DISCUSSION PAPER SERIES E

SHIGA UNIVERSITY

Discussion Paper No. E-23

Revealed Favorability and Indirect Utility : Applying the Samuelson Approach to the Price-Income Space

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April 2023

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Revealed Favorability and Indirect Utility :

Applying the Samuelson Approach to the Price-Income Space

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Abstract. This chapter studies the axiomatic foundations of the indirect utility function, based on a revealed preference approach a la Paul A. Samuelson. We look at a chain of comparisons of budgets as if it gives a relation on the normalized-price space (namely, a *revealed favorability relation*) rather than a relation on the commodity space (namely, a *revealed preference relation*). In analogy to the weak and strong axioms of revealed preference, the weak and strong axioms of revealed favorability are newly introduced, and a fundamental theorem concerning the relationship between the latter two axioms is established. Then, the indirect and direct utility functions are effectively derived on the basis of the strong axiom of revealed favorability. It is noted that neither the continuity of the demand function nor the convexity of its range is required for the approach taken here.

Keywords. revealed favorability, weak and strong axioms, indirect and direct utility functions, revealed preference, Paul A. Samuelson.

This research is a completely revised version of Sakai (1975), with many new figures added. I am grateful to Lionel W. McKenzie (Rochester), James W. Friedman (Rochester), Hiroshi Atsumi (Toronto), Koji Okuguchi (Tokyo Metropolitan) , Marcel K. Richter (Minnesota), Hugo Sonnenschein (Boston), and Werner Hildenbrand (Bonn) for valuable suggestions. I am also thankful to Masashi Tajima (Shiga) for technical assistance.

1 Revealed Preference and Revealed Favorability: Duality Relations in Consumer Demand

It is Paul A. Samuelson (1947, 1965,1967) who newly adopted a revealed preference approach to consumer choice theory, thus bravely departing from the ordinal utility approach, which had been traditional and dominant for a long time before Samuelson appeared on the economics stage. Such ordinal utility is conveniently shown by a *direct utility function of goods*, and has been well-developed by E. Slutsky (1915), R. D. D. Allen (1936), John R. Hicks (1939, 2nd ed. 1946), and others.

The purpose of this research is to carefully examine the axiomatic foundations of the *indirect utility function of normalized-prices,* which was later developed as a dual of revealed preference approach a la Paul A. Samuelson (1947, 7th ed. 1967) . Remarkably, there has been a strong revival of interest in the indirect utility function since it was first studied by Hotelling (1932) and others a long time ago.

It is now well-known that there exists the basic duality relations between the direct and indirect utility functions: namely, maximizing the direct utility of commodity is equivalent to minimizing the indirect utility of prices and income, with the identical budget constraint imposed on both instances. In conjunction with such duality relation, a number of propositions on the structure of utility functions and demand systems have been established by many economic theorists. It is of great interest, therefore, to reexamine the duality relations in the light of revealed preference theory, which have been long neglected so far.

In order to visually understand the duality relationship between the direct and indirect utility approaches, let us consider the simple world with two commodities, x_1 and x_2, and their prices, p_1 and p_2, for illustrative purpose. If we divide those prices by income, then we readily obtain normalized prices q_1 and q_2 in which $q_1 = p_1 / m$ and $q_2 = p_2 / m$. In the traditional consumer choice world, a representative consumer is supposed to maximize his (direct) utility $U(x_1, x_2)$ subject to his income budget $q_1 x_1 + q_2 x_2 \leqq 1$. It follows from the resulting consumer equilibrium that x_1 and x_2 are functions of the price pair (q_1, q_2), so that we may duly write $x_1 = x_1(q_1, q_2)$ and $x_2 = x_2(q_1, q_2)$.

If we substitute $x_1(q_1, q_2)$ and $x_2 = x_2(q_1, q_2)$ for the direct utility function $U(x_1, x_2)$, then we can rightly obtain the following indirect utility functions:

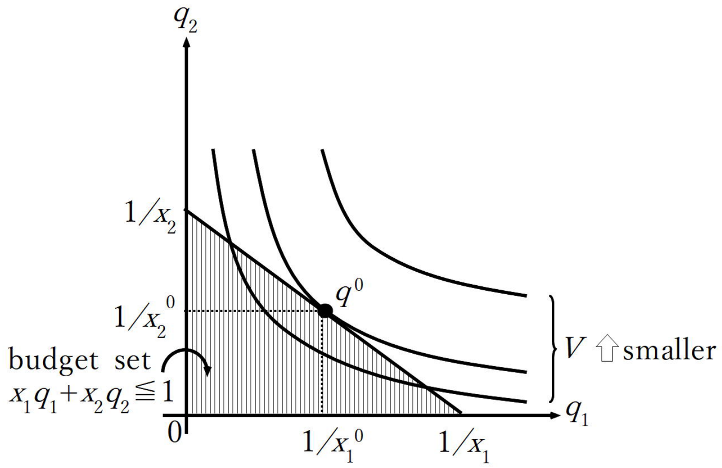
$$V(q_1, q_2) = U(x_1(q_1, q_2), x_2(q_1, q_2)). \tag{1}$$

It is reasonable for us to say that the function V is *indirect utility function* because the utility is *indirectly related to* the normalized-price pair (q_1, q_2) through the " medium " of the commodity pair (x_1, x_2). With the preparations aforementioned, we are ready to say that the consumer is supposed to minimize his indirect utility $V(q_1, q_2)$ subject to the budget set $x_1 q_1 + x_2 q_2 \leqq 1$.

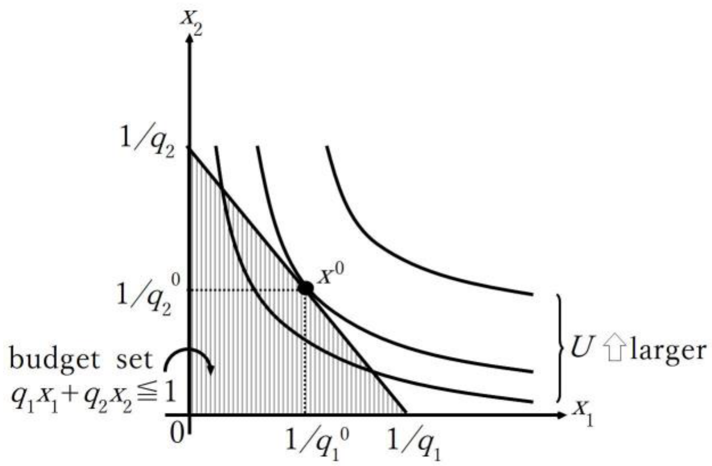
The dual and symmetrical relationship between the indirect and direct utility function approaches may vividly be visualized in Fig. 1. In the upper panel (A), the (normalized) price point q^0 *minimizes the indirect utility* $V(q_1, q_2)$ subject to the budget set $x_1 q_1 + x_2 q_2 \leqq 1$. In the lower panel (B), the commodity point x^0 *maximizes the direct utility* $U(x_1, x_2)$ subject to the budget set $q_1 x_1 + q_2 x_2 \leqq 1$.

. We are now in a position to exactly define the normalized-price analog of revealed preference relation as follows. First, let us first suppose that a commodity bundle x^1 is chosen from the budget set $b(q^1)$ associated with a normalized price vector q^1. Second, let us also assume that this bundle x^1 belongs to $b(q^0)$, namely the budget set associated with a different normalized-price vector q^0. Then, we duly say that *the new budget set* $b(q^0)$ *is directly revealed more favorable than the old budget set* $b(q^1)$. Or more simply, we cay say that *the normalized-price vector* q^0 *is directly revealed more favorable than the normalized-price vector* q^1. This is presumably because there is some commodity bundle, say x^0, in the new budget set $b(q^0)$ which is better than all commodity bundles in the old budget set $b(q^1)$. Such *revealed favorability relation* may be well-illustrated in Fig. 2 . [1)]

In the traditional revealed preference theory a la Samuelson (1947) and Houthakker (1950), there exist the famous two axioms. They are the weak and strong axioms of revealed preference. In a very similar way, we can rightly define the normalized-price analogs of those axioms. More specifically, *the weak axiom of revealed favorability* requires that the direct revealed favorability relation thus defined be asymmetric. Needless to say, this is nothing but the normalized-price counterpart of revealed preference on the commodity space. Likewise, analogous to the indirect revealed preference relation on the commodity space, the *indirect revealed favorability relation* may be defined as what we can call "transitive closure" of the direct revealed favorability relation on the price-income space. *The strong axiom of revealed favorability* requires that the indirect revealed favorability relation be asymmetric .



(A)　The normalized price space　(q_1, q_2)



(B)　The commodity space　(x_1, x_2)

Fig. 1　The dual relationship between the indirect and direct utility approaches

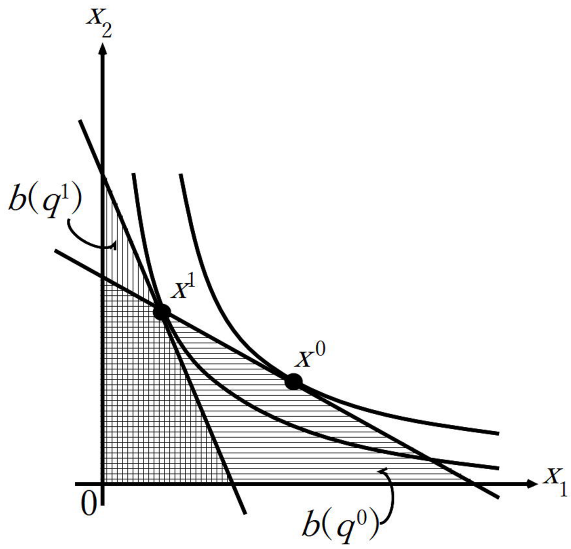


Fig. 2. Revealed favorability relation.
$b(q^0)$ is directly revealed more favorable than $b(q^1)$;
or simply, q^0 is directly revealed more favorable than q^1.

