infty$. Thus, $\{\overline{u}_n\}$ is a Cauchy sequence in $C$. As $C$ is closed in $H$, it is complete. Consequently, there exists $\overline{u} \in C$ such that $\overline{u}_n \to \overline{u}$ as claimed.

Next, we demonstrate that $\{x_n\}$ has the same limit point, that is,

$$x_n \to \overline{u}. \tag{5.8}$$

As the metric projection is nonexpansive, from (5.7) and the assumption $u_n \to u$, it follows that

$$\begin{aligned} \|x_n - \overline{u}\| &\leq \|x_n - \overline{u}_n\| + \|\overline{u}_n - \overline{u}\| \\ &= \|P_{C_n} u_n - P_{C_n} u\| + \|\overline{u}_n - \overline{u}\| \\ &\leq \|u_n - u\| + \|\overline{u}_n - \overline{u}\| \to 0. \end{aligned}$$

Thus, (5.8) holds true as claimed. This implies that $\{x_n\}$ is bounded. From (5.1), $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are also bounded.

As $\{x_n\}$ is convergent, the following expression holds:

$$x_n - x_{n+1} \to 0. \tag{5.9}$$

Next, observe that

$$X_n - x_{n+1} \to 0. \tag{5.10}$$

Indeed, as $x_{n+1} = P_{C_{n+1}} u_{n+1} \in C_{n+1}$, we have

$$\begin{aligned} &\|X_n - x_{n+1}\|^2 \\ \leq\ & a_n \|y_n - x_{n+1}\|^2 + b_n \|z_n - x_{n+1}\|^2 + c_n \|w_n - x_{n+1}\|^2 \\ \leq\ & a_n (\|y_n - x_n\| + \|x_n - x_{n+1}\|)^2 + b_n (\|z_n - x_n\| + \|x_n - x_{n+1}\|)^2 \\ & + c_n (\|w_n - x_n\| + \|x_n - x_{n+1}\|)^2 . \end{aligned} \tag{5.11}$$

From (5.2) and (5.9), we obtain $X_n - x_{n+1} \to 0$ as claimed. From (5.9) and (5.10), we have $x_n - X_n \to 0$. As $\{x_n\}$ is bounded, $\{X_n\}$ is also bounded.

Note that

$$\|Z_n - q\| \leq \|z_n - q\| \quad \text{and} \quad \|W_n - q\| \leq \|w_n - q\| \tag{5.12}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. The inequalities in (5.12) can be proved in a similar manner as (3.5). We aim to demonstrate that

$$y_n - Z_n \to 0 \ \text{ and } \ y_n - W_n \to 0. \tag{5.13}$$

Let $q \in F(S) \cap F(T)$. From (2.1), (5.12), and (5.1), we obtain the following expressions:

$$\begin{aligned}
& \|X_n - q\|^2 \\
= \; & \|a_n (y_n - q) + b_n (Z_n - q) + c_n (W_n - q)\|^2 \\
= \; & a_n \|y_n - q\|^2 + b_n \|Z_n - q\|^2 + c_n \|W_n - q\|^2 \\
& - a_n b_n \|y_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - y_n\|^2 \\
\leq \; & a_n \|y_n - q\|^2 + b_n \|z_n - q\|^2 + c_n \|w_n - q\|^2 \\
& - a_n b_n \|y_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - y_n\|^2 \\
\leq \; & a_n \|x_n - q\|^2 + b_n \|x_n - q\|^2 + c_n \|x_n - q\|^2 \\
& - a_n b_n \|x y_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - y_n\|^2 \\
= \; & \|x_n - q\|^2 \\
& - a_n b_n \|y_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - y_n\|^2 .
\end{aligned}$$

As $b_n c_n \|Z_n - W_n\|^2 \geq 0$, we have

$$\begin{aligned}
& a_n b_n \|y_n - Z_n\|^2 + a_n c_n \|y_n - W_n\|^2 \\
& \leq \|x_n - q\|^2 - \|X_n - q\|^2 \\
& \leq (\|x_n - q\| + \|X_n - q\|) \left| \|x_n - q\| - \|X_n - q\| \right| \\
& \leq (\|x_n - q\| + \|X_n - q\|) \|x_n - X_n\| .
\end{aligned}$$

Recall that $\{x_n\}$ and $\{X_n\}$ are bounded and $x_n - X_n \to 0$. Using the hypotheses $\underline{\lim}_{n\to\infty} a_n b_n > 0$ and $\underline{\lim}_{n\to\infty} a_n c_n > 0$, we obtain in the limit as $n \to \infty$ that $y_n - Z_n \to 0$ and $y_n - W_n \to 0$ as claimed.

