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Producer Decision and Input Demand Theory : An Axiomatic Approach

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Producer Decision and Input Demand Theory :

An Axiomatic Approach

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Abstract This chapter is concerned with an axiomatic approach to input demand theory. By help of general production possibility sets, we intend to derive decomposition equations in input demand theory, which have been rather neglected so far in the economics literature. Special attention is paid to important comparison between the firm's expansion effect and the consumer's income effect. We discuss the question how and to what extent the expansion effect is distinct from the income effect. In this connection, the LeChatelier -Samuelson principle is also discussed,

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Keywords producer decision, comparative static analysis, expansion effect, inferior input, LeChatelier-Samuelson principle

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I The Firm's Expansion Effect versus the Consumer's Income Effect

The purpose of this chapter is to rigorously analyze producer decision and production possibility sets, thus exploring the *axiomatic foundations* of production theory. In older days, Hicks (1946, 1953), Samuelson (1947), Morishima (1953a, 1953b), and others, elaborated upon the well-established analysis of profit-maximizing firm's demand for inputs. Their main mathematical tools were *classical differential calculus.* In more recent times, however, there have emerged a series of papers which bravely study the *axiomatic foundations of production theory* on the basis of *more modern topological tools.* For this point, see Scott (1962), Bear (1965), Ferguson (1966, 1868, 1969), Rader (1968), Basett and Borcherding (1969), Hirota & Sakai (1969), Syrquin (1970), Shephard (1970), Arrow & Hahn (1971), Sakai (1973, 1974, 1975), and Malinvaud (1985). After the 1990s until the present day, unifying the axiomatic foundations of both consumption and production theories, brand new approaches named *duality approaches to microeconomic theory* have emerged and widely flourished , with Diewert (1982, 2018), Varian (1999, 2009), and McKenzie (2002) being eminent accomplishments.

A number of those economists above mentioned have bravely attempted to clear up a matter of long-standing confusion concerning the analogy between *consumer's income effect* and *producer's* expansion *effect.* While consumer demand theory and input demand theory appear to be analogous, there are not the same at all. Even a small difference at start may produce a big distinction at goal. The question how and to what extent they are really different is our main concern here. [1)]

As was independently shown by Hirota & Sakai (1969) and also by Syrquin (1970) , the effect of a change in the price of a certain input on the demand for another input can be divided into the following two separate effects. They are *a substitution effect along the old isoquant* and *an expansion effect along the new expansion path.* Unfortunately, the mathematical tool they employed was simple calculus, thus lacking mathematical rigor and fineness. The main purpose of this paper is to make our input demand theory mathematically more sophisticated than the previous attempts, yet developing it in many possible applications to economic reality.

As will be seen below, we obtain a sort of decomposition equation in input demand theory which appears to correspond well to the famous Slutsky equation in consumer demand theory. The correspondence between the two decomposition equations, however, is not quite exact: indeed, they look similar but are not identical. We must pay special attention to the critical difference between the word " similar " and the word "identical." [2)]

In this paper, we want to widely apply the powerful method of McKenzie (1957), which was originally used for the development of consumption theory, to the new area of input demand theory. Then, we can successfully derive various kinds of decomposition equations. And in so doing. we develop the *new idea of a compensated change in output price when a certain input price varies.* This idea is seemingly analogous to, but not exactly the same as, the familiar concept of a compensated change in income when a certain commodity price varies in the traditional consumption theory a la Hicks (1946).

So far, input demand theory has been developed in connection with the problem of *inferior inputs*. An input is called *normal* (or *inferior*) if a rise in output price causes an increase (or a decrease) in the demand in that input. We can obtain the following results. (i) While it is possible that all inputs are normal, it is not possible that they are all inferior. (ii) In case a certain input is inferior, it is not possible that all other inputs are gross complements with it although it is possible that they are all gross substitute for it.

The contents of this paper are as follows. Section 2 is addressed to a system or a production technology. In Section 3, the definition of cost and profit functions will be given, with a careful discussion of their properties. The topological approach of Shephard (1970) to the duality principle between cost and production is developed in these two sections. The total effect of a change in the price of a certain input on the demand for another input is decomposed into substitution and scale effects. In Section 4, we are first concerned with the properties of substitution and total effects, and examine the question how and to what extent they are similar to, or different from, those of substitution and total effects in consumption theory. The problems of inferior inputs and of net and gross substitutability are also discussed. Section 5 is devoted to various types of decomposition equations in input demand theory — one finite increment and two differential versions. It is rigorously shown that the demand curve of an input is not positively sloping, and the substitution and expansion effects always go in the same direction. In Section 6, we will apply our analysis to show the validity of the LeChatelier-Samuelson principle in input demand theory.

II Production Possibility Sets

We are concerned with a firm that is faced with the problem of producing a single output from a combination of a finite number of inputs subject to a production technology. Let us suppose that there are n inputs and that input-output prices are competitively determined in the market, being independent of the firm's individual behavior.

In what follows, we will make full use of a powerful topological method. For this method, see McFadden (1966), Malinvaud (1985), Mas-Colell & Whinston & Green (1995). McKenzie (2002), and Mitra & Nishimura (2009).

Let us denote an input bundle is denoted by $x = (x_1, x_2, \ldots, x_n)$. The set of all conceivable input bundles is denoted by X. X is the set of all nonnegative n-vectors :

$$X = \{x = (x_1, x_2, \ldots, x_n) : x \geqq 0\} . \quad (1)$$

Let us assume that for any $x \in X$, the largest output is conceivable and conveniently denoted by a *production function* $f(x)$. Let Y be the range of the production function :

$$Y = \{ y : y \geqq 0 \text{ and } y = f(x) \text{ for some } x \in X \} \quad (2)$$

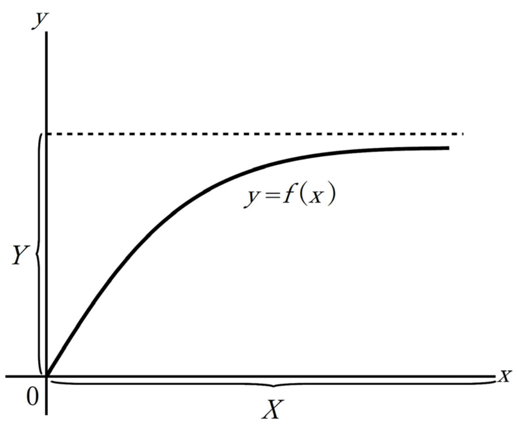

Fig. 1 A simple case of a single input and a single output.

We assume that Y is nonempty, convex, and open above. Y need not be the nonnegative real line. For a simple case with a single input and a single output, X and Y are illustrated in Fig. 1.

We now let define the following sets:

$$W = \{ w = (w_1, w_2, \ldots, w_n) : w > 0 \} , \tag{3}$$

$$P = \{ p : p > 0 \} . \tag{4}$$

Evidently, W is the input price space or the set of all input vectors, and P is the output price space or the set of all conceivable output prices.

There is an alternative useful description of the production function $f(x)$. To show this, let us define the following *production possibility sets*:

$$\bar{A}(y) = \{ \mathrm{x} : \mathrm{x} \in \mathrm{X} \text{ and } f(x) \geqq y \} , \tag{5}$$

$$\underline{A}(y) = \{ \mathrm{x} : \mathrm{x} \in \mathrm{X} \text{ and } f(x) \leqq y \} , \tag{6}$$

$$\begin{aligned} I(y) &= \bar{A}(y) \cap \underline{A}(y) \\ &= \{ \mathrm{x} : \mathrm{x} \in \mathrm{X} \text{ and } f(x) = y \} . \end{aligned} \tag{7}$$

It is noted that the *upper possibility set* $\bar{A}(y)$ is the set of all input vectors which are capable of producing *at least* the output y, whereas the *lower possibility set* $\underline{A}(y)$ is the set of all input vectors which are capable of producing *at most* the output y. The set $I(y)$, which is the intersection of $\bar{A}(y)$ and $\underline{A}(y)$, is clearly the familiar *isoquant frontier* corresponding to y. [2)]

For a simple case with two inputs, x_1 and x_2, those three production possibility sets, $\bar{A}(y)$, $\underline{A}(y)$ and $I(y)$, are well-illustrated in Fig. 2 .

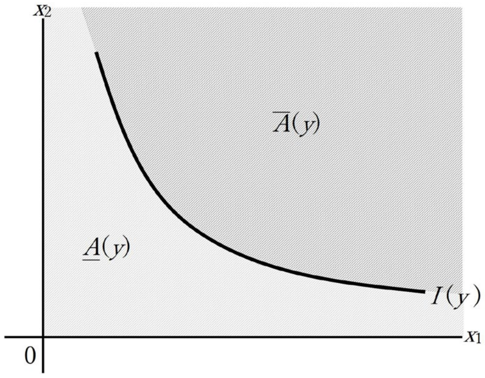

Fig. 2 Three production possibility sets: $\bar{A}(y)$, $\underline{A}(y)$, **and** $I(y)$.

We are ready to define a *production technology* as a family of those production possibility sets satisfying the following four assumptions: [3)]

Assumption (A1) If $0 \in I(y)$, then $y = 0$.

Assumption (A2) For each $y \in Y$, $\bar{A}(y)$ and $\underline{A}(y)$ are nonempty and closed in X.

Assumption (A3) Let us take $x^0 \in \bar{A}(y^0)$ and $x^1 \in \bar{A}(y^1)$ where $x^0 \neq x^1$. For any $t \in (0, 1)$, if we put $x^t = (1-t)x^0 + tx^1$ and $y^t = (1-t)y^0 + ty^1$, then we must find $x^t \in \text{Int}\,\bar{A}(y^t)$ or the interior of the set $\bar{A}(y^t)$. .

Assumption (A4) **If** $x^1 \geqq x^0$ and $x^0 \in \bar{A}(y)$, then $x^1 \in \bar{A}(y)$.

Those assumptions require detailed explanations. (A1) states that a positive output cannot be obtained from a null input bundle. In short, nothing comes from nothing. (A2) asserts that any output level y is attainable for some input bundle x , and that the production function $f(x)$ is continuous. Therefore, there should exist neither gaps nor jumps on the production curve .

Remarkably, (A3) together with (A2) clearly implies that the set $\bar{A}(y)$ is a "strictly convex set," meaning that for any two points, x^0 and x^1, of $\bar{A}(y)$, if $x^t = (1 - t)x^0 + tx^1$, then x^t must belong to the interior of the set $\bar{A}(y)$, not to the boundary $I(y)$. The essence of (A3) is well-illustrated in Fig 6.3. In terms of the production function $f(x)$, the production curve is a "strictly concave curve," with " no flat boundary."