In the traditional revealed preference approach, it is always assumed that the range of the demand function has the convex property. In case the new revealed favorability approach is adopted, however, it will be seen that such a stringent assumption can be discarded; therefore, the results obtained in this paper help to clarify the role by the convexity condition on the range of the demand function in the theory of consumer's demand. It should also be noticed that, in contrast to the previous results of Kuga (1969) , Weddepohl (1970), and others, the continuity or Lipschitz condition is not imposed at all on the demand function here. [2)]

The contents of the remaining sections may be outlined in the following way. In Section 2, a set of exact definitions and detailed assumptions used throughout this paper is carefully introduced. Section 3 thoroughly discusses the question whether there exists the duality relation between revealed favorability relations on the normalized-price space and reveled preference relations on the commodities. In Section 4, we establish an important theorem concerning the relationship between the weak and strong axioms of revealed favorability. As will be seen below, the strong axiom holds if and only if the weak axiom *plus a certain "regularity condition"* both hold. Interestingly enough, this regularity condition is a relationship between two definitions of *income compensated functions* that must hold if the strong axiom holds.

Section 5 deals with the *representation problem* of revealed favorability relations. It is shown that, given the strong axiom of revealed favorability together with some other usual assumptions on the demand function, there exists a real-valued function (namely, an indirect utility function) of normalized-price vectors, with the desired properties of minimality, lower semi-continuity, strict quasi-convexity, and monotonicity. A nice bridge between the indirect and direct utility functions will also be built in this research whenever the inverse of the demand function exists. Final several remarks will be made in Section 6.

2 Definitions and Assumptions

In this section, we will introduce certain definitions and basic assumptions to be used throughout this research. In particular, revealed favorability relations on the set of price-income vectors will rigorously be defined in terms of a system of axioms.

We are concerned with the behavior of a rational consumer, who chooses a set of commodity bundles subject to market prices and incomes. The *commodity space* Y is the set of all conceivable commodity bundles. For convenience, we assume that it is the non-negative quadrant of the n-dimensional vector space R^n :

$$Y = \{ y : y = (y_1, ..., y_n) \ \& \ 0 \leqq y \in R^n \} , \tag{2}$$

where y_i denotes the quantity of commodity i for i $= 1,..., n$. It should not be necessary that every $y \in Y$ is chosen subject to some normalized-price configuration.

The *price-income space* $P \times M$ is the positive orthant of the (n+1)-dimensional vector space:

$$P \times M = \{ (p, m) : (p, m) = (p_1, ..., p_n, m) \ \& \ 0 < (p, m) \in R^{n+1} \} , \tag{3}$$

where p_i denotes the price of commodity i ($i = 1, ... , n$) and m represents the consumer's income.

Henceforth, we will also use another price concept by means of " normalization."

To this end, let us define the *normalized-price space* Q is the positive orthant of the n - dimensional vector space:

$$Q = \{ q : q = (q_1, ..., q_n) \ \& \ 0 < q \in R^n \} , \tag{4}$$

where $q_i = (1/m)/p_i$ any $i = 1, ..., n$.

As can easily be seen, the normalized-price space Q is actually the set of all conceivable combinations of " normalized-prices," in the sense that "the sum of all x_i -weighted q_i' s over i " is always unity: namely, $\Sigma_i \, x_i q_i = 1$. Because of such normalization process, the income component is safely dropped out of our discussion. It is noted that both the commodity space Y and the normalized- price space Q have the same n-dimension . [3)]

For a given price vector $q \in Q$, the *budget set* $b(q)$ is defined as follows.

$$b(q) = \{ y \in Y : qy \leqq 1 \} . \tag{5}$$

Let B be the family of all budget sets:

$$B = \cup \{ b(q) : q \in Q \} . \tag{6}$$

Now, let h be a non-empty *demand correspondence* (*function*) on B , that is, a function which to each $b(q)$ assigns a non-empty subset. Such a subset may be called a *choice set,* being written as $h(b(q))$. In what follows, we will obey the following conventions. That is to say, for $h(b(q))$ and $b(q) \in B$, we will also simply write $h(q)$ and $q \in B$, respectively. Further, in the light of (5.5), we let X be the *range of the demand correspondence* h :

$$X = b(B) = \cup \{ b(q) : q \in Q \}$$

$$= \{ x \in Y : x \in h(q) \text{ for some } q \in Q \} . \tag{7}$$

In general, we find $X \subset Y$, but X need not be identical to Y ; indeed, X may be a proper subset of Y . This point may more clearly be seen by comparing the two figures (A) and (B) in Fig. 3.

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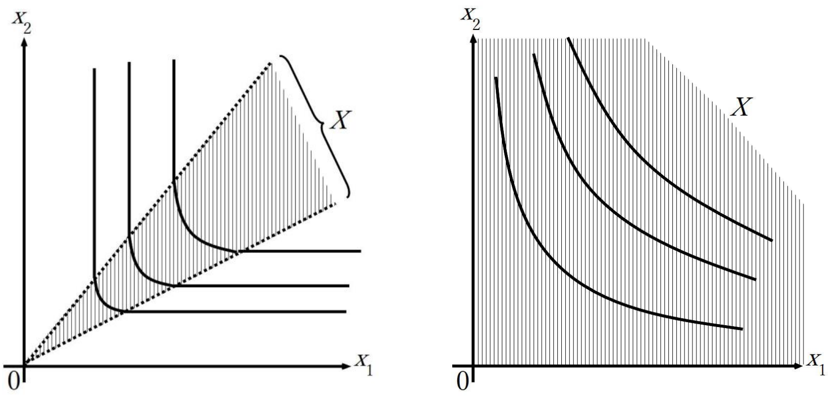


(A) $X \subset Y$ but $X \neq Y$ (B) $X = Y$

Fig. 3 The relationship between X and Y

Besides, we make the following usual assumption:

$(\boldsymbol{H})$ For all $q \in \boldsymbol{Q}$ and all $x \in h(q)$, we have $qx = 1$.

The assumption $(\boldsymbol{H})$ means that h is (positively) homogeneous of degree zero with respect to q and that the whole budget is spent.

Now, we are in a position to exactly define *revealed favorability relation* on Q in terms of h and b as follows. Suppose that the budget set $b(q^0)$ associated with a price vector q^0 contains the choice set $h(q^1)$ associated with a *distinct* price vector q^1. Then, we say that $b(q^0)$ is *directly revealed more favorable* than $b(q^1)$. For this relation, we will simply write $b(q^0)\,F^1\,b(q^1)$, since there is presumably some commodity bundle in $b(q^0)$ which is better than all commodity bundles in $b(q^1)$. We can formally write such relationship as follows.

$$b(q^0)F^1 b(q^1)$$

$$\Leftrightarrow \quad b(q^0) \supset h(q^1) \quad \text{and} \quad q^0 \neq q^1 ,$$

$$\Leftrightarrow \quad q^0 x^0 = 1 \geqq q^0 x^1 \quad \text{for } x^0 \in h(q^0), x^1 \in h(q^1), \text{and } q^0 \neq q^1 . \qquad (8)$$

. In order to avoid the lengthy expression $b(q^0)F^1 b(q^1)$, it is more convenient for us to simply write $q^0 F^1 q^1$. We believe that Fig. 4 is helpful for understanding the true meaning of Eq. (8). It is noted that although the essence of revealed favorability relation $q^0 F^1 q^1$ was already pointed out in the last Fig. 2 , it was quite unfortunate that it was shown on the *commodity space* rather than the *normalized-price* space. To correct such a mismatch, we can draw the new Fig. 4, in which the revealed favorability relation $q^0 F^1 q^1$ is now rightfully on the corresponding normalized-price space. For this point, see Kuga (1969), Weddepohl (1970, and Sakai (1977).

Now, suppose that for a finite sequence $r^1, \ldots, r^k \in Q$, we find the following sequence of relations:

$$q^0 F^1 r^1 F^1 \ldots F^1 r^k F^1 q^1 \qquad (9)$$

Then, we can say that q^0 is *revealed more favorable in* k *steps* than q^1, and we write $q^0 F^k q^1$. Moreover, if there exists a certain finite integer k for which the relation $q^0 F^k q^1$ holds, then we should say that q^0 is *indirectly revealed more favorable* than q^1, and we simply write $q^0 F\ q^1$. In other words, F is the "transitive closure" of F^1 , or the smallest transitive relation including F^1 on Q. It is easy to see that $q^0 \leq q^1$ implies $q^0 F^1\ q^1$, and hence $q^0 F\ q^1$.