Then,

$$x_n - Z_n \to 0 \text{ and } x_n - W_n \to 0. \tag{5.14}$$

In fact, using (5.2) and (5.13), the following expression can be derived:

$$\|x_n - Z_n\| \leq \|x_n - y_n\| + \|y_n - Z_n\| \to 0.$$

The second part in (5.14) can be similarly obtained.

From (5.8) and (5.14), it follows that $Z_n \to \overline{u}$ and $W_n \to \overline{u}$. As $S$ and $T$ satisfy the condition (4.1), we obtain $\overline{u} \in F(S) \cap F(T)$.

Our goal is to prove that $x_n \to \widehat{u}$. From (5.8), it is sufficient to show that

$$\overline{u} \left( = \lim_{n\to\infty} \overline{u}_n = \lim_{n\to\infty} x_n \right) = \widehat{u} \left( = P_{F(S)\cap F(T)} u \right) .$$

Applying (5.5) for $q = \widehat{u} \in F(S) \cap F(T)$, we have $\|u - \overline{u}_n\| \leq \|u - \widehat{u}\|$ for all $n \in \mathbb{N}$. From (5.7), we obtain $\|u - \overline{u}\| \leq \|u - \widehat{u}\|$. As $\overline{u} \in F(S) \cap F(T)$ and $\widehat{u} = P_{F(S)\cap F(T)} u$, this indicates that $\overline{u} = \widehat{u}$. According to (5.8), we can state that $x_n \to \widehat{u}$. This completes the proof. □

**Remark 5.2.** *We compare Theorems* 5.1 *and* 4.1 *focusing on conditions* (5.2) *and* (4.3)*. In Theorem* 5.1*, the additional conditions* $x_n - z_n \to 0$ *and* $x_n - w_n \to 0$ *are required. These assumptions are used in* (5.11) *when taking the limit as* $n \to \infty$*.*

From Theorem 5.1, the following corollary is obtained:

**Corollary 5.1** ([21]). *Let $C$ be a nonempty, closed, and convex subset of $H$. Let $S, T : C \to C$ be quasi-nonexpansive mappings satisfying the condition (4.1). Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\xi_n\}$, and $\{\theta_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $\lambda_n + \mu_n + \nu_n + \xi_n + \theta_n = 1$ for all $n \in \mathbb{N}$ and $\lambda_n \to 1$. Let $\{\lambda'_n\}$, $\{\mu'_n\}$, $\{\nu'_n\}$, $\{\xi'_n\}$, and $\{\theta'_n\}$ be sequences of real numbers in $[0,1]$ such that $\lambda'_n + \mu'_n + \nu'_n + \xi'_n + \theta'_n = 1$ for all $n \in \mathbb{N}$ and $\lambda'_n \to 1$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in $[0,1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$. Let $\{u_n\}$ be a sequence in $H$ such that $u_n \to u \, (\in H)$. Define a sequence $\{x_n\}$ in $C$ as follows:*

(5.15)

$$
\begin{aligned}
&x_1 = x \in C : \text{ given},\\
&C_1 = C,\\
&z_n = \lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n \frac{1}{n}\sum_{l=0}^{n-1} S^l x_n + \theta_n \frac{1}{n}\sum_{l=0}^{n-1} T^l x_n,\\
&w_n = \lambda'_n x_n + \mu'_n S x_n + \nu'_n T x_n + \xi'_n \frac{1}{n}\sum_{l=0}^{n-1} S^l x_n + \theta'_n \frac{1}{n}\sum_{l=0}^{n-1} T^l x_n,\\
&X_n = a_n x_n + b_n \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n + c_n \frac{1}{n}\sum_{l=0}^{n-1} T^l w_n,\\
&C_{n+1} = \left\{ h \in C_n : \|X_n - h\|^2 \le a_n \|x_n - h\|^2 + b_n \|z_n - h\|^2 + c_n \|w_n - h\|^2 \right\},\\
&x_{n+1} = P_{C_{n+1}} u_{n+1}
\end{aligned}
$$

*for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to an element $\widehat{u}$ in $F(S) \cap F(T)$, where $\widehat{u} = P_{F(S)\cap F(T)} u$.*

*Proof.* First, we ascertain that (a) $C_n$ is closed and convex, (b) $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$, and (c) the sequences $\{x_n\}, \{z_n\}, \{w_n\}, \{X_n\}$ in $C$ and $\{C_n\}$ are defined properly. To this end, we first consider the case of $n = 1$.

