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A TERM STRUCTURE INTEREST RATE MODEL WITH THE BROWNIAN BRIDGE LOWER BOUND

Kentaro Kikuchi

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The Institute for Economic and Business Research
Faculty of Economics
SHIGA UNIVERSITY

1-1-1 BANBA, HIKONE,
SHIGA 522-8522, JAPAN

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KENTARO KIKUCHI
SHIGA UNIVERSITY

Abstract. We propose a new short rate model with a stochastic lower bound, defined as the sum of a quadratic function of Gaussian state variables and a Brownian bridge that starts and ends at zero with a random interval. The start and end times of the bridge correspond to the start and end dates of a negative interest rate environment. Our model captures time series of the yield curve, including a negative interest rate environment. Within this framework, we derive the zero coupon pricing formula in a semi-analytical form and estimate the model using time series data on the Japanese government bond yield curve. Our estimation results indicate a good fit of the model to the observations. We also compute the expected excess returns on bonds and the posterior distribution of the duration of the negative interest rate environment, providing insights into the market's views on monetary policy developments.

Keywords– No-arbitrage condition, Quadratic Gaussian term structure model, Brownian bridge Negative interest rate policy.

JEL Classification– E43, E52, G12

1. Introduction

Numerous studies have attempted to estimate the term structure model of interest rates using historical yield curve data to extract market participants' expectations. The affine Gaussian term structure model (ATSM) is the most popular model due to its analytical simplicity and ease of estimation. For example, Ang and Piazzesi (2003) and Kim and Orphanides (2012) use the ATSM to extract market expectations from U.S. Treasury yield data. However, in the low-interest-rate environment that followed the global financial crisis of 2007–2008, the ATSM is inappropriate for time-series analysis of the yield curve because it is likely to overestimate the probability of negative future interest rates. Therefore, an alternative model, such as the Shadow Rate Model (SRM) or the Quadratic Gaussian Term Structure Model (QTSM), is needed for time-series analyses of the yield curve, including a low-interest-rate environment in the sample period.

In the SRM, the short rate is defined as the greater of a latent variable called the shadow rate and a threshold, where the threshold serves as a lower bound on interest rates. This model was introduced by Black (1995) and further developed by Gorovoi and Linetsky (2004). Empirical studies using the SRM include Kim and Singleton (2012), Krippner (2013), Bauer and Rudebusch (2016), Wu and Xia (2016), Kortela (2016), Lemke and Vladu (2017), Ueno (2017), and Wu and Xia (2020). These studies examine time-series data from the United States, Europe, or Japan, including periods of

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quantitative easing, in which a central bank purchases financial assets from the financial market to provide more liquidity and stimulate the economy.

The QTSM, studied by Ahn *et al.* (2002) and Leippold and Wu (2002), is another alternative to the ATSM. In the QTSM, the short rate is defined as a quadratic function of state variables, generating a term structure of interest rates with a lower bound. Nyholm and Vidova-Koleva (2012) estimate the QTSM using US yield curve data, while Kim and Singleton (2012) compare the QTSM and the SRM using Japanese yield curve data. Although Nyholm and Vidova-Koleva (2012) and Kim and Singleton (2012) set the lower bound of interest rates at zero, allowing a negative lower bound could result in a term structure model that allows negative interest rates.

Previous empirical studies utilizing the SRM and QTSM have made significant contributions to the time-series analysis of the yield curve in low-interest-rate environments that include negative interest rates. However, most of these studies, except for Kortela (2016), Lemke and Vladu (2017), Ueno (2017), and Wu and Xia (2020), use a constant lower bound of interest rates. One problem with this approach is that it fails to capture changes in the future interest rate probability distribution due to shifts in monetary policy, such as unconventional monetary policy strengthening or tapering. Thus, it is insufficient to extract market information using a term structure model with a constant negative lower bound for interest rates under a negative interest rate environment. To overcome this issue, a better approach is to incorporate a stochastic lower bound of interest rates, which can extract more accurate market information. Accordingly, this study aims to construct a new term structure model with a stochastic lower bound of interest rates.