And finally, (A4) insures that free disposal of inputs is possible. In other words, the producer can dispose of any extra inputs with no costs. Although this may not necessarily reflect the reality, it is theoretically a very convenient assumption.

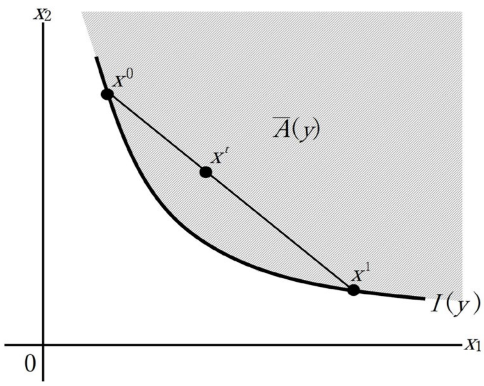

Fig. 3 The strict convexity of the upper production set $\bar{A}(y)$.
It is noted that x^t belongs to the interior of $\bar{A}(y)$.

III The Cost and Profit Functions: Definitions and Properties

In what follows,. we assume that Assumptions (A1) − (A4) are met for the production technology. In this section, the definition of cost and profit functions will be given and their properties will carefully be investigated.

First of all, for any $(w, y) \in W \times Y$, we define a *cost function* as follows:

$$c(w, y) = \textbf{Min} \{ wx : x \in \bar{A}(y) \} . \tag{8}$$

To see that the cost function $c(w, y)$ is well-defined, we take any arbitrary x^*

$\in \bar{A}(y)$ and let $H(w) = \{x : x \in X$ and $wx \leqq wx^*\}$. Since $\bar{A}(y)$ is closed by (A2) and $H(w)$ is clearly compact, it follows that $\bar{A}(y) \cap H(w)$ is compact as well. Therefore, as is seen in Fig. 6.4 , the continuous function wx takes on a minimum on $\bar{A}(y) \cap H(w)$. Note that for any $x \in \bar{A}(y) - H(w)$, we have $wx > wx^*$. Hence, the function wx attains a minimum on $\bar{A}(y)$.

In connection with the cost function $c(w, y)$, we define a *compensated input function* as follows:

$$u(w, y) = \{x : x \in \bar{A}(y) \text{ and } wx = c(w, y)\}. \qquad (9)$$

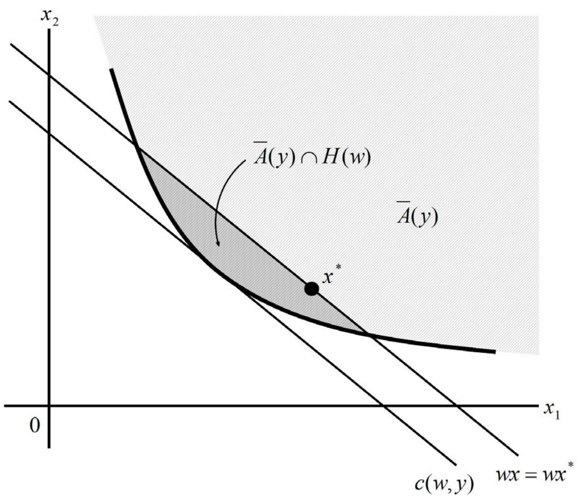

Fig. 6.4 The cost function is illustrated here

The newly-defined function $u(w, y)$ indicates the input bundle demanded by the firm when input prices are w and output is to be y. Under Assumptions (A1) – (A4), $u(w, y)$ is the unique element of X minimizing wx subject to $x \in \bar{A}(y)$. Consequently, we should have the following equation:

$$c(w, y) = w \cdot u(w, y). \qquad (10)$$

The relation between $c(w, y)$ and $u(w, y)$ is very important and clearly seen in Fig. 6.5. We are now in a position to establish the following useful lemma:

LEMMA 1. (Properties of $c(w, y)$ and $u(w, y)$)

For all $(w, y) \in W \times Y$, we have the following properties:

(1) $c(w, y)$ is continuous in (w, y).

(2) $c(w, y)$ is differentiable in w, and

$$\partial c(w, y) / \partial w_i = u_i(w, y). \qquad i = 1, 2, \ldots, n.$$

(3) $c(w, y)$ is homogeneous of degree one in w.

(4) $c(w, y)$ is concave in w.

(5) $c(w, y)$ is strictly convex in y.

(6) If $y^0 > y^1$, $y^0, y^1 \in Y$, then $c(w, y^0) > c(w, y^1)$.

(7) $u(w, y)$ is continuous in (w, y).

(8) $u(w, y)$ is homogeneous of degree zero in w.

Proof. We note that the present assumptions (A1) − (A4) are slightly stronger than the assumptions used by Uzawa (1964), and also by Friedman (1972): They did not assume the *strict* concavity of the production function $f(x)$, but merely the concavity of the upper production set $\bar{A}(y)$. Since the proof of Properties (3), (4), and (6) is found in the Uzawa (1964), and the proof of Property (1) is found in Friedman (1972), the proof of those properties are omitted here.

We will first prove Property (2). Let us pay attention to a small change of the input price vector Δw. For instance, if we have $\Delta w = (\Delta w_1, 0, \ldots, 0)$, then we must have $w + \Delta w = (w_1 + \Delta w_1, w_2, \ldots, w_n)$. In the light of Eq. (7.10) above, it is noted that we clearly have the following set of equations:

$$\begin{aligned} & c(w+\Delta w, y) - c(w, y) \\ &= (w+\Delta w)\, u(w+\Delta w, y) - w \cdot u(w, y). \\ &= w\, [\, u(w+\Delta w, y) - u(w, y)\,] + \Delta w \cdot u(w+\Delta w, y). \end{aligned} \tag{11}$$

$$c(w, y) = w \cdot u(w, y) < w \cdot u(w+\Delta w, y). \tag{12}$$

$$\begin{aligned} & c(w+\Delta w, y) = (w+\Delta w)\, u(w+\Delta w, y) \\ & \leqq (w+\Delta w)\, u(w, y) = c(w, y) + \Delta w \cdot u(w, y). \end{aligned} \tag{13}$$

Note here that for the simple case of two inputs, the validity of Eq. (12) is graphically illustrated in Fig. 5.

Taking advantage of Eqs. (11) and (12), we can derive the following:

$$\begin{aligned}
&【\ c(w+\Delta w, y) - c(w, y)\ 】 / |\Delta w| \\
&= w\ 【\ u(w+\Delta w, y) - u(w, y)\ 】 / |\Delta w| \\
&\quad + \Delta w \cdot u(w+\Delta w, y) / |\Delta w| \\
&\geqq \Delta w \cdot u(w+\Delta w, y) / |\Delta w| .
\end{aligned} \tag{14}$$

In the light of Eq. (6.13), we find the following:

$$\begin{aligned}
&【\ c(w+\Delta w, y) - c(w, y)\ 】 / |\Delta w| \\
&\leqq \Delta w \cdot u(w, y) / |\Delta w|.
\end{aligned} \tag{15}$$

Now, if we take care of Eqs. (14) and (13), then we obtain the following:

$$\begin{aligned}
0 &\leqq 【\ c(w+\Delta w, y) - c(w, y) - \Delta w \cdot u(w+\Delta w, y)\ 】 / |\Delta w| \\
&\leqq \Delta w\ 【u(w, y) - u(w+\Delta w, y)\ 】 / |\Delta w|.
\end{aligned} \tag{16}$$

Since $u(w, y)$ is continuous by Property (1), we can see that $u(w, y) - u(w+\Delta w, y) \rightarrow 0$ as $\Delta w \rightarrow 0$. Because $\Delta w / |\Delta w|$ is clearly bounded, the right-hand side of Eq. (6,16) goes to zero as $\Delta w \rightarrow 0$. Therefore, we must have the following:

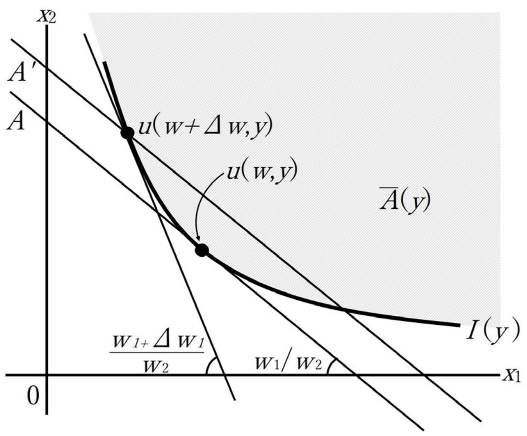

Fig. 5. For the simple case of two inputs, we clearly have the following :

$$c(w_1, w_2, y) = w_1 u_1(w_1, w_2, y) + w_1 u_2(w_1, w_2, y)$$

$$\leqq \; w_1 u_1(w_1+\Delta w_1, w_2, y) \; + w_1 u_2(w_1+\Delta w_1 \;, w_2, y)$$

$A'B'$ lies above AB, so that we find $w \cdot u(w, y) \; < \; w \cdot u(w+\Delta w, y)$.

$$\lim_{\Delta w \to 0} \left[c(w+\Delta w, y) - c(w, y) - \Delta w \cdot u(w, y) \right] / \; |\Delta w| .$$

$$= \lim_{\Delta w \to 0} \left[c(w+\Delta w, y) - c(w, y) - \Delta w \cdot u(w+\Delta w, y) \right] / \; |\Delta w| .$$

$$= \quad 0 \quad . \tag{17}$$

This clearly shows differentiability of $c(w, y)$ with respect to w. Now, let us specify $\Delta w = (0, ..., 0, \; \Delta w_i \;, 0, ..., 0)$. Then, it is easily seen that the following equation holds:

$$\partial c(w, y) / \partial w_i \;=\; u_i\,(w, y), \qquad i \;=\; 1,2, ..., n . \tag{18}$$

To prove Property (5), let us take $y^0, \; y^1 \in Y, \; y^0 \neq y^1$, and let $y^t = (1-t)\,y^0 + t\,y^1, \; t \in (0, 1)$. In the light of Assumptions (A2) and (A3), we find $(1-t)\,u(w, y^0) + t\,u(w, y^1) \in \mathrm{Int}\,\bar{A}(y^t)$. Hence, we must have the following:

$$\begin{aligned} &(1-t)\,c(w, y^0) + t\,c(w, y^1) \\ &= w \left[(1-t)u(w, y^0) + t\,u(w, y^1)\right] \; > \; c(w, y^t) \quad . \end{aligned} \tag{19}$$

To show Property (7), let us take a sequence $\{(w^k, y^k)\}$ in $W \times Y$ such that

$$\lim_{k\to\infty} (w^k, y^k) \;=\; (w, y) \in \; W \times Y \;. \tag{20}$$

Then, by definition, we have $c(w^k, y^k) = w^k \cdot u(w^k, y^k)$. Since $c(w^k, y^k)$ is continuous by Property (1), letting $k \to \infty$ yields the following:

$$c(w, y) \;=\; \lim_{k\to\infty} c(w^k, y^k) = \; w \cdot \lim_{k\to\infty} u(w^k, y^k) . \tag{21}$$

Since $u(w, y)$ is uniquely determined for (w, y), we must obtain the following:

$$\mathbf{Lim}_{k\to\infty}\ u(w^k, y^k) = u(w, y). \qquad (22)$$

This clearly shows that $u(w, y)$ is continuous in (w, y).