(i) Given $x_1 \in C_1 \, (= C)$, the elements $z_1$, $w_1$, and $X_1$ in $C$ and the set $C_2 \, (\subset C_1)$ are defined following the rule (5.15). As $C_1$ is closed and convex, $C_2$ is also closed and convex, as discussed in Theorem 5.1. We prove that $F(S) \cap F(T) \subset C_2$. Arbitrarily select $q \in F(S) \cap F(T) \, (\subset C_1)$. Then, it follows from (2.2) that

$$
\begin{aligned}
\|X_1 - q\|^2 &= \|a_1 (x_1 - q) + b_1 (z_1 - q) + c_1 (w_1 - q)\|^2\\
&\le a_1 \|x_1 - q\|^2 + b_1 \|z_1 - q\|^2 + c_1 \|w_1 - q\|^2,
\end{aligned}
$$

which implies that $q \in C_2$. Therefore, $F(S) \cap F(T) \subset C_2$ as claimed. From the assumption $F(S) \cap F(T) \neq \emptyset$, we have $C_2 \neq \emptyset$. From this, the metric projection $P_{C_2}$ is guaranteed to exist and $x_2 = P_{C_2} u_2$ is defined.

(ii) Given $x_2 \in C_2 \, (\subset C_1 = C)$, $z_2, w_2, X_2 \, (\in C)$ and $C_3 \, (\subset C_2 \subset C_1)$ are defined by the iterative rule (5.15). We can verify that $C_3$ is closed and convex and $F(S) \cap F(T) \subset C_3$. As $F(S) \cap F(T) \neq \emptyset$ is assumed, it follows that $C_3 \neq \emptyset$. Thus, the metric projection $P_{C_3}$ exists and $x_3 = P_{C_3} u_3$ is defined.

By repeating this reasoning, we can ascertain that (a), (b), and (c) hold true as claimed.

From Theorem 5.1, it is sufficient to demonstrate that $\|z_n - q\| \leq \|x_n - q\|$ and $\|w_n - q\| \leq \|x_n - q\|$ for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$, with $x_n - z_n \to 0$ and $x_n - w_n \to 0$. First, let us prove that $\|z_n - q\| \leq \|x_n - q\|$ and $\|w_n - q\| \leq \|x_n - q\|$. Choose $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. As $S$ and $T$ are quasi-nonexpansive, we can prove that

$$\left\| \frac{1}{n} \sum_{l=0}^{n-1} S^l x_n - q \right\| \leq \|x_n - q\| \quad \text{and} \quad \left\| \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n - q \right\| \leq \|x_n - q\| \tag{5.16}$$

in the same way as (3.5). Using these inequalities yields

$$\begin{aligned} & \|z_n - q\| \\ = & \left\| \lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n \frac{1}{n} \sum_{l=0}^{n-1} S^l x_n + \theta_n \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n - q \right\| \\ \leq & \lambda_n \|x_n - q\| + \mu_n \|S x_n - q\| + \nu_n \|T x_n - q\| \\ & + \xi_n \left\| \frac{1}{n} \sum_{l=0}^{n-1} S^l x_n - q \right\| + \theta_n \left\| \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n - q \right\| \\ \leq & \|x_n - q\| . \end{aligned}$$

Similarly, the expression $\|w_n - q\| \leq \|x_n - q\|$ can be proved.

Define $\overline{u}_n \equiv P_{C_n} u \in C_n$. Then, there exists $\overline{u} \in C$ such that $\overline{u}_n \to \overline{u}$, as indicated in (5.7) in the proof of Theorem 5.1. Furthermore, $\{x_n\}$ also converge to $\overline{u}$; see (5.8). Thus, $\{x_n\}$ is bounded. As $S$ and $T$ are quasi-nonexpansive, $\{Sx_n\}$ and $\{Tx_n\}$ are also bounded. Indeed, for $q \in F(S)$, it holds that

$$\begin{aligned} \|S x_n\| &\leq \|S x_n - q\| + \|q\| \\ &\leq \|x_n - q\| + \|q\| . \end{aligned} \tag{5.17}$$

As $\{x_n\}$ is bounded, $\{Sx_n\}$ is also bounded. Similarly, $\{Tx_n\}$ is also bounded. Furthermore, the inequalities in (5.16) imply that $\left\{ \frac{1}{n} \sum_{l=0}^{n-1} S^l x_n \right\}$ and $\left\{ \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n \right\}$ are bounded.