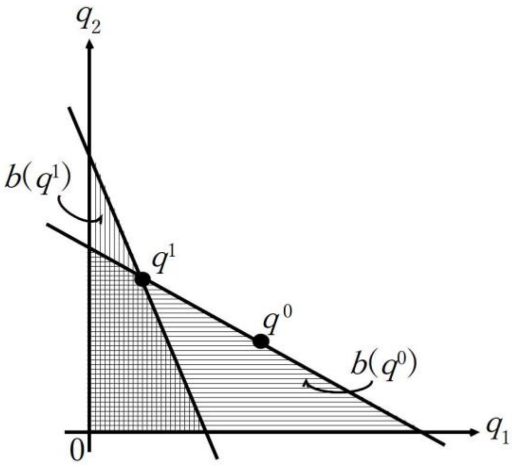


Fig. 4 Revealed favorability relation on the *normalized-price space*:

$$q^0 F^1 q^1 \Leftrightarrow b(q^0) F^1 b(q^1) \Leftrightarrow b(q^0) \supset h(q^1) \;\; \& \;\; q^0 \neq q^1 \; .$$

Corresponding to these two relations F^1 and F on Q, let us introduce the two axioms of revealed favorability, namely, the *weak axiom* **(*WF*)** *and the strong axiom* **(*SF*)** in the following fashion:

(WF) For $q^0, q^1 \in Q$, $q^0 F^1 q^1$ implies $\sim q^1 F^1 q^0$,

(SF) For $q^0, q^1 \in Q$, $q^0 F \; q^1$ implies $\sim q^1 F \; q^0$,

where, in general, the symbol "$\sim ABC$" stands for the " negation of ABC." These two axioms asserts that the consumer's behavior should be "directly" or "indirectly" consistent; therefore, they are quite analogous to the traditional revealed preference axioms of Samuelson (1947) and Houthakker (1950) on Y, the commodity space. The duality relationship between revealed favorability relations on Q and revealed preference relations on Y will be investigated in more details in the next section.

3 Revealed Favorability versus Revealed Preference

In this section, we will examine whether there exists the duality relationship between revealed favorability on the price space and revealed preference on the commodity space. We are also concerned with the question of how this duality relationship corresponds to the more familiar one between the indirect and direct utility functions.

As mentioned above, revealed preference relations on Y are defined in terms of h and b as follows. If there is $q \in Q$ such that $x \in h(q)$, $y \in b(q)$, and $x \neq y$, then we say that x is *directly revealed preferred* to y, and we write $x S y$. If there is some finite sequence $u^1, \ldots, u^k$ of elements of Y such that $x S u^1 S \ldots S u^k S y$, then we say that x is *indirectly revealed preferred* to y, and we write $x H y$.

On the one hand, the *weak axiom* **(*WP*)** *of revealed preference* requires that the direct revealed preference relation a la Samuelson S be non-symmetric:

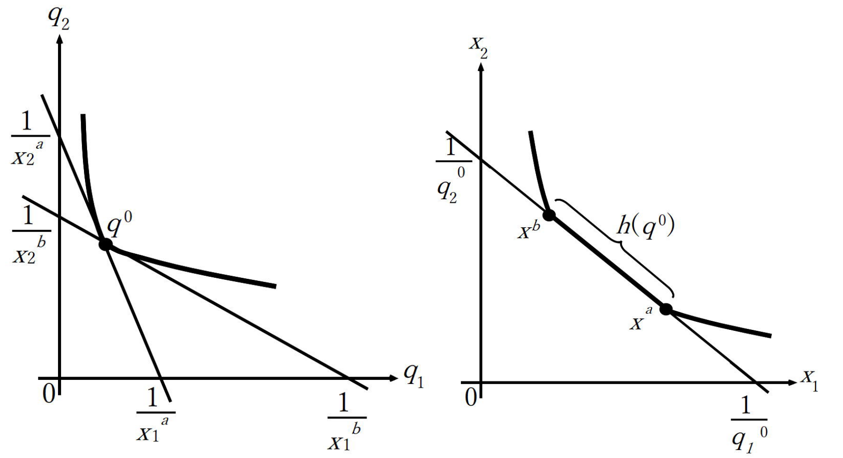
(WP) For x, y $\in Y$, $x S y$ implies $\sim y S x$.

On the other hand, the *strong axiom* ***(SP)*** *of revealed preference* requires that the indirect revealed preference relation H *be* non-symmetric:

(SP) For x, y $\in Y$, $x H y$ implies $\sim y H x$.

It is easy to show that if h satisfies ***(WF)*** on Q, then for any $x \in X$, the inverse image of x by h contains a single element; therefore, h is uniquely invertible. In contrast, it is also easily seen that if h satisfies ***(WP)*** on X, then for any $q \in Q$, the image of q by h is a singleton; therefore, h is single-valued. Hence, ***(WF)*** does not imply ***(WP)***, and vice versa. It is equally clear that ***(SF)*** does not imply ***(WF)***, and vice versa. This point will more sharply be understood by means of graphical illustrations.

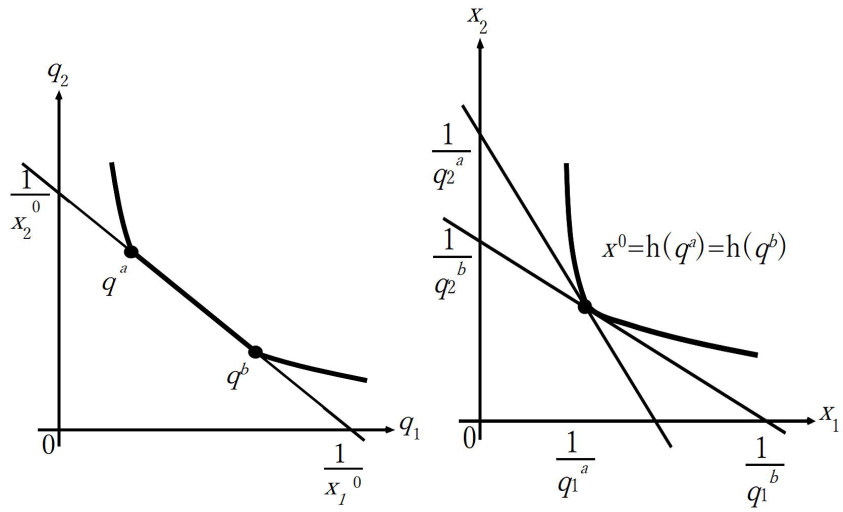
On the one hand, let us consider simple examples of indifference curves on Q with "pointed" portions, or equivalently, those of indifference curves on X with "flat" portions. As is seen in Fig. 5 (A) & (B), they indicate the consumer satisfying ***(WF)***, but not ***(WP)***. On the other hand, as is clear in Fig. 6 (A) & (B), symmetrical examples of indifference curves on Q with "flat" portions (and hence those of indifference curves on X with "pointed" portions) indicate the consumer satisfying, ***(WP)*** but not ***(WF)***. If h happens to be a one-to-one correspondence between Q and X, however, then it is readily seen that ***(WF)*** holds on Q if and only if ***(WP)*** holds on X, and also that (SP) holds on Q if and only if ***(SP)*** holds on X. [4)]



(A) Indifference curves have " pointed " sections on X.

(B) Indifference curves have " flat " sections on Q.

Fig. 5 The consumer satisfies *(WF)*, but not *(WP).*



(A) Indifference curves have " flat " sections on Q.

(B) Indifference curves have " pointed " sections on X.

Fig. 6 The consumer satisfies *(WP)*, but not *(WF).*

4 Relationship between the Weak and Strong Axioms of Revealed Favorability

In this section, we will study the relationship between the weak and strong axioms of revealed favorability. Let us suppose that the demand function h satisfies the homogeneity assumption ***(H)***. Then, it will be seen that the strong axiom ***(SF)*** of revealed favorability holds if and only if the weak axiom ***(WF)*** of revealed favorability together with a certain " regularity condition " holds. As will be seen, such a regularity condition is a relationship between two kinds of " income compensated functions " that must hold if the strong axiom holds.

To this end, we first establish the following important lemma.

LEMMA 5.1. (the closeness of the set $\{ q \in Q : \sim q F q^0 \}$).
Suppose that the demand function h satisfies ***(H)*** and ***(WF)***. Then, for any $q \in Q$, the set $\{ q \in Q : \sim q F q^0 \}$ is closed in Q.

Proof: [5] It suffices to show that the set $\{ q \in Q : q F q^0 \}$ is open in Q. Let us suppose $q^1 F q^0$. Then by the definition of F, there exists a certain sequence $q^1, q^2, \ldots, q^0$ such that $q^1 F^1 q^2 F^1 \ldots F^1 q^0$. In the light of Eq. (5.8), this surely implies that there exists a price vector $q^2 \in Q$ such that

$$q^1 x^1 = 1 \geqq q^1 x^2 \quad \text{for } x^1 \in h(q^1), x^2 \in h(q^2), \text{ and } q^1 \neq q^2 . \tag{10}$$

and

$$q^2 F q^0 \quad \text{or} \quad q^2 = q^0 . \tag{11}$$

Now, as is seen in Fig. 7, let us define $q^t = (1-t)\, q^1 + t \cdot q^2$, $t \in (0, 1)$, and $x^t \in h(q^t)$. Then, we must have the following:

$$q^t x^2 = (1-t)\, q^1 x^2 + t \cdot q^2 x^2 = (1-t)\, q^1 x^2 + t . \tag{12}$$

We easily note the following equation.

$$q^t x^t = 1 \geqq q^t x^2 , \quad q^t \neq q^2 . \tag{13}$$

In the light of Eq. (12) and the weak axiom ***(WF)***, we must obtain the following.

$$q^2 x^t > 1 = q^2 x^2 \quad . \qquad\qquad . \tag{14}$$

Now, we recall that by the choice of q^t , we must find the following equation.

$$(1-t)\,(q^1 x^t - 1) + t\,(q^2 x^t - 1)$$

$$= [\,(1-t)\,q^1 + t \cdot q^2\,]\; x^t + [\,(1-t)\,(-1) + t\,(-1)\,]$$

$$= q^t x^t - 1 = 0 \; . \tag{15}$$

If we compare Eqs. (12) and (13), we clearly obtain $q^1 x^t < 1$. We can choose a neighborhood $U(q^1)$ of q^1 so that the following equation holds.

$$q^1 x^t < 1 = q x \;, \quad x \in h(q) \quad \text{for any} \quad q \in U(q^1) \,. \tag{16}$$

Eqs. (14) , (11), and (10) yield the following relation.

$$q^2 F q^0 \qquad \text{for any} \quad q \in U(q^1) \;,$$

This ensures that the set $\{ q \in Q : q F q^0 \}$ is open in Q . *Q.E.D.*

As our experience often teaches us, a graphical illustration may be a great help for our complicated mathematical proof. So, we believe that Fig. 7 may well-illustrate the essence of the proof of Lemma 1 .