Finally,, to see Property (8), let $\lambda > 0$. Since $c(w, y)$ is homogenous of degree one in w by Property (3) , we must have the following:

$$\lambda w \cdot u(\lambda w, y) = c(\lambda w, y) = \lambda \cdot c(w, y) = \lambda w \cdot u(w, y). \qquad (23)$$

Thus, we must obtain $u(\lambda w, y) = u(w, y)$, showing that $u(w, y)$ is homogenous of degree zero in w. Q.E.D.

At first glance, the contents of Lemma 6.1 appear to be rather technical and even too mathematical. It should be noticed, however, that it discusses many important properties of the cost and compensated input functions which play critical roles in the theory of cost and production. According to Properties (1), the cost function $c(w, y)$ is continuous in (w, y), so that the *cost curve* as a graphical expression of the cost function is overall smooth and has neither gaps nor jumps throughout. Property (2) tells us that focusing on w only , $c(w, y)$ is extremely smooth, having no kinks at all. Moreover, it also indicates a very nice bridge between the two functions, namely the cost function $c(w, y)$ and the compensated input function $u(w, y)$. Exactly speaking, For any i, we should have $\partial c(w, y) / \partial w_i = u_i(w, y)$, demonstrating that the rate of change of total cost when the price of an input changes is equal to the amount of compensated demand for that input. This delicate relation between the two functions is well-illustrated in Fig. 6.

According to Properties (3), (4), (5) and (6), while the cost function $c(w, y)$ is homogeneous one and concave in w, it is strictly convex in y and also strictly increasing. Such concave and convex relations should be worthy of attention. Finally, in the light of Properties (7) and (8), whereas the compensated input demand function $u(w, y)$ is continuous in (w, y), it is homogeneous of degree zero in w only. This teaches us that $u(w, y)$ reflects the nice properties of $u(w, y)$.

In addition to Assumptions (A1) − (A4), we are now ready to postulate the following assumption.

Assumption (A5) Y is bounded from above. [4)]

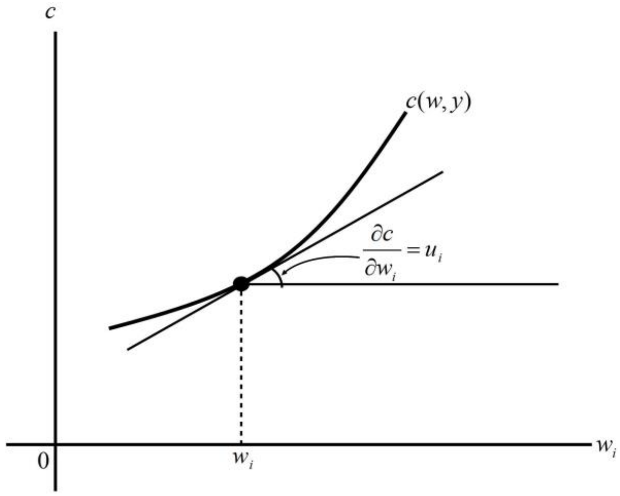

Fig. 6 A nice relation between c and u : $\partial c / \partial w_i = u_i$

As will be seen later, the purpose of Assumption (A5) is to force the profit function to be defined later for any $(w, p) \in W \times P$. The newly added Assumption (A5) together with the previous assumptions on Y implies that $Y = [0, y^*)$ for some $y^* < +\infty$. Let $E(w, p) = \{x : x \in X$ and $pf(x) - wx \geqq 0\}$.

Then, under Assumptions (A1) − (A5), the newly defined set E(w, p) is clearly bounded and closed; therefore, the continuous function pf(x) − wx takes on maximum on E(w, p). Note that by Assumption (A1), $pf(x) - wx = 0$ for $x = 0$, and maximum on E(w, p). Note that by Assumption (A1), $pf(x) - wx = 0$ for $x = 0$; $pf(x) - wx < 0$ for any $x \in X - E(w, p)$. Thus, the maximum attained on E(w, p) is actually the maximum on the whole X. Fig. 7 clearly illustrates the relation between the set $E(w, p)$ and X.

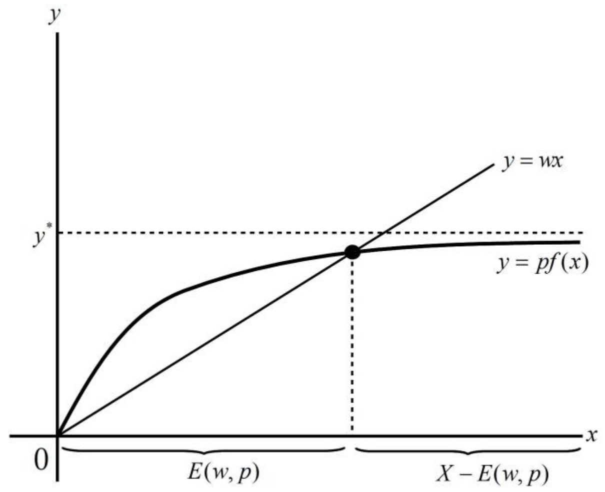

Fig. 7 The relationship between $E(w, p)$ and X.

Now, for any $(w, p) \in W \times P$, let us define a *profit function* $\pi(w, p)$ as follows:

$$\pi(w, p) = \textbf{Max} \{ p f(\mathrm{x}) - wx : x \in X \}. \qquad (24)$$

Clearly, for any given price vector (w, p), $\pi(w, p)$ stands for the maximum profit which can be obtained from the production technology. Since $f(x)$ is strictly concave by Assumption (A3), it is clear that $\pi(w, p)$ is positive for some $(w, p) \in W \times P$. An *optimum input function* and $x(w, p)$ and an *optimum output function* $y(w, p) \in W \times P$ are defined for all Evidently, $x(w, p)$ and $y(w, p)$ respectively indicate the input bundle as follows.

$$x(w, p) = \{ x : x \in X \text{ and } \pi(w, p) = p f(\mathrm{x}) - wx \}. \qquad (25)$$

$$y(w, p) = f\ (x(w, p)). \qquad (26)$$

Evidently, $x(w, p)$ and $y(w, p)$ respectively indicate the input bundle demanded by the firm, and the output bundle supplied by the firm. It is noted that the input-output vector $(x(w, p), y(w, p))$ is uniquely determined for each price vector $(w, p) \in W \times P$ since the profit $\pi(w, p)$ is strictly concave in x.

The properties of the profit function along with those of the optimum input-output function will be seen in the following lemma.

LEMMA 6.2. (Properties of $\pi(w, p)$) .

For all $(w, p) \in W \times P$, we have following properties:

(1) $\pi(w, p)$ is differentiable in (w, p), and

$$\partial \pi (w, p) / \partial w_i = - x_i (w, y). \quad i = 1,2, ..., n.$$
$$\partial \pi (w, p) / \partial p = y (w, y).$$

(2) $\pi(w, p)$ is homogeneous of degree one in (w, p).

(3) $\pi(w, p)$ is convex in (w, p).

(4) If $w^0 \geqq w^1 > 0$, then $\pi(w^0, p) \leqq \pi(w^1, p)$, with strict inequality if $x(w^0, p) > 0$.

(5) If $p^0 > p^1 > 0$, then $\pi(w, p^0) > \pi(w, p^1)$, with strict inequality if $y(w^0, p^1) > 0$.

(6) $x(w, p)$ and $y(w, p)$ are continuous in (w, p).

(7) $x(w, p)$ and $y(w, p)$ are homogeneous of degree zero in (w, p).

Proof. (a) For convenience, let us first show that $\pi(w, p)$, $x(w, p)$, and $y(w, p)$ are all continuous in (w, p).

To this end, take a sequence $\{(w^k, p^k)\}$ in $W \times P$ such that the following equation holds:

$$\mathbf{Lim}_{k\to\infty} (w^k, p^k) = (w, p) \in W \times P .$$

In the light of Eq. (6.24), we should have the following:

$$\begin{aligned} \pi(w, p) &= p\ y(w, p) - w x(w, p) \\ &\geqq p\ y(w^k, p^k) - w x(w^k, p^k) \ ; \end{aligned} \qquad (27)$$

$$\pi(w^k, p^k) = p^k \ y(w^k, p^k) - w^k x(w^k, p^k)$$
$$\geqq p^k \ y(w, p) - w^k x(w, p^k) \ . \qquad (28)$$

Because of Assumption (A5), it is clear that the sequences $\{\pi(w,p)\}$, $\{x(w,p)\}$, and $\{y(w,p)\}$ are all bounded, and contain convergent subsequences. Without loss of generality, we may assume that the original sequences themselves are convergent. By Eqs. (27) and (28), we derive the following:

$$\pi(w,p) = p \ y(w,p) - wx(w,p)$$
$$\geqq p \ \mathbf{Lim}_{k\to\infty} \ y(w^k, p^k) - w \ \mathbf{Lim}_{k\to\infty} \ x(w^k, p^k)$$
$$= \mathbf{Lim}_{k\to\infty} p^k \ \mathbf{Lim}_{k\to\infty} \ y(w^k, p^k) - \mathbf{Lim}_{k\to\infty} w^k \ \mathbf{Lim}_{k\to\infty} x(w^k, p^k)$$
$$= \mathbf{Lim}_{k\to\infty} \ \pi(w^k, p^k)$$
$$\geqq \mathbf{Lim}_{k\to\infty} p^k y(w,p) - \mathbf{Lim}_{k\to\infty} w^k x(w, p^k)$$
$$= p \ y(w,p) - wx(w,p). \qquad (29)$$

Therefore, we find $\mathbf{Lim}_{k\to\infty} \ \pi(w^k, p^k) = \pi(w,p)$, showing continuity of $\pi(w,p)$. Further, in the light of Eq. (6.29), we also obtain the following:

$$p \ \mathbf{Lim}_{k\to\infty} \ y(w^k, p^k) - w \ \mathbf{Lim}_{k\to\infty} x(w^k, p^k)$$
$$= p \ y(w,p) - wx(w,p). \qquad (30)$$

By the uniqueness property of $x(w,p)$ and $y(w,)$, it follows from Eq. (6.29) that $\mathbf{Lim}_{k\to\infty} \ y(w^k, p^k) = (w,p)$ and $\mathbf{Lim}_{k\to\infty} x(w^k, p^k) = x(w,p)$, thus assuring Property (6).