We demonstrate that $x_n - z_n \to 0$ and $x_n - w_n \to 0$. As $\lambda_n \to 1$, it follows that $\mu_n, \nu_n, \xi_n, \theta_n \to 0$. Therefore,

$$\begin{aligned} & \|x_n - z_n\| \\ & = \left\| x_n - \left( \lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n \frac{1}{n} \sum_{l=0}^{n-1} S^l x_n + \theta_n \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n \right) \right\| \\ & \leq (1 - \lambda_n) \|x_n\| + \mu_n \|S x_n\| + \nu_n \|T x_n\| + \xi_n \left\| \frac{1}{n} \sum_{l=0}^{n-1} S^l x_n \right\| + \theta_n \left\| \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n \right\| \\ & \to 0. \end{aligned}$$

As $\lambda_n' \to 1$, we obtain that $x_n - w_n \to 0$ as claimed. From Theorem 5.1, the desired result follows. □

## 6. Derivative Results

This section presents two convergence results as applications of Theorem 5.1. Although Theorems 3.1 and 4.1 can also be applied, we exclusively refer to Theorem 5.1 to save space. First, the following corollary is obtained:

**Corollary 6.1.** *Let $C$ be a nonempty, closed, and convex subset of a real Hilbert space $H$. Let $S, T : C \to C$ be quasi-nonexpansive mappings satisfying the condition (4.1). Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be sequences of real numbers in $[0,1]$ such that $\lambda_n \to 1$ and $\mu_n \to 1$. Let $\{u_n\}$ be a sequence in $H$ such that $u_n \to u\,(\in H)$. Define a sequence $\{x_n\}$ in $C$ as follows:*

$$\begin{aligned} x_1 &= x \in C : \text{ given}, \\ C_1 &= C, \\ z_n &= \mu_n x_n + (1-\mu_n) T x_n, \\ y_n &= \lambda_n z_n + (1-\lambda_n) S z_n, \\ X_n &= a_n y_n + b_n \frac{1}{n}\sum_{l=0}^{n-1} S^l y_n + c_n \frac{1}{n}\sum_{l=0}^{n-1} T^l z_n, \\ C_{n+1} &= \left\{ h \in C_n : \|X_n - h\|^2 \le (a_n + b_n)\|y_n - h\|^2 + c_n \|z_n - h\|^2 \right\}, \\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned} \tag{6.1}$$

*for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to an element $\widehat{u}$ in $F(S) \cap F(T)$, where $\widehat{u} = P_{F(S)\cap F(T)} u$.*

*Proof.* First, we verify that (a) $C_n$ is closed and convex, (b) $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$, and (c) the sequences $\{x_n\}, \{z_n\}, \{y_n\}, \{X_n\}, \{C_n\}$ are properly defined. We begin with the case of $n = 1$.

(i) Given $x_1 \in C_1\,(= C)$, the elements $z_1, y_1, X_1 \in C$ and set $C_2\,(\subset C_1)$ are defined following the iterative rule outlined in (6.1). Note that

$$\begin{aligned} &\|X_1 - h\|^2 \le (a_1 + b_1)\|y_1 - h\|^2 + c_1\|z_1 - h\|^2 \\ \iff &0 \le (a_1 + b_1)\|y_1\|^2 + c_1\|z_1\|^2 - \|X_1\|^2 - 2\langle (a_1+b_1) y_1 + c_1 z_1 - X_1,\ h\rangle . \end{aligned} \tag{6.2}$$

As $C_1$ is closed and convex, from (2.5) and (6.2), $C_2$ is also closed and convex. We demonstrate that $F(S) \cap F(T) \subset C_2$. Let $q \in F(S) \cap F(T)\,(\subset C_1)$. It follows from (2.2) that

$$\begin{aligned} \|X_1 - q\|^2 &= \|a_1(y_1 - q) + b_1(y_1 - q) + c_1(z_1 - q)\|^2 \\ &\le a_1\|y_1 - q\|^2 + b_1\|y_1 - q\|^2 + c_1\|z_1 - q\|^2 \\ &= (a_1 + b_1)\|y_1 - q\|^2 + c_1\|z_1 - q\|^2 , \end{aligned}$$

which implies that $q \in C_2$. Hence, $F(S) \cap F(T) \subset C_2$ as claimed. From the assumption $F(S) \cap F(T) \neq \emptyset$, we have $C_2 \neq \emptyset$. Consequently, the metric projection $P_{C_2}$ exists and $x_2 = P_{C_2} u_2$ is defined.