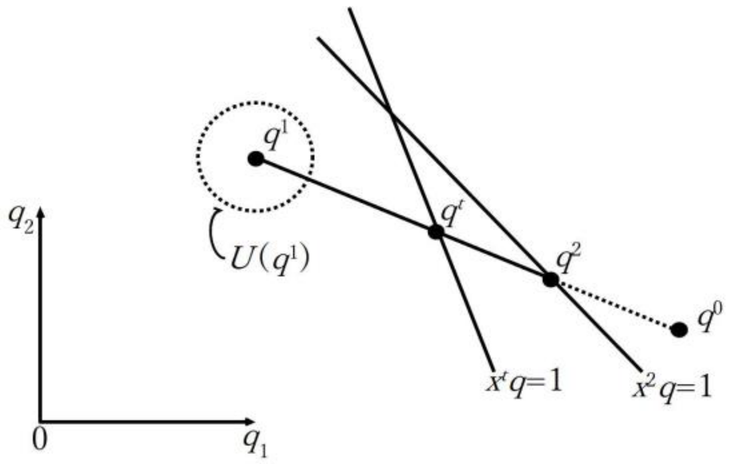


Fig. 7 The proof of LEMMA 5.1 is well-illustrated here.

Now, for a given $p \in \boldsymbol{Q}$ and a given $q^0 \in \boldsymbol{Q}$, let us define the " M-plus and M- minus sets " of incomes as follows :

$$\boldsymbol{M+}(\boldsymbol{p}, \boldsymbol{q}^0) = \{ \; m \in M \; : \; \sim \; (1/\mathrm{m})\, pFq^0 \; \} \quad ; \tag{17}$$

$$\boldsymbol{M-}(\boldsymbol{p}, \boldsymbol{q}^0) = \{ \; m \in M \; : \; \sim \; q^0F \; (1/\mathrm{m})\, p \; \} \quad . \tag{18}$$

While the " M–plus set " $\boldsymbol{M+}(\boldsymbol{p}, \boldsymbol{q}^0)$ represents the set of incomes m such that $(1/\mathrm{m})\, p$ is *not* indirectly revealed more favorable than q^0, the "M–minus set " $\boldsymbol{M-}(\boldsymbol{p}, \boldsymbol{q}^0)$ represents the set of incomes m such that q^0 is *not* indirectly revealed more favorable than $(1/\mathrm{m})\, p$. Then, the " m - plus and m -minus income compensation functions " , $\boldsymbol{m+}$ and $\boldsymbol{m-}$, can respectively be defined in terms of $\boldsymbol{M+}$ and $\boldsymbol{M-}$ in the following way : 6)

$$\boldsymbol{m+}(\boldsymbol{p}, \boldsymbol{q}^0) = \boldsymbol{sup} \; \{ \; m \; : \; m \in \boldsymbol{M+}(\boldsymbol{p}, \boldsymbol{q}^0) \; \} \quad ; \tag{19}$$

$$\boldsymbol{m-}(\boldsymbol{p}, \boldsymbol{q}^0) = \boldsymbol{inf} \; \{ \; m \; : \; m \in \boldsymbol{M-}(\boldsymbol{p}, \boldsymbol{q}^0) \; \} \quad . \tag{20}$$

We can give economic interpretations to those two functions $\boldsymbol{m}+$ and $\boldsymbol{m}-$ in the following fashion. Let p and q^0 respectively be a given "absolute" price vector and a given "relative" price vector q^0 . Then, whereas the function $\boldsymbol{m}+(\boldsymbol{p}, \boldsymbol{q}^0)$ stands for the *supremum* (or the least upper bound) of incomes m such that (1/m) p is *not* indiretly revealed more favorable than q^0, the function $\boldsymbol{m}-(\boldsymbol{p}, \boldsymbol{q}^0)$ stands for the *infimum* (or the greatest lower bound) of incomes m such that q^0 is *not* indirectly revealed more favorable than (1/m) p . It is clear by definition that $\boldsymbol{M}+(\boldsymbol{p}, \boldsymbol{q}^0) \supset (0, \boldsymbol{m}+(\boldsymbol{p}, \boldsymbol{q}^0))$ and $\boldsymbol{M}-(\boldsymbol{p}, \boldsymbol{q}^0) \supset (\boldsymbol{m}-(\boldsymbol{p}, \boldsymbol{q}^0), +\infty)$.

While the " m-plus income compensation function " $\boldsymbol{m}+(\boldsymbol{p}, \boldsymbol{q}^0)$ corresponding to the " M-plus set " $\boldsymbol{M}+(\boldsymbol{p}, \boldsymbol{q}^0)$ is illustrated in Fig. 8, the " m-minus income compensation function" $\boldsymbol{m}-(\boldsymbol{p}, \boldsymbol{q}^0)$ corresponding to the " M-minus set " $\boldsymbol{M}-(\boldsymbol{p}, \boldsymbol{q}^0)$ **is** shown in Fig. 9. .

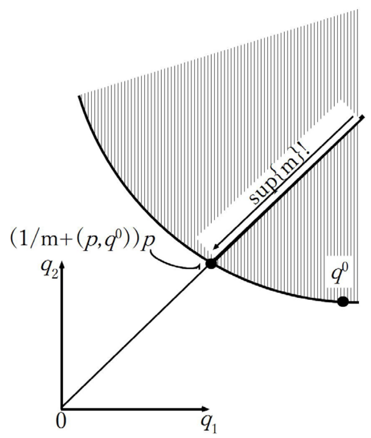


Fig. 8 The " m-plus income compensation function" : $\boldsymbol{m}+(\boldsymbol{p}, \boldsymbol{q}^0)$.

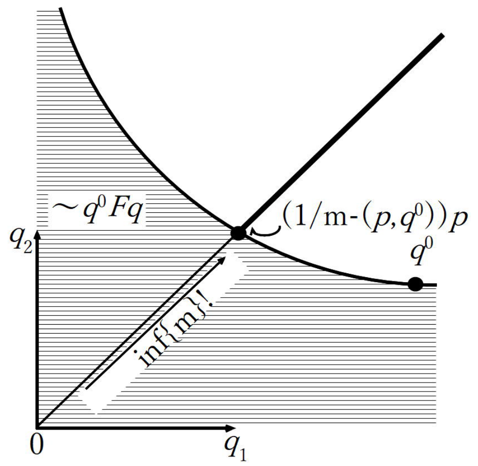


Fig. 9 The " m-minus income compensation function" : m - $(p\,,\,q^0)$.

Then, we are ready to establish the following useful lemma.

LEMMA 2 (the closeness of the M-plus set $M+(p\,,\,q^0)$).
Let us suppose that the demand function h satisfies the homogeneity assumption ***(H)*** and the weak axiom ***(WF)***. Then, for any $(p\,,\,q^0) \in P \times Q$, the M-plus set $M+(p\,,\,q^0)$ is closed in M.

Proof. By definition, it is evident that $M+(p\,,\,q^0) = M - \{\, m \in M$: (1/m) $p\,F\,q^0$ }. So, to prove that the set $M+(p\,,\,q^0)$ is closed in M , it suffices to show that its complementary set { $m \in M$: (1/m) $p\,F\,q^0$ } is open in M.

Suppose $(1/m)\,p\,F\,q^0$. In the light of Lemma 1 , the set $\{\, q \in Q$: $q\,F\,q^0$ } is open in Q . Therefore, there exists a neighborhood $U((1/m)\,p)$ of $(1/m)\,p \in Q$ so that $q^1\,F\,q^0$ for any $q^1 \in U((1/m)\,p)$.

Choose a sufficiently small $e > 0$ such that $(1/(m+e))$, $(1/(m-e)) \in U((1/m)\,p)$. Then, for any $m' \in (m-e\,,\,m+e)$, we obtain $(1/\,m')\,F\,q^0$. This indicates that the set { $m \in M$: (1/m) $p\,F\,q^0$ } is open in M. Q.E.D.

According to Lemma 2, we find $M+(p\,,\,q^0) = (\,0,\ m+(p\,,\,q^0)\,]$ whenever h satisfies $(\,H\,)$ and $(\,WF\,)$. We are now approaching to the final goal of the equivalence theorem. To reach it safely, we need to pass through the following midpoint.