(b) By help of (a), we will next prove Property (1), In the light of Eqs. (28) and (6.29), we find, for any change $(\Delta w, \Delta p)$ in the price vector, we obtain the following equations.

$$\pi(w,p) \geqq p \cdot y(w+\Delta w, p+\Delta p) - w \cdot x(w+\Delta w, p+\Delta p)$$
$$= \pi(w+\Delta w, p+\Delta p) + \Delta w \cdot x(w+\Delta w, p+\Delta p)$$
$$-\Delta p \cdot y(w+\Delta w, p+\Delta p), \qquad (31)$$

$$\pi(w,p) \geqq (p+\Delta p) y(w,p) - (w+\Delta w) x(w,p)$$
$$= \pi(w,p) - \Delta w \cdot x(w,p) + \Delta p \cdot y(w,p). \qquad (32)$$

In the light of Eqs. (6.31) and (6.32), we can easily find the following:

$$- \Delta w \cdot x(w,p) + \Delta p \cdot y(w,p).$$
$$\leqq \pi(w+\Delta w, p+\Delta p) - \pi(w,p)$$
$$\leqq - \Delta w \cdot x(w+\Delta w, p+\Delta p) + \Delta p \cdot y(w+\Delta w, p+\Delta p).$$

Therefore, we can derive the following inequalities. [5)]

$$0 \leqq \frac{\pi(w+\Delta w, p+\Delta p) - \pi(w,p) + \Delta w \cdot x(w,p) - \Delta p \cdot y(w,p).}{|(\Delta w, \Delta p)|}$$

$$\leqq \frac{- \Delta w 【 x(w+\Delta w, p+\Delta p) - x(w,p) 】 + \Delta p 【 y(w+\Delta w, p+\Delta p) - y(w,p) 】}{|(\Delta w, \Delta p)|} \qquad (33)$$

In the light of (a) above, $【 x(w+\Delta w, p+\Delta p) - x(w,p) 】 \to 0$ and $【 y(w+\Delta w, p+\Delta p) - y(w,p) 】 \to 0$ as $(\Delta w, \Delta p) \to 0$. Since $\Delta w / |(\Delta w, \Delta p)|$ and $\Delta p / |(\Delta w, \Delta p)|$ are bounded, the extreme right-hand side of Eq. (6.33) goes to zero as $(\Delta w, \Delta p) \to 0$. Therefore, we can derive the following.

$$\lim_{(\Delta w, \Delta p)\to 0} \frac{\pi(w+\Delta w, p+\Delta p) - \pi(w,p) + \Delta w \cdot x(w,p) - \Delta p \cdot y(w,p).}{|(\Delta w, \Delta p)|} = 0, \qquad (34)$$

assuring differentiability of $\pi(w,p)$.

Now, letting $(\Delta w, \Delta p) = (0, \ldots, \Delta w_i, 0, \ldots, 0)$, Eq. (6.34) yields the following.

$$\partial \pi(w,p) / \partial w_i = - x_i(w,p). \qquad i = 1,2, \ldots, n.$$

Similarly, letting $(\Delta w, \Delta p) = (0, \ldots, \Delta p)$, it yields the following.

$$\partial \pi(w,p) / \partial p = y(w,p).$$

(c) To see Properties (2) and (7) together, let $\pi > 0$. Then, we find the following：

$$\lambda p \cdot y(\lambda w, \lambda p) - \lambda w \cdot x(\lambda w, \lambda p) = \pi(\lambda w, \lambda p)$$
$$= \mathbf{Max} \{ \lambda p \ f(x) - \lambda w x : x \in X \}$$
$$= \lambda \ \mathbf{Max} \{ p \ f(x) - w x : x \in X \}$$
$$= \lambda \pi (w, p) = \lambda p \cdot y(w, p) - \lambda w \cdot x(w, p) .$$

By the uniqueness property of $x(w, p)$ and $y(w, p)$, we find $x(\lambda w, \lambda p) = x(w, p)$ and $y(\lambda w, \lambda p) = y(w, p)$, thus assuring (7). We also have $\pi(\lambda w, \lambda p) = \lambda \pi (w, p)$, thereby assuring (2).

(d) To see (3), select $(w^0, p^0), (w^1, p^1) \in W \times P$, $(w^0, p^0) \neq (w^1, p^1)$. Let $(w^t, p^t) = (1-t)(w^0, p^0) + t (w^0, p^0)$, $t \in (0, 1)$. Then, we have the following.

.

$$\pi(w^t, p^t) = p^t y(w^t, p^t) - w^t x(w^t, p^t)$$
$$= (1 - t)\{p^0 y(w^t, p^t) - w^0 x(w^t, p^t)\}$$
$$+ t \{p^1 y(w^t, p^t) - w^1 x(w^t, p^t)\}$$
$$\leqq (1 - t) \pi(w^0, p^0) + t \pi(w^1, p^1)$$

Clearly, this assures Property (3).

(c) To see Property (4), let $w^0, w^1 \in W$, $w^0 \geq w^1$ 、and $p \in P$. Then, we have the following.

$$\pi(w^0, p^t) = p \cdot y(w^0, p) - w^0 \cdot x(w^0, p)$$
$$\leqq p \cdot y(w^0, p) - w^1 \cdot x(w^0, p),$$

with strict inequality if $x(w^0, p) > 0$. This assures Property (4). The proof of Property (3) should be parallel to the above. Q.E.D.

Lemma 6.2 is quite interesting in making a bridge between the profit function and the optimum input-output function. The essence of Property (1) is expressed by the two equations; $\partial \pi / \partial w_i = - x_i (w, p)$ and $\partial \pi / \partial p = y (w, y)$. Graphically, it can easily be seen in Fig. 8.

In plain English, on the one hand, the rate of change of total profit when the price of an input price changes is equal to the amount of the input demanded by the firm, multiplied by (−1) . On the other hand, the rate of change of total profit when the

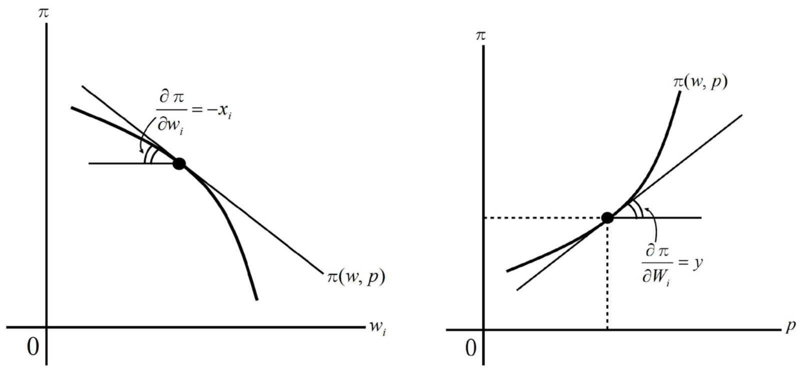

Fig. 8 Nice relationship between profit and input-output :
$\partial \pi / \partial w_i = - x_i$ **and** $\partial \pi / \partial p = y$ . .

price of an output changes is equal to the amount of the output supplied by the firm.

According to Property (1), the profit function $\pi(w, p)$ is not only continuous but also differentiable in (w, p), so that the *profit curve* as a graphical expression of the profit function is overall very smooth, with no kinks, having neither gaps nor jumps throughout. Property (2) implies that when (w, p) doubles, π also doubles. It follows from Property (3) that for any $(w, p) \in W \times P$, and for any fraction t $\in (0, 1)$, the following inequality holds:

$$(1-t)\ \pi(w^0, p^0) + t\ \pi(w^1, p^1)$$
$$> \pi\left((1-t)w^0 + t\ w^1, (1-t)p^0 + t\ p^1 \right). \quad (35)$$

According to Properties (3) and (4), while π tends to decrease as w rises, but it tends to increase as p rises, thus agreeing with common sense.

Finally, by help of Properties (6) , inputs and outputs continuously change in response to a small change in (w, p) . And Property (7) shows us that when all input prices and all output prices change in the same proportion, all inputs and all output are expected to remain unaffected, agreeing our common sense.

Now, we are ready to establish the following interesting lemma.

LEMMA 6.3. (Marginal cost and output price)

For all $(w, p) \in W \times P$, we obtain the following :

$$\partial c(w, y(w, p)) / \partial y = p, \quad \text{almost everywhere}.$$

Proof. By the definition of $c(w, y(w, p))$, we note that the following inequality must hold:

$$\begin{aligned} & p \cdot y(\mathrm{w}, \mathrm{p}) - c(w, y(\mathrm{w}, \mathrm{p})) \\ \geqq\ & p\ y - c(w, y) \quad \text{for all } (w, y) \in W \times Y. \end{aligned} \tag{36}$$

Since $c(w, y)$ is concave in y by Lemma 7.1 (4) , it must be almost everywhere differentiable in y .[6]

Further, Eq. (36) asserts that the function $p\ y - c(w, y)$ is maximized at $y = y(w, p)$. Therefore, the first-order characterization for the maximum yields the following :

$$\partial (py - c(w, y)) / \partial y = 0, \quad \text{almost everywhere}.$$

This implies that $p - \partial c(w, y) / \partial y = 0$ at $y = y(w, p)$ almost everywhere , from which immediately $\partial c(w, y(w, p)) / \partial y = p$ almost everywhere . Q.E.D.

Lemma 6.3 is an important lemma, saying that the marginal cost of an output is equal to output price Surely, it demonstrates profit-maximizing behavior of the firm from a different angle. The essence of Lemma 6.3 is graphically illustrated in Fig. 9 .