(ii) Given $x_2 \in C_2\,(\subset C_1 = C)$, $z_2, y_2, X_2\,(\in C)$ and $C_3\,(\subset C_2 \subset C_1)$ are defined by the iterative rule (6.1). We can thus verify that $C_3$ is closed and convex. Furthermore, $F(S) \cap F(T) \subset C_3$, given that $S$ and $T$ are quasi-nonexpansive. As $F(S) \cap F(T) \neq \emptyset$, $C_3$ is also nonempty. Thus, the metric projection $P_{C_3}$ exists and $x_3 = P_{C_3} u_3$ is defined.

Through a similar analysis, we can prove (a), (b), and (c).

From Theorem 5.1, it is sufficient to demonstrate that $\|y_n - q\| \leq \|x_n - q\|$ and $\|z_n - q\| \leq \|x_n - q\|$ for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$, with $x_n - y_n \to 0$ and $x_n - z_n \to 0$.

First, observe that $\|y_n - q\| \leq \|x_n - q\|$ and $\|z_n - q\| \leq \|x_n - q\|$. Let $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. As $T$ is quasi-nonexpansive, the following expression can be derived:

$$\begin{aligned} \|z_n - q\| &= \|\mu_n (x_n - q) + (1 - \mu_n)(Tx_n - q)\| \\ &\leq \mu_n \|x_n - q\| + (1 - \mu_n)\|Tx_n - q\| \\ &\leq \mu_n \|x_n - q\| + (1 - \mu_n)\|x_n - q\| = \|x_n - q\|. \end{aligned} \tag{6.3}$$

Using this, we obtain

$$\begin{aligned} \|y_n - q\| &= \|\lambda_n (z_n - q) + (1 - \lambda_n)(Sz_n - q)\| \\ &\leq \lambda_n \|z_n - q\| + (1 - \lambda_n)\|Sz_n - q\| \\ &\leq \lambda_n \|z_n - q\| + (1 - \lambda_n)\|z_n - q\| \\ &= \|z_n - q\| \leq \|x_n - q\| \end{aligned}$$

as claimed.

Define $\overline{u}_n = P_{C_n} u \in C_n$. Similar to the proof of Theorem 5.1, we can show that there exists $\overline{u} \in C$ such that $\overline{u}_n \to \overline{u}$ and $x_n \to \overline{u}$, as indicated in (5.7) and (5.8) in the proof of Theorem 5.1. As $\{x_n\}$ is convergent, it is bounded. Moreover, as $T$ is quasi-nonexpansive, $\{Tx_n\}$ is also bounded, as indicated in (5.17) in the proof of Corollary 5.1. Furthermore, from (6.3), $\{z_n\}$ is bounded. Therefore, $\{Sz_n\}$ is also bounded as $S$ is quasi-nonexpansive.

We show that $x_n - y_n \to 0$ and $x_n - z_n \to 0$. As $\mu_n \to 1$, it follows that

$$\begin{aligned} \|x_n - z_n\| &= \|x_n - (\mu_n x_n + (1 - \mu_n) Tx_n)\| \\ &\leq (1 - \mu_n)\|x_n - Tx_n\| \to 0. \end{aligned} \tag{6.4}$$

Using $\lambda_n \to 1$ and (6.4), we have

$$\begin{aligned} \|x_n - y_n\| &= \|x_n - (\lambda_n z_n + (1 - \lambda_n) Sz_n)\| \\ &= \|\lambda_n (x_n - z_n) + (1 - \lambda_n)(x_n - Sz_n)\| \\ &\leq \lambda_n \|x_n - z_n\| + (1 - \lambda_n)\|x_n - Sz_n\| \to 0 \end{aligned}$$