LEMMA 3 (the existence of $m+(p, q^0)$ and $m-(p, q^0)$)
Suppose that the demand function h satisfies the homogeneity assumption ***(H)*** and the strong axiom *(SF)*. Then, for any $(p, q^0) \in P \times Q$, the income compensated functions $m+(p, q^0)$ and $m-(p, q^0)$ exist and are finite.

Proof: To see that $m+$ exists x^0 and is finite, it suffices to show that the set $M+(p, q^0)$ is non-empty and bounded above.

Let $(p, q^0) \in P \times Q$. Then, as is seen in Fig. 10, we can surely choose a sufficiently *small* $m^1 \in M$ such that $(1/m^1)p > q^0$. Let $x^0 \in h(q^0)$ and $x^1 \in h((1/m^1)p)$. Then, we find

$$q^0 x^0 = 1 = (1/m^1)p \ x^1 > q^0 \ x^1 , \tag{21}$$

.

so that we must obtain $q^0 F^1 (1/m^1)p$, implying $q^0 F \ (1/m^1)p^1$. By virtue of the strong axiom ***(SF)***, this gives $\sim (1/m^1)p^1 F \ p^0$, so that $m^1 \in M+(p, q^0)$. Therefore, $M+(p, q^0)$ is nonempty.

Now, as is seen in Fig. 11, we can choose a sufficiently *large* m^2 such that $m^2 \in M$ such that $(1/m^2)p < q^0$. Letting $x^0 \in h(q^0)$, we have

$$1 = q^0 x^0 > (1/m^2)p \ x^0 \ ., \tag{22}$$

whence $(1/m^2)p \ F \ q^0$. This shows that m^2 is an upper bound to $M+(p, q^0)$.

The proof that $m-(p, q^0)$ exist and is finite proceeds in a similar way. Q.E.D.

Now, we are ready to formulate a *regularity condition* in terms of income compensation functions $m+(p, q^0)$ and $m-(p, q^0)$:

(***R***) For any $(p, q^0) \in P \times Q$, $m+(p, q^0) \geqq m-(p, q^0)$.

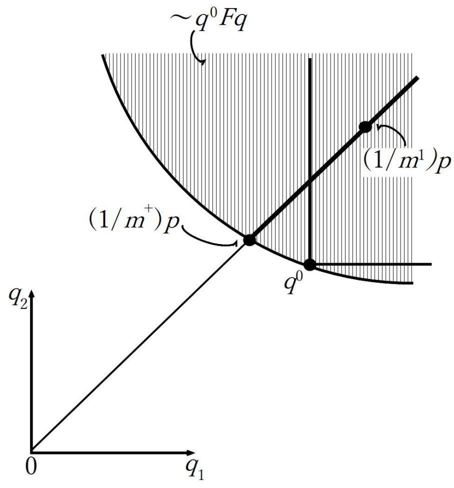


Fig. 10 We can choose a *sufficiently small* $m^1 \in M$ such that $(1/m^1)p > q^0$.

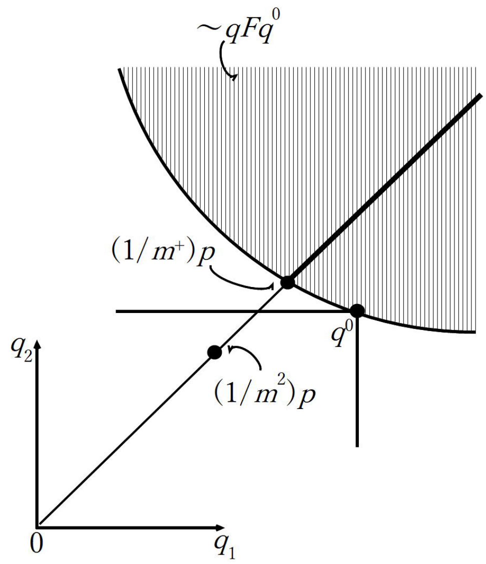


Fig. 11 We can choose a *sufficiently large* $m^2 \in M$ such that $(1/m^2)p < q^0$.

As will be seen, the regularity condition ($\boldsymbol{R}$) plays a critical role in making a bridge between the weak and strong axioms of revealed favorability, ***(WF)*** and ***(SF).*** In plain English, it states that for any given price vector (p, q^0), the supremum of incomes m such that $(1/m)\,p$ is "no better than " q^0 is greater than or equal to the infimum of incomes m such that q^0 is " no better than " $(1/m)\,p$. Saying over again, seeing is really believing ! Fig. 12 represents the case in which ***(R)*** is satisfied, while Fig. 13 indicates the case in which ***(R)*** is *not* satisfied. . As is quite clear from comparison of these two figures, condition ***(R)*** requires that that there be no " $\sim F$- gaps " in the ray $\{(1/m)\,p : m \in M\}$ for any given price vector p, yet allowing for the existence of " $\sim F$- overlaps " in the relevant ray. It is noted that in these two figures, the upper-shaded area denotes the set $\{q \in \boldsymbol{Q} : \sim q\,F\,q^0\}$ and the lower-shaded area the lower-shaded area represents the set $\{q \in \boldsymbol{Q} : \sim q^0\,F\,q\}$.

Now, it is high time for us to establish a very important theorem concerning the relation the weak and strong axioms of revealed favorability.

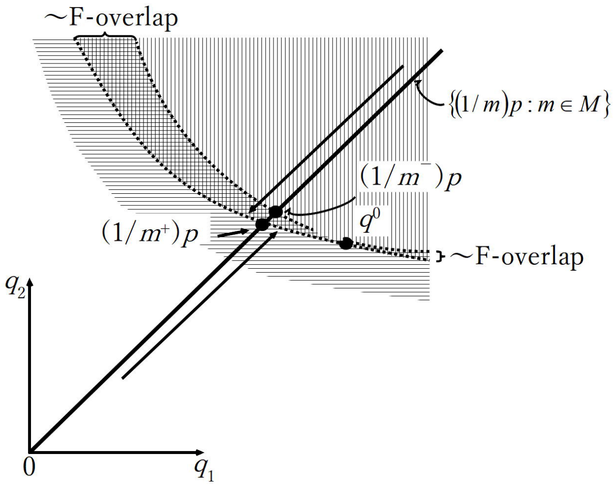


Fig. 12 The regularity condition *(R)* is satisfied : $m^+(p, q^0) \geqq m^-(p, q^0)$.

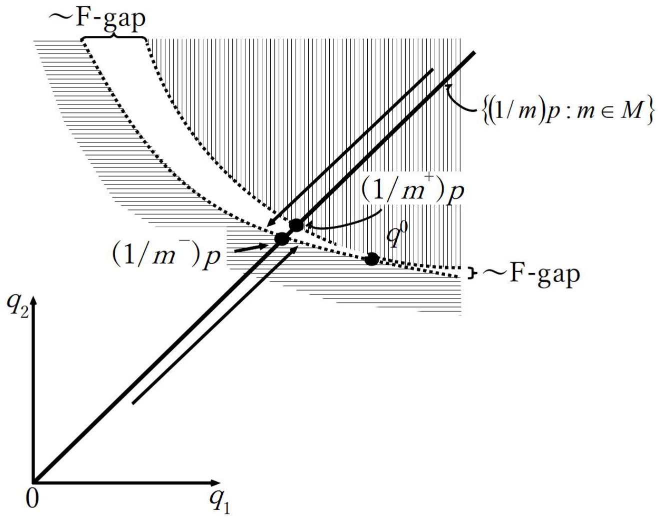


Fig. 13 The regularity condition *(R)* is *not* satisfied: $m^{+}(p, q^{0}) < m^{-}(p, q^{0})$.

THEOREM 4 (the main equivalence theorem)
Suppose that the demand function h satisfies ***(H)***. Then, the strong axiom ***(SF)*** of revealed favorability holds if and only if both the weak axiom ***(WF)*** of revealed favorability and the regularity condition ***(R)*** hold.

Proof . (a) (Necessity) Suppose that ***(SF)*** holds. Then, ***(WF)*** is obviously implied. Besides, by virtue of LEMMA 3, the two income compensation functions $m^{+}(p, q^{0})$ and $m^{-}(p, q^{0})$ surely exist and are finite.

Now, assume by way of contradiction that the regularity condition ***(R)*** does NOT hold, so we should find $m^{+}(p, q^{0}) < m^{-}(p, q^{0})$ for some $(p, q^{0}) \in P \times Q$. Choose a " middle point " $m \in M$ so that the following inequalities hold:

$$m^{+}(p, q^{0}) < m < m^{-}(p, q^{0}) . \tag{23}$$

By the definitions of $m^{+}(p, q^{0})$ and $m^{-}(p, q^{0})$, Eqs. (5.23) implies that $(1/m)\, p\, F\, q^{0}$ and $q^{0}\, F\ (1/m)\, p$. This clearly contradicts ***(SF)*.** So, to get rid of a contradiction, we must conclude that the regularity condition ***(RF)*** must hold.