Consider a hypothetical country that has adopted a negative interest rate policy (NIRP) but is now planning to exit from it[1]. In such a case, the lower bound on interest rates would gradually approach zero towards the end of NIRP. With this in mind, we model the lower bound on interest rates as a Brownian bridge. The initial time of the Brownian bridge used in the model corresponds to the introduction of NIRP (or the first observation of negative interest rates in the market). The end time of the Brownian bridge corresponds to the end of NIRP (or the time when negative interest rates are no longer observed in the market). During the NIRP period, people do not know the policy's end date. Hence, our model assumes that the final time of the Brownian bridge representing the lower bound of interest rates is a random variable. In other words, the duration of NIRP is treated as random[2]. It should be noted that while a Brownian bridge generally has a fixed interval, the Brownian bridge with a random interval is studied in detail in Bedini *et al.* (2017). The work of Ajevskis and Vitola (2010) inspires our modeling of the lower bound of interest rates. They use a Brownian bridge to model the spread of short-term interest rates between the European Monetary Union (EMU)'s candidate and member countries. The final time of their Brownian bridge corresponds to the date when the candidate countries officially join the EMU, and their spreads converge to zero as the final time approaches.

In this study, we construct a term structure model where the short rate is defined as the sum of positive parts representing a quadratic function of state variables and a

[1]In this study, we use the term "NIRP" to refer to the situation where some interest rates become negative under unconventional monetary policies, even after the end of a negative interest rate policy, as other policies such as QE or zero interest rate policy may still be in effect.

[2]This assumption is similar to that of Marumo *et al.* (2003), who assume that the short rate remains at zero until the end of a central bank's zero interest rate policy (ZIRP) and follows the Vasicek model once ZIRP ends. Additionally, they model the exit time from ZIRP as a random variable and derive the bond pricing formula under the no-arbitrage condition.

stochastic lower bound provided by a Brownian bridge, as explained above. In other words, we build a QTSM-based term structure model with a Brownian bridge lower bound on rates. We then derive the zero coupon bond pricing formula under the no-arbitrage condition. One advantage associated with the QTSM-based model is that it can represent stochastic volatilities of bond prices through changes in state variables.

Since our proposed model is formulated in both risk-neutral and physical probability measures, it enables us to perform a time-series analysis of yield curves that include periods when negative interest rates have been observed in the market, such as in Europe and Japan. By using time-series data on the yield curve, we can estimate the state variables and parameters of our model, allowing us to estimate the lower bound of interest rates that can not be directly observable in the market. Additionally, we can obtain the posterior probability distribution of the number of years during which a negative interest rate environment will persist. These implications are valuable for monetary policy and investor portfolio management.

The paper is organized as follows. Section 2 presents the model setup. Section 3 and 4 provide the derivation of the zero coupon bond pricing formula for the cases where the end date of NIRP is deterministic or random, respectively. Section 5 describes the estimation methodology, including the state space representation, parameter setting, data set, and formulation of the posterior distribution of the NIRP duration. Section 6 presents the estimation results, and Section 7 concludes the paper.

2. Setup

We define a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t}, \mathbb{P})$ where the filtration $(\mathcal{F}_t)_{0\leq t}$ satisfies the usual conditions of right-continuity and completeness and is the natural filtration generated by two stochastic processes X_t and y_t^τ as defined below. $\mathbb{P}$ denotes the physical measure. $W_{t,x}^{\mathbb{P}} \in \mathbb{R}^n$ and $W_{t,y}^{\mathbb{P}} \in \mathbb{R}^1$ are independent standard Brownian motions under $\mathbb{P}$.

We assume that the market is complete and has no-arbitrage opportunities. This implies the existence of the unique risk-neutral measure $\mathbb{Q}$.

The state variable X_t satisfies the following stochastic differential equation under $\mathbb{P}$:

$$dX_t = K_X^{\mathbb{P}}(\theta^{\mathbb{P}} - X_t)dt + \Sigma_X dW_{t,x}^{\mathbb{P}}, \tag{1}$$

where all eigenvalues of the mean reversion coefficient matrix $K_X^{\mathbb{P}} \in \mathbb{R}^{n\times n}$ are assumed to be positive.