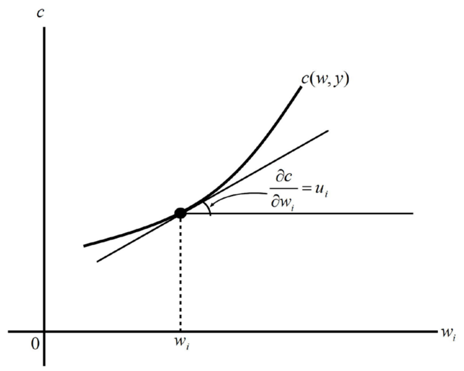

Fig. 9　　At equilibrium, the following relation holds almost everywhere:

$$\partial c(w, y) / \partial y = p \quad \text{at} \quad y = y(w, p).$$

We are now in a position to discuss how the optimum and compensated input functions are related with each other.

LEMMA 4. (Properties of $x(w, p)$ and $y(w, p)$)

For all $(w, p) \in W \times P$, we obtain the following equations :

(1) $x(w, p) = u(w, y(w, p))$.

(2) $c(w, y(w, p)) = w \cdot x(w, p)$.

(3) $\pi(w, p) = p \cdot y(w, p) - c(w, y(w, p))$. .

Proof. To see Property (1), we first note $x(w, p) \in \bar{A}(y(w, p))$. Let us next take $x^s \in \bar{A}(y(w, p))$, and $x^s \neq x(w, p)$. This is possible by Assumptions (A2), and (A4). Now, let a "middle point" $x^t = (1 - t) x(w, p) + x^s$, $t \in (0, 1)$. Then, by means of Assumptions (A2) and (A3), we obtain $x^t \in \text{Int}\, \bar{A}(y(w, p))$, so that $w x^t > w \cdot x(w, p)$ as is well-illustrated in Fig. 10.

Letting $t \rightarrow 0$ yields $x^t \rightarrow x(w, p)$, and $w x \geqq w \cdot x(w, p)$. Since $x(w, p) \in \bar{A}(y(w, p))$, and $u(w, y(w, p))$ is uniquely determined for each (w, p), it must follows that $u(w, y(w, p)) = x(w, p)$. This proves Property (1).

From (1), we find $c(w, y(w, p)) = w \cdot x(w, p) = u(w, y(w, p))$, and $\pi(w, p) = p \cdot y(w, p) - c(w, y(w, p))$, assuring Properties (2) and (3). Q.E.D.

.

From this lemma, the economic significance of $u(w, y(w, p))$ is quite clear, For an arbitrary price vector (w, p), the optimum input-output combination $(x(w, p))$ is uniquely determined from the production technology. Then, $c(w, y(w, p))$ denotes the cost level which governs the output level $y(w, p)$, and $u(w, y(w, p))$ is the corresponding input bundle. It is also seen that the profit gained by the firm is equal to the difference between $p \cdot y(w, p)$ and $c(w, y(w, p))$.

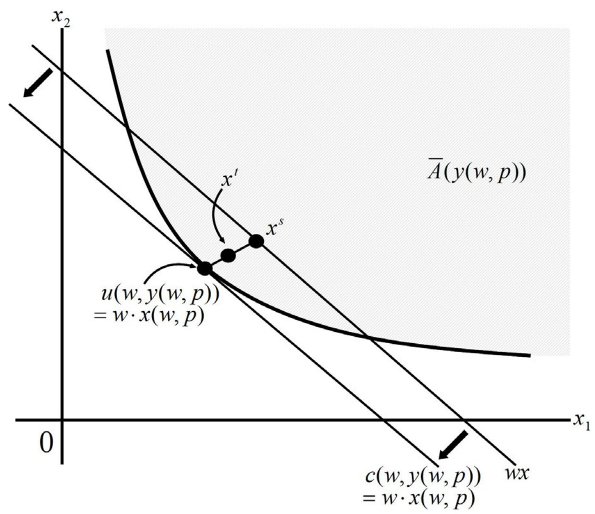

Fig. 10 The proof of Lemma 6.10 (1) is graphically illustrated here.

6. 4 Properties of the Substitution and Total Effects

In the last section, we sought the information about conditions governing inputs demanded by the firm at given input-output prices. In this section, we are now ready to use it to discover how the inputs will change when these prices vary.

Now, let us consider a change in w_j. Then, we will see that its impact on x_i can be divided into the following two effects: a *substitution effect* along the old isoquant and an *expansion effect* (or a *scale effect)* along the new expansion path (or scale path). Although the naming of the first effect is fairly traditional, that of the second effect might be quite new. The composition of the total effect into the substitution and the expansion (or scale) effect in production theory was first noticed by Hirota and Sakai (1969), and later developed by Sakai (1973), Sakai (1974), and others. Such composition in production seems to be similar to, yet not as the same as the composition into the substitution and income effects in consumption theory. How and to what extent the expansion effect differs from the income effect is a very important question, thus composing the main theme of the present chapter. A detailed discussion on this point will be made in the next section. Although "the expansion effect" has been frequently used in the previous literature and also in present paper, we would like to say that "the scale effect" may be an equally appealing name because it deals with "contraction" (larger scale) as well as "expansion" (smaller scale).

The properties of the substitution effect can be summarized in the following theorem.

THEOREM 5. (Differential properties of $u(w, y)$)

For all $(w, y) \in W \times Y$, we have the following properties :

(1) $(\partial u_i/\partial w_1) w_1 + ... + (\partial u_i/\partial w_n) w_n = 0$ for any i, almost everywhere.

(2) $w_1(\partial u_1/\partial w_j) + .. + w_n (\partial u_n/\partial w_j) = 0$ for any j, almost everywhere.

(3) $\partial u_i/\partial w_j = \partial u_j/\partial w_i$ for any i, j, almost everywhere.

(4) The following matrix is negative semi-definite, almost everywhere.

$$\begin{pmatrix} \partial u_1/\partial w_1 & \cdots & \partial u_1/\partial w_n \\ \vdots & & \vdots \\ \partial u_n/\partial w_1 & \cdots & \partial u_n/\partial w_n \end{pmatrix}$$

Proof. Recalling that $c(w, y)$ is concave in w by Lemma 1 (4) , it should be *almost everywhere twice differentiable* in w. Note that by Lemma 1 (2) , we find that $\partial c(w, y) / \partial w_i = u_i(w, y)$ for all i. Therefore, it follows that for any i, $u_i(w, y)$ is almost everywhere differentiable in w. [7)]

With those preparations in mind, we also recall that $u(w, y)$ is homogeneous of degree zero in w by Lemma 6.1 (8). Here, if we apply the famous Euler theorem on homogeneous equations. we immediately find the following equation.

$(\partial u_i/\partial w_1) w_1 + ... + (\partial u_i/\partial w_n) w_n = 0$ for any i, almost everywhere ,.

assuring Property (1).

To prove Property (2), it is noted that by definition of the compensated function u, the following inequality holds :

$$w \cdot u(w^s, y) \geqq w \cdot u(w, y) \quad \text{for all} \quad w^s \in W . \qquad (37)$$

Note that the validity of Eq. (7.37) is well-illustrated in Fig. 11. Eq. (37) tells us that the function $w \cdot u(w, y)$ attains minimum among all the functions of the form $w \cdot u(w^s, y)$ at $w^s = w$.

Hence, the first order characterization for the minimum yields the following .

$$\partial(w \cdot u(w^s, y)) / \partial w^s_j = 0 \quad \text{at} \quad w^s = w, \quad j = 1, \ldots, n,$$

almost everywhere, from which Property (2) follows.

To see Property (3), we recall the well-known Young theorem on differential calculus, which says that whenever a function is twice differentiable, the order of differentiation is not important. [8)]

Therefore, by Lemma 1 (2) above, we obtain the following.

$$\partial u_i / \partial w_j = \partial^2 c / \partial w_j \partial w_i = \partial^2 c / \partial w_i \partial w_j = \partial u_j / \partial w_i$$
$$\text{for any } i, j, \quad \text{almost everywhere,}$$

which proves Property (3).

Finally, since the Hessian matrix of the concave function $c(w, y)$ with respect to w is almost everywhere negative semi-definite, Property (4) is easily derived from Lemma 1 (2). Q.E.D.

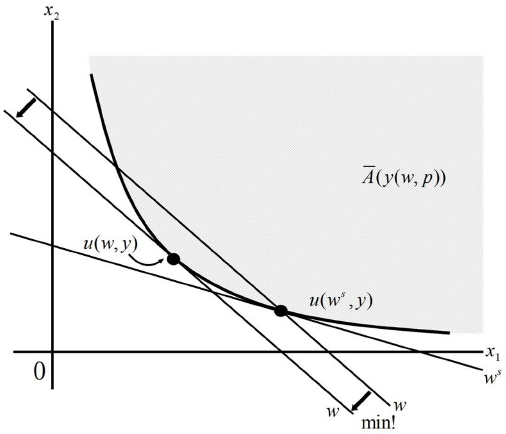

Fig. 6.11. $w \cdot u(w^s, y) \geqq w \cdot u(w, y)$ **for all** $w^s \in W$.

We are in a position to discuss a new concept of *net substitution and net complementarity* in contrast to a more traditional concept of *gross substitution and gross complementarity*. We say that the input i is a *net substitute for the input* j if $\partial u_i / \partial w_j > 0$, and a *net complement for it if* $\partial u_i / \partial w_j < 0$. Possibly as a rare case, we could have the special situation in which $\partial u_i / \partial w_j \leqq 0$ *for all* i . Generally speaking, when the price of a certain input price rises, the demand for that input tends to decrease if output is to be constant. In other words, the compensated demand curve tends to be negatively sloping. Coupled with this result, Property (1) of Theorem 6.5 implies that whereas it is possible that all other inputs are net substitutes for an input, it is not possible that they are net complements for it. In short, although net substitutability can be seen everywhere, net complementarity is a rare phenomenon. Property (2) also shows further limits on the possibility of net complementarity. Property (3) asserts that the substitution effect should be symmetrical between two inputs. Naturally, this agrees with common sense.

The properties of the total effect will be summarized in the following theorem.

Theorem 6 . (Differential properties of $x(w, p)$ and $y(w, p)$)

For all $(w, p) \in W \times P$, we have the following properties :

(1) $(\partial x_i / \partial w_1) w_1 + \ldots + (\partial x_i / \partial w_n) w_n + (\partial x_i / \partial p) p = 0$ for any i, almost everywhere .