(b) (Sufficiency) Suppose that ***(WF)*** and ***(R)*** both hold. Let q^{1}, q^{0}

$\in Q$ be such that $q^1 F q^0$ Then, we will show the following relation . .

$$m^{+} \ (q^1, q^0) \ < \ 1 \ . \tag{24}$$

. To this end, let us dare to assume otherwise: namely, $m^{+} \ (q^1, q^0) \geqq 1$. Since *(WF)* yields $M^{+}(q^1, q^0) = (0, \ m^{+}(q^1, q^0) \,]$ by means of Lemma 5.2, it would follow that $1 \in M^{+}(q^1, q^0)$, meaning that $\sim (1/1) \ q^1 \ F q^0$, or simply $\sim q^1 F q^0$. Clearly, this contradict our initial assumption. To get rid of a contradiction, we must conclude that Eq. (24) must hold.

Now, let us recall the regularity condition *(R)* which says that $m^{+}(p, q^0) \geqq m^{-}(p, q^0)$. If we combine this inequality and Eq. (5.23), then we immediately find the following relation.

$$m^{-} \ (q^1, q^0) \ < \ 1 \ . \tag{25}$$

Let us recall that $M^{-}(q^1, q^0) \supset (\ m^{-}(q^1, q^0), \ +\infty \ )$. Then, this clearly implies $1 \in M^{-}(q^1, q^0)$, meaning that $\sim q^0 F \ (1/1) \, q^1$, or simply $\sim q^0 F \ q^1$. For the outline of proof here, see Fig. 14.

To conclude, we have thus seen that under *(WF)* and *(R)*, the revealed favorability relation F is non-symmetric, meaning that $q^1 F q^0$ implies $\sim q^0 F \ q^1$. The proof is now complete. Q.E.D.

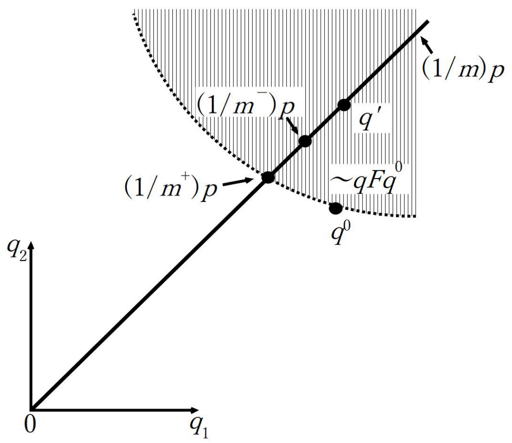


Fig. 14 The proof outline of Theorem 5.14 (the sufficiency part) :
$q^1 F q^0$ implies $\sim q^0 F \ q^1$, ensuring *(SF)* .

As far as we know, the relation between the weak and strong axioms of *revealed favorability* on the normalized-price space has hardly been investigated in the economics literature. Needless to say, this is exactly the dual to the more popular relation between the weak and strong axioms of *revealed preference* on the commodity space, which has been so intensively explored in the literature. Such non-symmetric treatment of the two approaches —— one almost neglected and another extensively explored —— seems to be very strange from common sense [7].

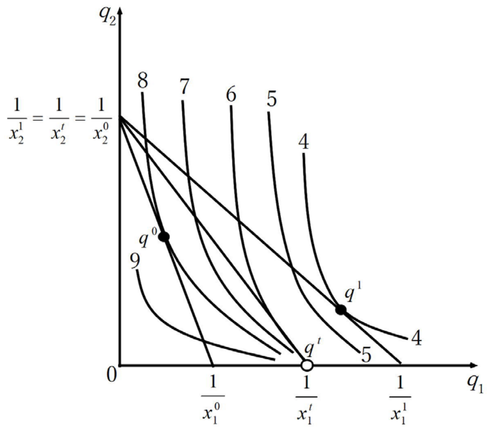
We should point it out that the hypothesis of Theorem 5.4 is not so strong, and indeed amazingly rather weak. This is because (i) *no* continuity condition is imposed on the demand function h, and (ii) X, the range of h, need NOT be convex in Y, the whole commodity space.

In particular, the above conditions (i) and (ii) are *not* implied by the budget assumption ***(H)*** and the strong axiom ***(SF)***. This important point can be demonstrated by the two examples, illustrated in Fig. 15 and Fig. 16.

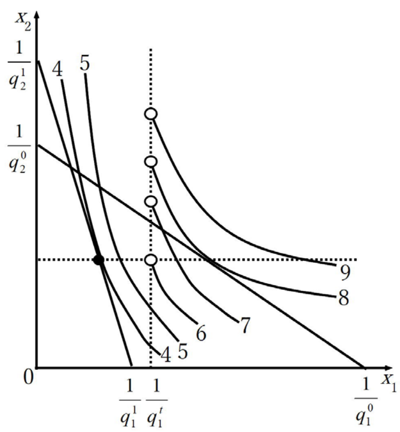
In Fig. 15, let h be generated by *indirect* preferences indicated by the upper diagram (A) , or equivalently, by *direct* preferences indicated by the lower diagram (B) . Then, as can easily seen, h satisfies ***(SF)*** and is continuous. However, the range of h is *not* convex; indeed, we find the middle point $x^t = (1\text{-}t)\,x^0 + t\,x^1$ does *not* belong the set X.

In Fig. 16, similar yet different diagrams are drawn in (A) and (B). Here again, h satisfies ***(SF)***, but it is *not* continuous and its range is *not* convex; indeed, we find that the middle point $x^t = (1\text{-}t)\,x^0 + t\,x^1$ does *not* belong the set X. [8]

We must keep in mind that situations like Fig. 15 and Fig. 16 are less than extraordinary than they seem, but might happen in many ordinary circumstances.



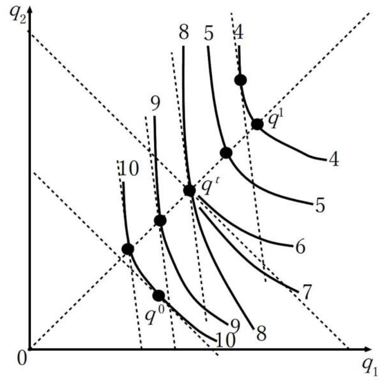
(A) The indirect preferences in the (q^1, q^2) space.



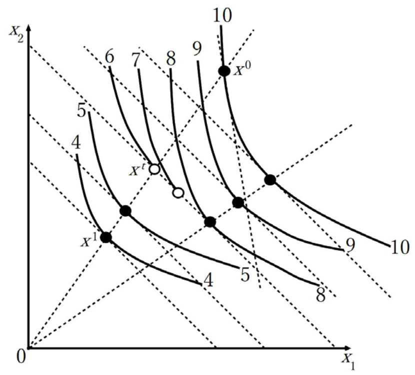
(B) The direct preferences in the (x^1, x^2) space.

Fig. 15 h satisfies *(SF)* and is continuous. However, its range is *not* convex;

$x^t = (1 - t)\, x^0 + t\, x^1$ does *not* belong to X.



(A) The indirect preferences in the (q^1, q^2) space.



(B) The direct preferences in the (x^1, x^2) space.

Fig. 16 *h* satisfies *(SF)*, but it is NOT continuous. The range of *h* is *not* convex either; in fact, $\boldsymbol{x}^t = (1 - t)\,\boldsymbol{x}^0 + t\,\boldsymbol{x}^1$ does *not* belong to X .

.

5 Derivation of the Indirect and Direct Utility Functions

The duality relation between the *direct* and *indirect* utility functions has especially been studied in connection with special functional forms of *separability.* Although such a separability approach is of some mathematical interest, its deeper and wider significance in economic theory remains to be still debatable. [9)]

In this section, we would like to investigate more fundamental questions than the separability question aforementioned. First, we will discuss the problem whether the indirect revealed favorability relation on the normalized space can be represented by a real-valued function, i.e., the *indirect* utility function. Second, we will explore the related problem whether there is a way of making a bridge between the *indirect* and *direct* utility functions from the perspective of the revealed favorability approach taken in the present research.

In this connection, we will state and prove the following result of very rich substance.

THEOREM 5.5 (the derivation of the *indirect* utility)
Suppose that the demand function h satisfies the budget assumption ***(H)*** and the strong axiom of revealed favorability ***(SF)*.** Then, there exists a real-valued function v on Q, namely the *indirect utility function*, such that the following series of properties hold.

(i) (minimality) For any $q \in Q$,
$h(q) = \{ x \in X : qx \leqq 1$, and
$v(r) \geqq v(q)$ for any $r \in Q$ such that $rx \leqq 1 \}$.

(ii) (closeness) For any $q \in Q$, the set $\{ r \in Q : v(q) \geqq v(r) \}$ is closed in Q.

(iii) (strict concavity) If $v(q^1) \geqq v(q^0)$, $q^1, q^0 \in Q$, $q^1 \neq q^0$, and $q^t = (1-t)\, q^1 + t\, q^0$, then $v(q^1) > v(q^t)$.,

(iv) (monotonicity) If $q^1 \leq q^0$ and $q^1, q^0 \in Q$, then $v(q^1) > v(q^0)$.,

(v) (heritability) For any $q^1, q^0 \in Q$, $q^1 F q^0$ implies $v(q^1) > v(q^0)$ ..,

Proof . The proof of this theorem is rather long and occasionally mathematically advanced. So, we carefully carry it out in a step-by-step fashion. [10)]

(Step 1) Homomorphism :

Since F is a non-symmetric and transitive relation on Q by means of the strong axiom ***(SF)*** of revealed favorability, and definition itself, it should be non-reflexive and transitive on Q.