as claimed. From Theorem 5.1, the desired result follows. □

The iterative scheme in Corollary 6.1 is a three-step type. For three-step iterative methods, see Noor [31], Dashputre and Diwan [7], Phuengrattana and Suantai [32], and Chugh *et al.* [6]. Set $\mu_n = 1$ for all $n \in \mathbb{N}$ in Corollary 6.1. Then, $z_n = x_n$, and the following iterative scheme can be obtained:

$$\begin{aligned} y_n &= \lambda_n x_n + (1 - \lambda_n) Sx_n, \\ X_n &= a_n y_n + b_n \frac{1}{n} \sum_{l=0}^{n-1} S^l y_n + c_n \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n, \\ C_{n+1} &= \left\{ h \in C_n : \|X_n - h\|^2 \leq (a_n + b_n)\|y_n - h\|^2 + c_n \|x_n - h\|^2 \right\}, \\ x_{n+1} &= P_{C_{n+1}} u_{n+1}, \end{aligned}$$

where $x_1 = x \in C$ is given and $C_1 = C$. This scheme represents a two-step iterative scheme. The next corollary also provides a two-step iterative method to approximate a common fixed point:

**Corollary 6.2.** *Let $C$ be a nonempty, closed, and convex subset of $H$. Let $S, T : C \to C$ be quasi-nonexpansive mappings satisfying the condition (4.1). Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in $[0,1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$. Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, and $\{\xi_n\}$ be sequences of real numbers in $[0,1]$ such that $\lambda_n + \mu_n + \nu_n + \xi_n = 1$ for all $n \in \mathbb{N}$ and $\lambda_n \to 1$. Let $\{u_n\}$ be a sequence in $H$ such that $u_n \to u\,(\in H)$. Define a sequence $\{x_n\}$ in $C$ as follows:*

$$
\begin{aligned}
& x_1 = x \in C : \text{ given}, \\
& C_1 = C, \\
& y_n = \lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n T^2 x_n, \\
& X_n = a_n y_n + b_n \frac{1}{n} \sum_{l=0}^{n-1} S^l x_n + c_n \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n, \\
& C_{n+1} = \left\{ h \in C_n : \|X_n - h\|^2 \leq a_n \|y_n - h\|^2 + (b_n + c_n) \|x_n - h\|^2 \right\}, \\
& x_{n+1} = P_{C_{n+1}} u_{n+1}
\end{aligned} \tag{6.5}
$$

*for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to an element $\widehat{u}$ in $F(S) \cap F(T)$, where $\widehat{u} = P_{F(S)\cap F(T)} u$.*

*Proof.* At the outset, we verify that (a) $C_n$ is closed and convex, (b) $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$, and (c) the sequences $\{x_n\}$, $\{y_n\}$, $\{X_n\}$, and $\{C_n\}$ are properly defined.

(i) Given $x_1 \in C_1\,(= C)$, the elements $y_1$ and $X_1$ in $C$ and the set $C_2\,(\subset C_1)$ are defined following the iterative rule (6.5). In the definition of $C_2$,

$$
\begin{aligned}
& \|X_1 - h\|^2 \leq a_1 \|y_1 - h\|^2 + (b_1 + c_1) \|x_1 - h\|^2 \\
\Longleftrightarrow\ & 0 \leq a_1 \|y_1\|^2 + (b_1 + c_1) \|x_1\|^2 - \|X_1\|^2 - 2 \langle a_1 y_1 + (b_1 + c_1) x_1 - X_1,\ h \rangle .
\end{aligned}
$$

From (2.5), $C_2$ is closed and convex as $C_1$ is closed and convex. We demonstrate that $F(S) \cap F(T) \subset C_2$. Let $q \in F(S) \cap F(T)\,(\subset C_1)$. It follows from (2.2) that

$$
\begin{aligned}
\|X_1 - q\|^2 &= \|a_1 (y_1 - q) + b_1 (x_1 - q) + c_1 (x_1 - q)\|^2 \\
&\leq a_1 \|y_1 - q\|^2 + b_1 \|x_1 - q\|^2 + c_1 \|x_1 - q\|^2 \\
&= a_1 \|y_1 - q\|^2 + (b_1 + c_1) \|x_1 - q\|^2 .
\end{aligned}
$$

This means that $q \in C_2$. Thus, we obtain $F(S) \cap F(T) \subset C_2$ as claimed. From the assumption $F(S) \cap F(T) \neq \emptyset$, $C_2$ is also nonempty. We can thus conclude that the metric projection $P_{C_2}$ exists and $x_2 = P_{C_2} u_2$ is defined.