(2) $(\partial y / \partial w_1) w_1 + \ldots + (\partial y / \partial w_n) w_n + (\partial y / \partial p) p = 0$, almost everywhere.

(3) $\partial x_i / \partial w_j = \partial x_j / \partial w_i$ for all i , j, almost everywhere.

(4) $\partial x_i / \partial p + \partial y / \partial w_i = 0$ for all i, almost everywhere.

(5) The following matrix is positive semi-definite, almost everywhere.

$$\begin{pmatrix} -\partial x_1/\partial w_1 & \cdots & -\partial x_1/\partial w_n & \partial x_1/\partial p \\ \vdots & & \vdots & \vdots \\ -\partial x_n/\partial w_1 & \cdots & -\partial x_n/\partial w_n & \partial x_n/\partial p \\ -\partial y/\partial w_1 & \cdots & -\partial y/\partial w_n & \partial y/\partial p \end{pmatrix}$$

Proof. We first note that the profit function $\pi(w, p)$ is almost everywhere twice differentiable since it is convex by Lemma 2 (3). Since $\partial \pi / \partial w_i = - x_i$ for any i, and $\partial \pi / \partial p = y$, this obviously implies that both input function $x(w, p)$ and the output function $y(w, p)$ are almost everywhere differentiable. Such differential properties are quite important for further derivations.

From Lemma 2 (7), we see that $x(w, p)$ and $y(w, p)$ are homogeneous of degree one. Therefore, by applying the famous Euler theorem on homogenous functions here , we can immediately obtain Properties (1) and (2).

Next, Let us recall the famous Young theorem, which says that if a function is twice differentiable, the order of differentiation does not matter, leading to the same result. So, if we apply the Young theorem to the profit function $\pi(w, p)$ which is convex by Lemma 2 (7) , then we can obtain the following equations .

$$\partial x_i / \partial w_j = - \partial^2 \pi / \partial w_j \partial w_i = - \partial^2 \pi / \partial w_i \partial w_j$$
$$= \partial x_j / \partial w_i \quad \text{for all } i, j, \text{ almost everywhere.,}$$

$$\partial x_i / \partial p + \partial y / \partial w_i = - \partial^2 \pi / \partial p\, \partial w_i + \partial^2 \pi / \partial w_i\, \partial p$$
$$= - \partial^2 \pi / \partial w_i\, \partial p + \partial^2 \pi / \partial w_i\, \partial p = 0 \quad \text{for all } i \text{, almost everywhere .}$$

Therefore, Properties (3) and (4) are surely assured.

. Finally, since the risk function $\pi(w, p)$ is convex, its Hessian matrix must be positive semi-definite, almost everywhere. We note that the following set of equations hold.

$$\partial^2 \pi / \partial w_j\, \partial w_i = \partial x_i / \partial w_j, \quad \partial^2 \pi / \partial p\, \partial w_i = \partial x_i / \partial p,$$
$$\partial^2 \pi / \partial w_j\, \partial p = \partial y / \partial w_j, \quad \partial^2 \pi / \partial p^2 = \partial y / \partial p .$$

Hence, Property (5) is definitely assured. Q.E.D.

To see the economic significance of Theorem 7.6, let us introduce several useful concepts here. We say that the input i is a *gross substitute for the input* j if $\partial x_i / \partial w_j > 0$; a *gross complement for it if* $\partial x_i / \partial w_j < 0$. We also say that the input i is a *normal input* if $\partial x_i / \partial p > 0$; an *inferior input* if $\partial x_i / \partial p < 0$. If we employ these concepts, we can give new light on the relationship between inputs and outputs on the one hand and input prices and output prices on the other hand.

First of all, Property (4) of Theorem 6.4 tells us that the "supposedly normal situation" under which $\partial x_i / \partial w_i \leqq 0$ for any i and $\partial y / \partial p \geqq 0$ is certainly plausible but not inevitable. As can easily be expected, while the input demand curve tends to be negatively sloping, the output supply curve tends to e positively sloping. Property (4) means that $\partial x_i/\partial p > 0$ (or <0) if and only if $\partial y/ \partial w_i < 0$ (or > 0). In plain English, thus means that the input i is a normal input (or am inferior input) if and only if a fall in the price of the input leads to an increase (or a decrease) in output.

In the light of Property (5), we find $\partial y/\partial p \geqq 0$. So, Property (2) implies that although it is likely that all of inputs are normal, it is a mission impossible that they are all inferior. Now suppose that a certain input, say x_i , is an inferior input, so that $\partial x_i/\partial p < 0$. It follows from Property (1) that, in such a case, it is not possible at all that all other inputs are gross complements for the input i although it is truly possible that they are all gross substitutes for it. Because of Property (3), the *total* effect is symmetrical between two inputs. However, we already know from Theorem 6.5 (3) above that the *substitution* effect is symmetrical between them. Therefore, the *expansion* effect, as the difference of the total and substitution effects, must be symmetrical as well. [9)]

Finally, in the light of Properties (3) and (5), we obtain the following inequalities.

$$(\partial x_i / \partial w_i)(\partial x_j/\partial w_j) \geqq (\partial x_i / \partial w_j)^2 \quad \text{for all } i \neq j .$$

$$(-\partial x_i / \partial w_i)(\partial y/\partial p) \geqq (\partial x_i / \partial p)(-\partial y/\partial w_1) \quad \text{for all } i \neq j .$$

Clearly, these inequalities indicate dominance of the *own effects* over the *cross effects.* Summing up, the differential properties of $x(w, p)$ and $y(w, p)$ are so important that they will be utilized for further discussions on input demand theory .

V Decomposition Equations in Input Demand Theory

As stated above, the *total effect* of a change in the price of an input on the demand for another input can be split up into the two separate effects, namely the *substitution and scale effects.* The purpose of this section is to use the previous results to derive various types of *decomposition equations in input demand theory*. Our attention will be mainly devoted to seeing how and to what extent they are analogous to , or distinct from, the famous Slutsky equations in consumer demand theory (see Slutsky (1915) and McKenzie (1957)).

Let us attempt to decompose the change of the demand for the input i responding to a change in the price of the input j . For that purpose, let us 1 consider the following increment .

$$\Delta x_i\ (w, p) \ = \ x_i\ (w + \ \Delta_j w, p\ ) \ - \ x_i\ (w, p) \ ,$$

where $\Delta_j w$ is defined as follows .

$$\Delta_j w \ = \ (0, \ldots, 0, \ \Delta w_j, 0, \ldots, 0) .$$

Then, by Lemma 6.4 (1) above, we obtain the following.

$$\begin{aligned} \Delta x_i(w, p) = &\ u_i(w + \Delta_j w, \ y(w + \Delta_j w, p)) - u_i(w, y(w, p)) . \\ = &\ u_i(w + \Delta_j w, \ y(w, p)) - u_i(w, y(w, p)) . \\ &+ u_i(w + \Delta_j w, y(w + \Delta_j w, p)) - u_i(w + \Delta_j w, \ y(w, p)) . \end{aligned} \tag{38}$$

Now, let us newly define the quantities SE_{ij} and EE_{ij} as follows.

$$SE_{ij} \ = \ 【\ u_i(w + \Delta_j w, \ y(w, p)) - u_i(w, y(w, p))\ 】 / \ \Delta w_j \ .$$

$$EE_{ij} \ = 【\ u_i(w + \Delta_j w, y(w + \Delta_j w, p)) - u_i(w + \Delta_j w, y(w, p))\ 】 / \ \Delta w_j \ .$$

Then, it is easily seen from Eq. (6.38) that the following equation holds.

$$\Delta x_i / \ \Delta w_j \ = \ SE_{ij} \ + \ EE_{ij} \ . \tag{39}$$

In our opinion, Eq. (6.39) has very important implications. It is noted that SE_{ij} and EE_{ij} respectively show the *substitution effect* and the *expansion effect.* At first appearance, it seems to be the *finite increment version* of the famous Slutsky equation in consumer demand theory. In fact, a change Δw_j in the price of the input j affects the behavior of the firm in two different ways. While it causes a change in input price ratios which induces technical substitution among inputs *along the old isoquant*, it entails a change in the profit maximizing output *along the new scale path.*

On the one hand, the first substitution effect stands for the *variation in the optimum combination of inputs within the isoquant class to which the original* $x(w, p)$ *belongs.* Then, $c(w + \Delta_j w, y(w, p))$ represents the corresponding level of cost which *keeps the firm within the same isoquant class as before the input price change, even in the new input price situation* $w + \Delta_j w$. On the other hand, the scale effect represents the *shift of the optimum input combination in the new input price situation* $w + \Delta_j w$*, responding to the change of output level from the old level* $w + \Delta_j w$ *to the new level* $y(w + \Delta_j w, p)$.

These two distinct effects are well-illustrated in Fig. 12. There, the movement from Point A to Point B indicates the substitution effect, and the movement from Point B to Point C demonstrates the expansion effect.

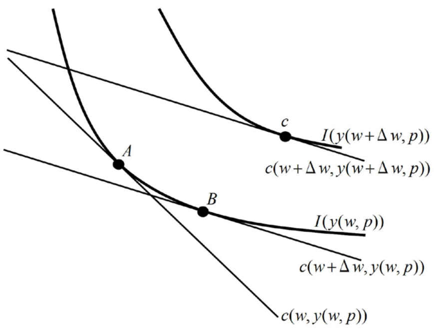

Fig. 12. The substitution and expansion effects graphically illustrated

It is recalled that Eq. (6.39) merely stands for the *difference version of the decomposition equation.* We are now ready to derive its *differential version,* which is perhaps more interesting than the difference version. To this end, it is necessary to make an additional assumption as follows.

Assumption (A6). For all $(w, p) \in W \times P$, $x(w, p)$ and $y(w, p)$ are differentiable in (w, p).

This newly added assumption (A6) is not so far from the previous assumptions (A1) − (A5). In fact, as was seen above, the latter ones already assure *almost everywhere differentiability of* $x(w, p)$ and $y(w, p)$. The only difference between the previous assumptions and the newly added assumption comes down to the difference between " *almost* everywhere " and " everywhere ." Moreover, when (A6) is assumed, it is not hard to see from Lemma 6.4 that $u(w, y)$ and $c(w, y)$ are also differentiable.

Now, we are in a position to establish and prove one of the most important theorems in this chapter.

THEOREM 7 (The first kind of decomposition equations)

For all $(w, p) \in W \times P$, we have the following equations.