Let C denote the set of all elements of Q having all coordinates *rational .* Then, we will show that this C is a countable partially F-dense subset of Q. To this end, let us assume that $q^1, q^0 \in Q - C$ satisfy $q^1 F q^0$. Then, by virtue of

Lemma 5.1, we can choose a neighborhood $U(q^1)$ of q^1 so that $s\,F\,q^0$ for all $s \in U(q^0)$. Take a $c \in U(q^1) \cap C$ for which $c > q^1$. We then have $q^1\,F\,c\,F\,q^0$; therefore C is F-dense in Q. Now, we pay special attention to the following important theorem： .

Theorem (Richter 1971)

Let ɜ be a non-reflective and transitive relation on a set Q. If there is a countable partially ɜ - dense subset of Q , then there exists a weak homomorphism from the partially-ordered space (Q, ɜ) into the real straight line ($(-\infty, +\infty)$, $>$).

The proof of this theorem can be seen in Richter (1971), p. 49, and omitted here. Fig. 17 may give us a rough idea of homomorphism from the partially-ordered space (Q, ɜ) into the real straight line ($(-\infty, +\infty)$,. $>$) If we apply the Richter theorem to the present case, then we can see that there exists a non-negative-valued and bounded function f, $f : Q \rightarrow (-\infty, +\infty)$, satisfying the following relation:

$$\text{For any } q^1, q^0 \in Q, \quad q^1 F q^0 \Rightarrow f(q^1) > f(q^0). \qquad (26)$$

(Step 2) Property (ii)：

Since the function f introduced above is bounded, we can newly define a real-valued function v on Q by $v = \liminf f$; namely, for $q \in Q$

$$v(q) = \sup \{ \inf \{ f(s) : s \in U \} : q \in U \in \text{Ц} \}, \qquad (27)$$

where Ц denotes the family of open sets in Q. Because v is the lower limit function of f , it must be lower semi-continuous. For this point, see McShane & Botts (1952), Theorem 3.6, p. 7.5. Therefore, we can see that the set $\{ r \in Q : v(q) \geqq v(r) \}$ is closed in Q , successfully establishing Property (ii) .

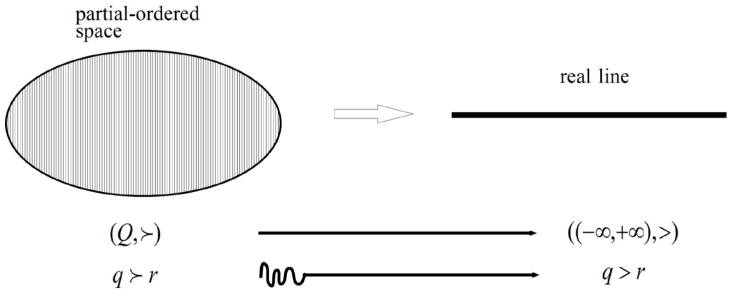


Fig. 17 Homomorphism from $(X, \succsim)$ into $((-\infty, +\infty), >)$.

(Step 3) Property (v) :

To see Property (v), let us choose $q^1, q^0 \in Q$ such that $q^1 F q^0$. Since F is lower semi-continuous by Lemma 5.1 above, we can choose $q^2 \in Q$ for which $q^1 F q^2 F q^0$, as was shown in (Step 1) above. By the lower semi-continuity of F again, $q^1 F q^2$ implies that there exists a neighborhood $U(q^1)$ of q^1 so that $q F q^2$ for all $q \in U(q^1)$. Consequently, taking advantage the Richter theorem again, we find from (5.25) that the following inequalities hold:

$$inf \ \{f(q) : q \in U(q^1)\} \geqq f(q^2) > f(q^0) \ \} \ . \tag{28}$$

Then, it follows from (26) that the following relations must be met:

$$\begin{aligned} v(q^1) &= sup \ \{inf \ \{f(s) : s \in U\} : q^1 \in U \in Ц \ \}, \\ &\geqq inf \ \{f(q) : q \in U(q^1)\} \geqq f(q^2) \\ &> f(q^0) \geqq v(q^0) \ . \end{aligned} \tag{29}$$

In short, we have thus shown that $q^1 F q^0$ implies $v(q^1) > v(q^0)$, ensuring Property (v).

(Step 4) Property (i) :

To prove Property (i), let us define the following new function :

$$w(q) \equiv \{x \in X : qx \leqq 1, \text{ and } v(r) \geqq v(q) \text{ for any } r \in Q \text{ such that } rx \leqq 1\}. \tag{30}$$

If we want to prove Property (i), then it suffices for us to show that for any $q \in Q$, $h(q) = w(q)$. First of all, from **Step 3** above, we clearly find $h(q) \subset w(q)$ for all $q \in Q$.

Now, by way of contradiction, let us suppose $x \in w(q) - h(q)$. Since $x \in X$, we have $x \in h(r)$ for some $r \in Q$, $r \neq q$. But we then obtain qFr, which implies $v(q) > v(r)$ by help of **Step 3** again. On the other hand, we have $rx = 1$, so that if $x \in w(q.)$ we must have $v(r) \geqq v(q)$; which gives a contradiction. Therefore, to get rid of a contradiction, we must have $h(q) = w(q)$ for all $q \in Q$.

(Step 5) Property (iii) :

To see Property (iii), let $q^1, q.^0 \in Q$ be such that $v(q^1) \geqq v(q^0)$ and $q^1 \neq q^0$, and let $q^t = (1-t)q^1 + tq^0$ for any $t \in (0, 1)$. Then, letting $x^t \in h(q^t)$, we find $(1-t)q^1x^1 + tq^0x^1 = q^tx^t = 1$, whence $q^1x^1 \leqq 1$ or $q^0x^1 \leqq 1$. On the one hand, if $q^1x^1 \leqq 1$, then we find q^1Fq^t, implying that $v(q^1) > v(q^t)$ by means of **Step 3**. On the other hand, if $q^0x^1 \leqq 1$, then we find q^0Fq^t, implying that $v(q^1) \geqq v(q^0) > v(q^t)$. In either case, we thus obtain , $v(q^1) > v(q^t)$. This ensures Property (iii).

(Step 6) Property (iv) :

Finally, to prove Property (iv), let $q^1, q^0 \in Q$ be such that $q^1 \leq q^0$. Then clearly, we obtain q^1Fq^0. In the light of **Step 3** above, this gives $v(q^1) > v(q^0)$. The is now complete. Q.E.D.

Theorem 5 is a very important theorem, deriving the *indirect* utility function and its many properties. Property (i) indicates that for a given normalized-price vector q, the choice set $h(q)$ constitutes bundles x that *minimize* the *indirect* utility of normalized-price vectors r subject to the budget constraint $rx \leqq 1$. Property (ii) means that for a given q, the " inferior " set $\{r : v(q) \geqq v(r)\}$ is closed in Q ; namely,, the *indirect* utility function is lower semi-continuous. Property (iii) shows that v is strictly quasi-concave, whereas Property (iv) says that v is monotonous. Finally, it follows from Property (v) that the revealed favorability relation F can be well-represented by the *indirect* utility v. In other words, there is a sort of heritability relationship between revealed favorability and *indirect* utility,

Now, we are in a position to make a bridge between the *indirect* and *direct* utility functions. To carry out such a nice task, we find it necessary to introduce an additional *strong assumption* on the demand function h :

(D) For any $q \in Q$, $h(q)$ is a singleton, or a set which contains exactly one element.

Let us recall that the strong axiom ***(SF)*** of reveled favorability implies the unique invertibility of h . Therefore, if h satisfies both *(SF)* and ***(D)***, then h is a *one-to-one correspondence* between Q and X.

Let us define that a function g on X as follows:

For any $x \in X$, $g(x) = h^{-1}(x)$. (31)

Then, it is clear that $g(h(q)) = q$ for all $q \in Q$, and that $h(g(x)) = x$ for all $x \in X$. Since g is the inverse of h on X, it immediately follows from THEOREM 5 that for any $x \in X$, the following equation holds:

$$g(x) = \{ q \in Q : qx \leqq 1 \text{, and } v(r) \geqq v(q) \text{ for any } r \in Q \text{ such that } rx \leqq 1 \} . \tag{32}$$

Now, we are in a position to define a real-valued function (a *direct* utility function) u on X *,* i.e. that h uniquely *maximizes* u over B . More specifically, we would like to establish the following important theorem.

THEOREM 6 (the derivation of the *direct* utility)

Suppose that the demand function h satisfies the budget assumption ***(H)*** and the strong axiom of revealed favorability ***(SF)*** *together with the strong demand assumption* ***(D)*** ***.*** Then, there exists a real-valued function u on Q, namely the *direct utility function*, such that the following series of properties hold.

(i) (maximality) For any $q \in Q$,

$$h(q) = \{ x : x \in b(q) \cap X \text{, and } u(r) \geqq u(q) \text{ for any } y \text{ such that } y \in b(q) \cap X \} .$$

(ii) (closeness) NOT DERIVABLE

(iii) (strict concavity) If $u(x^1) \geqq u(x^0)$, $x^1, x^0 \in X, x^1 \neq x^0$, and $x^t = (1-t)\,x^1 + t\,x^0$, then $u(x^1) > u(x^t)$.,

(iv) (monotonicity) If $x^1 \geq x^0$ and $x^1, x^0 \in X$, then $u(q^1) > u(q^0)$.

,

(v) (heritability) For any $x^1, x^0 \in X, x^1 H x^0$ implies $u(q^1) > u^0(q^0)$.,

Proof. While the proof seems to be analogous to, yet not exactly as the same as, the proof of THEOREM 5.5 , it will carefully be carried out in a step-by-step fashion.