(ii) Given $x_2 \in C_2\,(\subset C_1)$, two elements $y_2$ and $X_2$ in $C$ and the set $C_3\,(\subset C_2 \subset C_1)$ are defined by the rule (6.5). We can prove that $C_3$ is closed and convex and $F(S) \cap F(T) \subset C_3$. According to the assumption $F(S) \cap F(T) \neq \emptyset$, $C_3 \neq \emptyset$. Therefore, the metric projection $P_{C_3}$ exists and $x_3 = P_{C_3} u_3$ is defined.

Through a similar analysis, we can ascertain that (a), (b), and (c) hold true.

Our aim is to prove that $\|y_n - q\| \leq \|x_n - q\|$ for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ and that $x_n - y_n \to 0$. Choose $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ arbitrarily. As $S$ and

$T$ are quasi-nonexpansive, it holds that

$$\begin{aligned}
& \|y_n - q\| \\
= \ & \left\|\lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n T^2 x_n - q\right\| \\
\leq \ & \lambda_n \|x_n - q\| + \mu_n \|S x_n - q\| + \nu_n \|T x_n - q\| + \xi_n \left\|T^2 x_n - q\right\| \\
\leq \ & \lambda_n \|x_n - q\| + \mu_n \|x_n - q\| + \nu_n \|x_n - q\| + \xi_n \|x_n - q\| \\
= \ & \|x_n - q\|.
\end{aligned}$$

Therefore, the part $\|y_n - q\| \leq \|x_n - q\|$ is proved.

Define $\overline{u}_n = P_{C_n} u \in C_n$. Similar to the proof of Theorem 5.1, we can show that there exists $\overline{u} \in C$ such that $\overline{u}_n \to \overline{u}$ and $x_n \to \overline{u}$; see (5.7) and (5.8). As $\{x_n\}$ is convergent, it is bounded. As $S$ and $T$ are quasi-nonexpansive, $\{S x_n\}$ and $\{T x_n\}$ are also bounded. Furthermore, $\left\{T^2 x_n\right\}$ is also bounded. Indeed, as $T$ is quasi-nonexpansive,

$$\begin{aligned}
\left\|T^2 x_n\right\| &\leq \left\|T^2 x_n - q\right\| + \|q\| \\
&\leq \|T x_n - q\| + \|q\| \\
&\leq \|x_n - q\| + \|q\|.
\end{aligned}$$

As $\{x_n\}$ is bounded, $\left\{T^2 x_n\right\}$ is also bounded as claimed.

We demonstrate that $x_n - y_n \to 0$. As $\lambda_n \to 1$, it follows that $\mu_n, \nu_n, \xi_n \to 0$. Therefore,

$$\begin{aligned}
\|x_n - y_n\| &= \left\|x_n - \left(\lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n T^2 x_n\right)\right\| \\
&\leq (1 - \lambda_n) \|x_n\| + \mu_n \|S x_n\| + \nu_n \|T x_n\| + \xi_n \left\|T^2 x_n\right\| \to 0.
\end{aligned}$$

From Theorem 5.1, we obtain the desired result. □

For sequences like $y_n = \lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n T^2 x_n$, see Maruyama *et al.* [29], Kondo and Takahashi [24], and Singh *et al.* [34].

## 7. Concluding Remarks

In this study, we investigated iterative scheme generating methods using mean-valued sequences for finding common fixed points of nonlinear mappings. Our contributions include enhancements to several results from prior research. This method is applied for the first time to the Martinez-Yanes and Xu approximation method. The proposed method can generate various types of iterative schemes, including two- and three-step iterative schemes.

The key observations and remarks are as follows: First, our analysis highlights the differences between the shrinking projection method of Takahashi, Takeuchi, and Kubota (Theorem 4.1) and the Martinez-Yanes and Xu (Theorem 5.1). The required conditions differ slightly depending on the technical circumstances of the proofs; see Remark 5.2. Second, Nakajo and Takahashi's CQ method can be extended in a similar manner, although this study focuses on the shrinking projection method and Mann type method. Third, our emphasis is on quasi-nonexpansive and mean-demiclosed mappings. This class of mappings contains more general types of mappings than nonexpansive mappings. For further details regarding this aspect, readers may refer to the Appendix in the work of Kondo [22]. Finally, although this article addresses common fixed point theorems for two nonlinear mappings, the methods can be extended to scenarios involving finitely many mappings.

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