(1) $\partial x_i/\partial w_j = \partial u_i/\partial w_j + (\partial u_i/\partial y)(\partial y/\partial w_j)$ for all i, j.

(2) $\partial x_i/\partial p = (\partial u_i/\partial y)(\partial y/\partial p)$ for all i.

Proof. In the light of Lemma 4 (1), the proof of (1) and (2) is easy and straightforward. Indeed, we know that the following equation holds. .

$$x(w, p) = u(w, y(w, p)).$$

If we apply the well-known rule on the differentiation of a composite function, then we can immediately obtain the desired Properties (1) and (2). Q.E.D.

Property (1) indicates the *first version* of decomposition equation in input demand theory in *differential terms .* Exactly speaking, we find that $\partial x_i/\partial w_j = \partial u_i/\partial w_j + (\partial u_i/\partial y)(\partial y/\partial w_j)$. The first term $(\partial u_i/\partial w_j)$ on the right-hand side tells us to what extent a change in w_j influences u_i. And, the second term $(\partial u_i/\partial y)(\partial y/\partial w_j)$ on the right-hand , representing the remarkable scale effect term, shows the *composite effect* containing the following two partial derivative terms :

(i) The partial derivative term $(\partial y/\partial w_j)$, representing the change in the optimum output corresponding to the change in the price of the input j,

(ii) The partial derivative term $(\partial u_i/\partial y)$, showing the change in the optimum input responding to the variation in the output above mentioned. [10]

Property (2) demonstrates that the total effect of a change in input price on the input i can be decomposed into the following two partial derivative terms :

(i) The partial derivative term $(\partial y/\partial p)$, representing the change in the optimum output as a result of the change in output price,

(ii) The partial derivative term $(\partial u_i/\partial y)$, showing that the change in the optimum input corresponding to the variation in output above mentioned.

At first appearance, the combination of the firm's substitution and expansion effects seems to be analogous that of the consumer's well-known substitution and income effects. We must say, however, that *they are not exactly the same*.

Whereas in consumption theory we have simply to consider the relations between those commodities which can be regarded as being essentially similar, in production theory we have *two different kinds of commodities to consider --- inputs and outputs.* Therefore, their mutual relations and their cross-relations take a little more disentangling. We remind readers that, to obtain a decomposition equation in consumption theory, it is necessary to imagine a compensated change in consumer's income so as to keep the same preference level as before, in spite of a change in the price of a certain commodity. In contrast to such a compensated change in income in consumption theory, in production theory we have to introduce the idea of *a compensated change in output price*, which was first introduced by Hirota & Sakai (1969) and later developed by Sakai (1973), Otani (1982), Diewert (1993), and others. More exactly speaking, when the price of a certain input price varies, we imagine such a compensated change in output price as would induce the firm to maintain the output level as before the input price change.

The similarity and difference between decomposition in consumer demand and that in input demand are well-illustrated in Fig. 13. In Panel (A) , the movement of point from A to B, and that from B to C respectively indicate the substitution effect and the income effect in consumer demand decomposition. In Panel (B), similar movements $A^* \rightarrow B^*$ and $B^* \rightarrow C^*$ respectively show the substitution effect and the expansion effect in input demand decomposition. On one hand, in Panel (A), when p_1 falls, consumer's budget line twists from $a_1 a_2$ to $c_1 c_2$. Since consumer's income stays fixed in spite of a fall in p_1, the two point a_2 and c_2 on the vertical axis must coincide. On the other hand. in Panel (B), when w_1 falls, producer's cost line shifts from $a_1{}^* a_2{}^*$ to $c_1{}^* c_2{}^*$. Note that there are no fixed budget lines in input demand. Since producer's cost has to change responding to a fall in w_1, the two points $a_2{}^*$ and $c_2{}^*$ must be apart, and indeed $c_2{}^*$ lies above $c_2{}^*$.

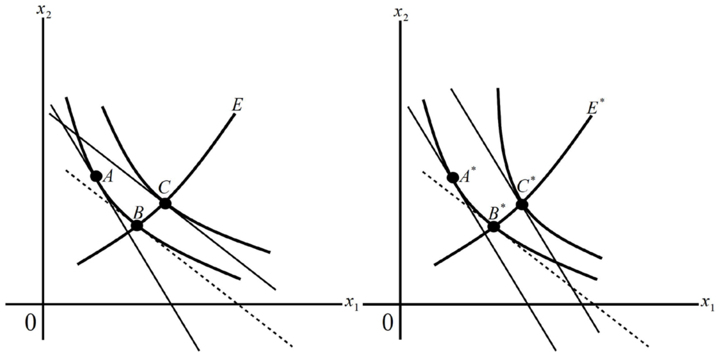

Fig. 13. Decomposition in consumer demand vs. decomposition in input demand : Namely, （A → B → C） vs. （A* → B* → C*）

By making use of the above-mentioned idea peculiar to production theory, we can now derive the *second version of decomposition equation.* Compared with the first version, this second one will turn out to be more analogous to the decomposition equation in consumption theory. For that purpose, Assumption (A6) needs to be a bit strengthened to the following assumption.

Assumption (A6'). For all $(w, p) \in W \times P$, $x(w, p)$ and $y(w, p)$ are *twice differentiable* in (w, p).

When Assumption (A6') is made, it is obvious by Lemmas 4 that $u(w, p)$ and $c(w, y)$ are also twice differentiable. We are now ready to derive the following theorem.

THEOREM 8. (The second kind of decomposition equations)

For all $(w, p) \in W \times P$, we have the following equations.

$$\partial x_i/\partial w_j = \partial u_i/\partial w_j - (\partial x_i/\partial p)(\partial p/\partial w_j) \quad \text{for all } i, j,$$

where $\partial p/\partial w_j \equiv [dp/dw_j]_{dw_i=0\,(i \neq j),\, dy=0}$.

Proof. We first note that under Assumption (A6'), $c(w, y)$ is now twice differentiable. Then, by help of Lemma 6.3 above, we find the following.

$$\partial c/\partial y = p \quad . \tag{40}$$

Total differentiation of this equation yields the following.

$$\Sigma_i (\partial^2 c/\partial w_i \partial y)\, dw_i + (\partial^2 c/\partial y^2)\, dy = dp \quad ,$$

in which we have

$$dy = \Sigma_i (\partial y/\partial w_i)\, dw_i + (\partial y/\partial p)\, dp \quad .$$

In particular, let $dw_i = 0$ *for* $i \neq j$, *and* $dy = 0$. Then, we obtain the following.

$$(\partial^2 c/\partial w_j \partial y)\, dw_j = dp \quad , \tag{41}$$

, $(\partial y/\partial w_j)\, dw_j + (\partial y/\partial p)\, dp = 0$. (42) .

If we differentiate Eq. (40) with respect to p only, we must have the following.

$$(\partial c^2/\partial y^2)(\partial y/\partial p) = 1 \quad ,$$

which must imply the following.

$$(\partial y/\partial p) \neq 0 \quad .$$

Therefore, it is clearly seen by Eq. (42) that for an arbitrary change dw_j , a *compensated change* dp , *making* $dy = 0$, is uniquely determined as the following quantity.

$$dp = - dw_j \, (\partial y/\partial w_j) / (\partial y/\partial p) \quad . \tag{43}$$

If we take account of Eqs. (43) and (41), then we can thus derive the following equation .

$$\begin{aligned} \partial p/\partial w_j &\equiv [\, dp/dw_j \,]_{dw_i = 0\ (i \neq j),\ dy = 0} \\ &= - (\partial y/\partial w_j)/(\partial y/\partial p) \quad . \\ &= \partial^2 c/\partial w_j \partial y \quad . \end{aligned} \tag{6.44}$$

In the light of Theorems 6 (3) and 7 (1), we obtain the following.

$$\begin{aligned} \partial x_i / \partial w_j &= \partial x_j / \partial w_i \quad . \\ &= \partial u_j/\partial w_i + (\partial u_j/\partial y)(\partial y/\partial w_i) \quad . \end{aligned} \tag{6.45}$$

Note that by Theorem 5 (3), we have $\partial u_j/\partial w_i = \partial u_i/\partial w_j$. Moreover, taking advantage of Lemma 1 (2), Theorem 6 (4) and Eq. (44), we can derive the following.

$$(\partial u_j/\partial y)(\partial y/\partial w_i) = - (\partial^2 c/\partial y \partial w_j)(\partial x_i/\partial p)$$
$$- (\partial x_i/\partial p)(\partial^2 c/\partial w_j \partial y) = - (\partial x_i/\partial p)(\partial p/\partial w_j) .$$

Therefore, in the light of Eq. (45), we obtain the following.

$$\partial x_i / \partial w_j = \partial u_j/\partial w_i - (\partial x_i/ \partial p)(\partial p/\partial w_j).$$

Thus, we have derived the desired result. Q.E.D.

Theorem 8 gives us the *second version of decomposition equations in input demand theory.* The scale effect here is meant to represent the composite effect of the following two terms:

(i) The change in output price so as to maintain the same output level as before, in spite of the change in the price of the input j .

.

(ii) The change in the input i corresponding to the variation in output price above mentioned. [11)]

This second version of decomposition equations are as important as the first version, presumably being even more comparable to the famous Slutsky equations in consumption theory (see Slutsky (1915)). As an immediate result of this theorem, we can finally derive the following.

THEOREM 9. (Property of the expansion effect)

For all $(w, p) \in W \times P$, we have the following property.

$$EE_{ii} \equiv -(\partial x_i/\partial p)(\partial p/ \partial w_i) \leqq 0 , \quad i = 1, \ldots, n . \qquad (46)$$

Proof. By making use of Eq. (44) , Theorem 7 (2), and Lemma 1 (2), we can obtain the following sequence of equations..

$$-(\partial x_i/\partial p)(\partial p/ \partial w_i) = . -(\partial x_i/\partial p)(\partial^2 c/\partial y\, \partial w_i)$$
$$= -(\partial u_i/\partial y)(\partial y/\partial p)(\partial u_i/\partial y) = -(\partial y/\partial p)(\partial u_i/\partial y)^2 ,$$

which should be non-positive by Theorem 6 (5) . Q.E.D.

The economic significance of this theorem is quite clear. It is recalled by Theorem 5 (4) that the following inequality holds.

$$SE_{ii} \equiv \partial u_i / \partial w_i \leqq 0 . \qquad (47),$$

In the light of Eqs. (46) and (47), we find that the substitution effect SE_{ij} and the scale effect EE_{ii} should always go in the same direction. Surely, this is also a remarkable contrast to consumption theory a la Hicks (1946), in which the substitution and income effects may go in opposite directions.