(Step 1) Property (iv):

Under the strong axiom of revealed favorability ***(SF)*** and the strong demand assumption ***(D)***, the demand function h is a one-to-one correspondence between the normalized-price space Q and the range of the demand correspondence X . It is recalled that X need not be identical to the whole commodity space Y : X may be a proper subset of Y.

Letting $q, r \in Q$, $x = h(q)$, and $y = h(r)$, this immediately implies that xHy if and only if qFr . Suppose that $x^1, x^0 \in X$ are such that $x^1 H x^0$, and let $q^1 = g(x^1)$ and $q^0 = g(x^0)$. Then, we have $q^1 F q^0$, so that $v(q^1) > v(q^0)$ by virtue of the last THEOREM 5(v) above. Consequently, we find the following:

$$u(x^1) = v(g(x^1)) > v(g(x^0)) = u(x^0). \qquad (33)$$

This establishes Property (iv).

(Step 2) Property (i):

To prove Property (i), for any $q \in Q$, let us define the following equation:

$$k(q) = \{x : x \in b(q) \cap X \text{ , and } u(x) \geqq u(y) \text{ for any } y \text{ such that } y \in b(q) \cap X \}. \qquad (34)$$

Then evidently, it suffices to show that for any $q \in Q$, $h(q) = k(q)$. First of all, it is obvious by Property (iv) that $h(q) \subset k(q)$. Next, by way of contradiction, assume $x \in k(q) - h(q)$. Then, $y = h(q)$ for some $y \neq x$. This yields $y H x$, so that $u(y) > u(x)$ by means of **Step 1** above. On the other hand, since $x \in k(q)$ and $y \in b(q) \cap X$, it follows Eq. (5.34) that $u(x) \geqq u(y)$, which is a contradiction. Therefore, to get rid of a contradiction, we must conclude that $h(q) = k(q)$ for all $q \in Q$.

(Step 3) Property (ii):

To show Property (ii), let us suppose that $u(x^1) \geqq u(x^0)$, $x^1, x^0 \in X$, $x^1 \neq x^0$, $t \in (0, 1)$, and $x^t = (1-t)x^1 + t x^0 \in X$. Then we find $1 = g(x^t)x^t = (1-t)g(x^t)x^1 + t g(x^t)x^0$. Hence, we have $g(x^t)x^1 \leqq 1$ or $g(x^0)x^0 \leqq 1$. On the one hand, if $g(x^t)x^1 \leqq 1$, then $x^t H x^1$, so that $u(x^1) > u(x^1) \geqq u(x^0)$ by Property (iv) and hypothesis. On the other hand, if $g(x^t)x^0 \leqq 1$, then we have $x^1 H x^1$ by Property (iv) again. In either case, we thus obtain $u(x^1) > u(x^t)$, establishing Property (ii).

(STEP 4) Property (iii):

Finally, to prove Property (iii), let $x^1, x^0 \in X$ be such that $x^1 \geq x^0$. Then clearly, we have $x^1 H x^0$, which gives $u(x^1) > u(x^0)$ by means of **Step 1** above.. Thus, the proof is complete. Q.E.D.

Comparison of the last THEOREMS 5 and the present THEOREM 6 indicates the fundamental duality that exists between the *indirect* and *direct* utility functions.

(i) First of all, minimizing the *indirect* utility v of normalized prices q is equivalent to the *direct* utility u of commodities x, with the identical budget constraint $qx \leqq 1$ being imposed in both instances.

(ii) Second, v is strictly quasi-convex on Q whereas u is stricly quasi-concave on X. v is decreasing on Q whereas u is increasing on X.

(iii) Third, v represents the *indirect* revealed favorability relation F on Q whereas u represents the *direct* revealed preference relation H on X.

It should be noticed, however, that such nice symmetry between the *indirect* and *direct* utilities is not perfect, and may possibly break down under the present assumptions including ***(H)***, ***(SF)*** and ***(D)***.

Let us pay attention to Property (ii) of the present THEOREM 6, which is unfortunately not derivable under the present assumptions. Indeed, although the *indirect* utility function v is lower semi-continuous (see the last THEOREM 5(ii), the *direct* utility function u may not be upper semi-continuous under the present assumptions only.

Such an " inconvenient truth " is well-illustrated by Sonnenscein's ingenious Example 3 (Sonnenschein 1971, pp. 274-275). As the proverb goes, seeing is believing here. Although Fig. 5.18 appears to be intentionally distorted by a charting device , we believe that it grasps the essence of things. There, the demand function h generated by the indicated *direct* utility function u clearly satisfies the assumptions ***(H)*** , ***(SF)*** and ***(D)*** , but no upper semi-continuous function can represent h : for instance, the set $\{ x : u(x) \geqq 10 \}$ is not closed.

It should be remarked that X, the range of h , is not convex in the Sonnenschein example : indeed, the line segment 【 9, 9 】 does not belong to the range of h . . This once again clarifies the critical role played by the convexity assumption on the range of the demand function in exploring the exact theory of consumer's demand.

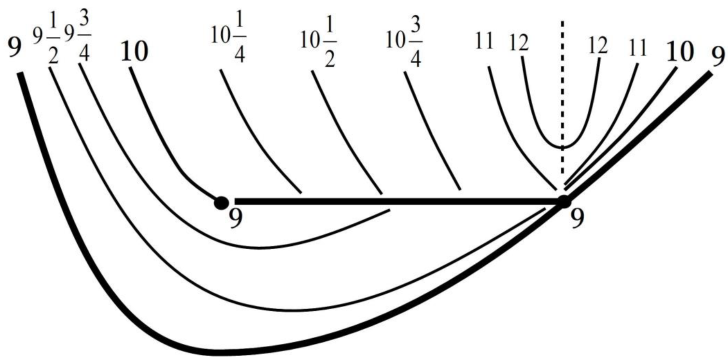


Fig. 18 Sonnenschein's ingenious example: (1) the set { x : $u(x)$ ≧10 } is *not* closed;

(2) the line segment 【 9, 9 】 does *not* belong to the range of h either.

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6 Choice and Rationality: Effectiveness and Limitations

In the above, we have been assumed that the consumer's choice behavior is *always* *rational and consistent* . In fact, the weak and strong axioms of revealed favorability on the normalized-price space and those of revealed preference on the commodity space may well-represent such *rationality and consistency.* Besides, as was shown above, the *convexity of the range of the demand function* is also a good indicator of the *wise and sensible human judgment.* It is recalled that Richter (1971) discussed many possible kinds of the consumer's rationality including " transitive rational" and " reflexive rational." In line with his way of argument, we may also add that the consumer is " convex rational" if his demand range is wide enough to satisfy the convex condition.

In the light of the long history of economic theory, there lies the academic struggle between "Econs" and "Humans." According to Richard H. Thaler (2015), our good friend and respected Rochester graduate, many standard models tend to use a fictional creature called *homo economics* , or simply Econs. Econs are generally supposed in the majority of economic books and papers including Samuelson (1955, 7th edition 1967) and the present book per se. However, we have to be very careful of the possible danger of going too far or too much. If we may put it in the strongest terms, we are allowed to regard Econs as a sort of "rational fools" a la Amartia Sen (1987) in the sense that people rationally choose goods and services by following the weak and strong axioms of revealed favorability or those of revealed preference. Or equivalently, people are fictionally assumed to optimize their indirect or direct utilities.

In contrast, Humans are just human beings, or *homo sapience.* Compared with fictional Econs, Humans are supposed to have a lot of non-rational feelings such as envy, hatred, optimism, pessimism, sympathy, compassion, and the like. In the world in which many Humans live, the traditional economic theory might be far from satisfactory. We need to establish a more comprehensive model of human behavior including a variety of complicated psychologies. As is stressed by George A. Akerlof and Robert J. Shiller (2009), a nice bridge between economic theory and human psychology must urgently be built.

It is true that getting economic theory back on its feet again will not an easy task, presumably requiring a very long arduous way ahead. We believe, however, that where there is a strong will, there is a nice way out.

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Footnotes

1) .The term "more favorable" was first used by Weddepohl (1970)

2) For example, see Hurwicz & Richter (1971), Richter (1966), Richter (1971), Uzawa (1960, revised 1971), and many others. Strangely enough, it is rather common to assume that the demand function has the *convex property*. One of main purposes in this paper is to do away with such deep-rooted tradition.

3) It is noted that in the present case, Ω is nothing but the positive orthant of R^n and hence happens to be equal to P.

4) I am indebted to Richter (1966) for clarifying this line of argument.

5) I am thankful to Uzawa (1960) for suggesting the proof presented here.

6) Income compensation functions were first introduced by McKenzie (1957) and Yokoyama (1953), independently, in terms of preference orderings on the commodity space.

7) Even in the twenty-first century, the relationship between the weak and strong axioms of *revealed favorability on the normalized-price space has rarely been* mentioned in the micro-economics literature. Here again, we see extreme difficulty to break through the hard crust of convention.

8) I am thankful to Sonnenschen for suggesting Fig 5.15 and Fig. 5.16. In my opinion, he is a very skillful drawer of indifference figures.

9) For the duality results, see Lau (1969).

10) I am grateful to Richter (1966, 1971) and Hurwicz & Richter (1971) for giving a hint for the proof presented here.