In short, input demand theory is input demand theory, thus being distinct from consumer demand theory. Although those two theories look somewhat similar, they are definitely different. We have to understand exactly how and to what extent they are analogous or distinct.

6.6 Final Remarks on the LeChatelier - Samuelson Principle

In this chapter, we have been manly concerned with the axiomatic foundations of input demand theory. While the approach taken here looks mathematical and rigorous, it has useful economic implications. In particular, it is noted that the total effect of a change in an input can be decomposed into the substitution and expansion effect. How and to what extent such decomposition in input demand is comparable to that in consumer demand is certainly a very important question to ask. To our surprise, such comparison has been rather neglected for long time in the economics literature. We do believe, however, that as the saying goes, it is better late than never.

We are ready here to do some economic applications and make final remarks. First, we note that our decomposition in input demand is closely related to the famous *LeChatelier -Samuelson principle.* Although this connection was pointed out by our friend Yoshihiko Otani (1982), we are going our own way to confirm it below.

Henri Louis LeChatelier (1850-1936) was a noted French scientist. He was best known for his work on his chemical equilibrium, which was to be called leChatelier principle in the academia. In his classical work contained in Samuelson (1947, enlarged edition 1983), Samuelson boldly applied the principle to economic equilibrium, so that naming of the *LeChatelier - Samuelson principle* has been so popular in the academic world, especially in the economic profession (see Stiglitz (1966)). [12)]

Seeing is believing ! Samuelson (1947) once remarked as follows.

We have the following general theorem:

$$(dx_i / d\alpha_i)_0 \leqq (dx_i / d\alpha_i)_1 \leqq \ldots \leqq (dx_i / d\alpha_i)_{n-1} \leqq 0 .$$

> While the change in an x with respect to its own parameter is always negative regardless of the number of constraints, it is most negative when there are no constraints, only less so when there is a single constraint, and so forth, until the number of auxiliary constraints reaches the maximum possible, namely (n − 1).
>
> (Samuelson 1947 & revised 1987, p. 38)

In our setting of input demand theory discussed so far, we have only to compare the case with no isoquant constraint and the one with a single constraint, namely the constraint that output y remains constant regardless a change in w_i . So, in terms of the input demand setting, we must have the following inequality.

$$(dx_i / dw_i)_0 \leqq (dx_i / dw_i)_1 \leqq 0 . \tag{48}$$

If we rather want to follow the decomposition equation formula (6.45) above mentioned, then in the light of Eq. (45), (46) and (47) , we must find the following set of inequalities.

$$\partial x_i / \partial w_i = \partial u_i / \partial w_i - (\partial x_i / \partial p)(\partial p / \partial w_i),$$

or equivalently

$$TE_{ii} = SE_{ii} + EE_{ii} .$$

Since the terms TE_{ii}, SE_{ii}, and EE_{ii} are all non-positive, we must have the following.

$$TE_{ii} \leqq SE_{ii} \leqq 0 . \tag{49}$$

Needless to say, this equation clearly demonstrates the validity of the famous *LeChatelier - Samuelson principle* in input demand equilibrium. For this point, also refer to Rader (1968). .

We now turn to the application of our input demand theory to some numerical examples. Hopefully, the following example will be helpful in understanding the fundamental difference between *the net effect and the gross effect* in input demand theory.

Let us consider the following production function of Cobb-Douglas type :.

$$y = f(x_1, x_2, x_3) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}, \quad \alpha_1 + \alpha_2 + \alpha_3 < 0.$$

Let X and Y be the domain and range of f, respectively. If we assume that X is bounded above, so is Y, assuring assumption (A5). It can be proved that Assumptions (A1-A4) and (A6 ') are also satisfied. It can be shown without difficulty that the matrix of the *total effect terms* is given as follows.

$$\begin{pmatrix} \partial x_1/\partial w_1 & \partial x_1/\partial w_2 & \partial x_1/\partial w_3 \\ \partial x_2/\partial w_1 & \partial x_2/\partial w_2 & \partial x_2/\partial w_3 \\ \partial x_3/\partial w_1 & \partial x_3/\partial w_2 & \partial x_3/\partial w_3 \end{pmatrix}$$

$$= 1 / [(1-\alpha_1-\alpha_2-\alpha_3)py] \times G, \tag{50}$$

where

$$G = \begin{pmatrix} -[(1-\alpha_2-\alpha_3)/\alpha_1]x_1^2 & -x_1x_2 & -x_1x_3 \\ -x_2x_1 & -[(1-\alpha_1-\alpha_3)/\alpha_2]x_2^2 & -x_2x_3 \\ -x_3x_1 & -x_3x_2 & -[(1-\alpha_1-\alpha_2)/\alpha_3]x_3^2 \end{pmatrix}$$

It is clearly seen from matrix G that any two inputs are *gross complements.*

Now, turning our attention to the matrix of *substitution effects* , we can easily derive the following.

$$\begin{pmatrix} \partial u_1/\partial w_1 & \partial u_1/\partial w_2 & \partial u_1/\partial w_3 \\ \partial u_2/\partial w_1 & \partial u_2/\partial w_2 & \partial u_2/\partial w_3 \\ \partial u_3/\partial w_1 & \partial u_3/\partial w_2 & \partial u_3/\partial w_3 \end{pmatrix}$$

$$= \quad 1/[(\alpha_1+\alpha_2+\alpha_3)py] \quad \times \quad H \quad , \tag{51}$$

where

$$H = \begin{pmatrix} -[(\alpha_2+\alpha_3)/\alpha_1]x_1^2 & x_1x_2 & x_1x_3 \\ x_2x_1 & -[(\alpha_1+\alpha_3)/\alpha_2]x_2^2 & x_2x_3 \\ x_3x_1 & x_3x_2 & -[(\alpha_1+\alpha_2)/\alpha_3]x_3^2 \end{pmatrix} .$$

It is evidently seen that any two inputs are *net substitutes* .

Finally, in the light of Eqs. (6.50) and (6.51) , we can show that the matrix of the *expansion effect terms* is provided as follow.

$$\begin{pmatrix} \partial x_1/\partial w_1 & \partial x_1/\partial w_2 & \partial x_1/\partial w_3 \\ \partial x_2/\partial w_1 & \partial x_2/\partial w_2 & \partial x_2/\partial w_3 \\ \partial x_3/\partial w_1 & \partial x_3/\partial w_2 & \partial x_3/\partial w_3 \end{pmatrix} - \begin{pmatrix} \partial u_1/\partial w_1 & \partial u_1/\partial w_2 & \partial u_1/\partial w_3 \\ \partial u_2/\partial w_1 & \partial u_2/\partial w_2 & \partial u_2/\partial w_3 \\ \partial u_3/\partial w_1 & \partial u_3/\partial w_2 & \partial u_3/\partial w_3 \end{pmatrix}$$

$$= \quad 1/[(\alpha_1+\alpha_2+\alpha_3)(1-\alpha_1-\alpha_2-\alpha_3)py] \quad \times \quad J \quad , \tag{6.52}$$

where

$$J \quad = \quad \begin{pmatrix} -x_1^2 & -x_1x_2 & -x_1x_3 \\ -x_1x_2 & -x_2^2 & -x_2x_3 \\ -x_1x_2 & -x_2x_3 & -x_3^2 \end{pmatrix}$$

Let us compare the corresponding diagonal elements of the matrix G and the matrix J . Then, we will immediately find that the substitution and expansion effects of the price of an input on the demand of the same input must go in the same direction, thus intensifying each other. The famous the LeChatelier-Samuelson principle is surely valid as expected.

The name of Chemist LeChatelier is brilliantly shining not only in the chemical world but also in the economic world as well. Life may be short, but science is long and spreading indeed !

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Footnotes

1) In the 1960s and the 1970s, I myself was a graduate student at the University of Rochester. And completing my Ph.D. thesis, I became an assistant professor at the University of Pittsburgh. In hindsight, I was so fortunate to be surrounded by many outstanding professors such as Lionel McKenzie, Edward Zabel, Ronald Jones, Hugh Rose, James Friedman, and Akira Takayama, and by many brilliant students including Jerry Green, Jose Sheinkman, Masayoshi Hirota, and Michihiro Ohyama. Though perhaps too late, I would like to say my sincere thanks to all of them.

2) Historically speaking, the fundamental difference between consumer demand theory and input demand theory was first noticed by Hicks (1946), Chapter 7, and mathematically sophisticated by Samuelson (1947), Chapter 4. It seems, however, that the *similarity* between them was not fully developed by these authors, resulting later development of input demand theory.

3) For a topological and convex approach to production theory, see Uzawa (1964), Nikaido (1968), Shephard (1970), and others. .

4) The purpose of Assumption (A5) is to force the profit function $\pi(w, p) = \{p \cdot f(x) - wx : x \in X\}$ to be defined for any $(w, p) \in W \times P$.

5) According to the definition of the length of a vector, we find the following:

$$|(\Delta w, \Delta p)| = 【\Sigma_i (\Delta w_i)^2 + (\Delta p)^2】^{1/2} .$$

6) For the properties of convex and concave functions, see Fenchel (1953) , Rockafellar (1970), Shephard (1979) and Varian (1999, 2009)..

7) Note that any concave or convex function is *almost everywhere twice differentiable.* See Alexandroff (1939).

8) See Takagi (1961), pp. 57-58.

9) This markedly contrasts with the income effect in consumption theory since the latter may be non-symmetrical.

10) The *first* version of decomposition equation was merely referred to by Basett & Borcherding (1969) without any proof, and later was given a formal proof by Hirota & Sakai (1969) and Syrquin (1970) by using simple calculus. It is noted that the most exact form of decomposition equations in input demand theory is given here in terms of the compensated input functions.

11) By relying on the traditional calculus method, Hirota & Sakai (1969) succeeded in deriving the *second* version of decomposition equations. To our regret, however, their proof was rather sketchy and based on the Hessian matrix of the production function per se. We believe that a topological approach taken here is more general, and more elegant, than the previous method. This may look a small step, but surely a giant jump indeed.

12) According to Samuelson (1972), the LeChatelier Principle was discovered more than one hundred years ago by LeChatelier, French chemist. Samuelson found, however, that it was a rather vaguely stated principle. So, he decided to make it more perfect, and boldly applied to economic theory. This is the reason why this principle is now called the LeChatellier-Samuelson principle. Also see Dietzenbacher (1992) and Alexandrov & Bedre-Defolie(2017).