We assume that the risk-free short rate r_t is the sum of a quadratic function of X_t and y_t^τ:

$$r_t = X_t' \Psi X_t + y_t^\tau, \tag{2}$$

where X_t' represents the transposition of X_t and Ψ is positive–definite. Since $X_t'\Psi X_t > 0$, Eq.(2) implies that y_t^τ is the lower bound of r_t.

We define y_t^τ by the Brownian bridge process with $y_0^\tau = 0$, $y_\tau^\tau = 0$, and $y_t^\tau = 0$ for $t \geq \tau$. y_t^τ can be represented as

$$y_t^\tau = \sigma_y W_{t,y}^{\mathbb{P}} - \frac{\sigma_y t}{\tau \vee t} W_{\tau\vee t,y}^{\mathbb{P}}, \tag{3}$$

where $\tau \vee t = \max(\tau, t)$. Since $W_{t,x}^{\mathbb{P}}$ and $W_{t,y}^{\mathbb{P}}$ are independent as described above, y_t^τ is independent from X_t. For the time being, we assume that τ is a strictly positive constant value. Eq.(3) is equivalent to the following Eq.(4) in the stochastic differential equation form:

$$dy_t^\tau = 1_{\{t\le\tau\}}\left(-\frac{y_t^\tau}{\tau-t}dt+\sigma_y dW_{t,y}^{\mathbb{P}}\right). \tag{4}$$

Since Eq.(2) indicates that the short rate will always become positive after time τ, we can interpret τ as the date when the unconventional monetary policy ends and interest rates with all maturities become positive. Therefore, in this study, we refer to this date as the end date of NIRP.

The stochastic differential equation of X_t under $\mathbb{Q}$ is supposed to be

$$dX_t = K_X^{\mathbb{Q}}(\theta^{\mathbb{Q}}-X_t)dt+\Sigma_X dW_{t,x}^{\mathbb{Q}}, \tag{5}$$

where $W_{t,x}^{\mathbb{Q}}\in\mathbb{R}^n$ is a standard Brownian motion under $\mathbb{Q}$ and $K_X^{\mathbb{Q}}\in\mathbb{R}^{n\times n}$ is a matrix with all eigenvalues being positive.

From Eqs.(1) and (5), we have the following relationship between $W_{t,x}^{\mathbb{Q}}$ and $W_{t,x}^{\mathbb{P}}$:

$$dW_{t,x}^{\mathbb{Q}} = dW_{t,x}^{\mathbb{P}}+\Lambda(X_t)dt, \tag{6}$$

where $\Lambda(X_t) = (K_X^{\mathbb{P}}\theta^{\mathbb{P}}-K_X^{\mathbb{Q}}\theta^{\mathbb{Q}})-(K_X^{\mathbb{P}}-K_X^{\mathbb{Q}})X_t$. $\Lambda(X_t)$ can be interpreted as the market price of factor risks. This affine form was first introduced in Duffee (2002) as the essentially affine market price of risk.

We assume that the market price of risk for y_t^τ is zero. While it is possible to set the market price of risk for y_t^τ to a non-zero value, doing so would result in y_t^τ at time τ not being able to take the value zero under $\mathbb{Q}$, making the interpretation of y_t^τ difficult. Therefore, the dynamics of y_t^τ under $\mathbb{Q}$ follows

$$dy_t^\tau = 1_{\{t\le\tau\}}\left(-\frac{y_t^\tau}{\tau-t}dt+\sigma_y dW_{t,y}^{\mathbb{Q}}\right). \tag{7}$$

3. Bond pricing in the case where τ is deterministic

In this section, we derive a bond pricing formula in the case where τ is deterministic. We assume that τ is a strictly positive constant. Hereinafter, we denote a normal policy period, $\tau\le t$ (the post-NIRP period) with a superscript of 'n' and an unconventional policy period, $t<\tau$ (the NIRP period) with a superscript of 'unc'.

3.1. **Bond pricing in a normal policy period, the post-NIRP.**

In this subsection, we derive a zero coupon bond pricing formula in a normal policy period, $\tau\le t$. This period corresponds to the post-NIRP period.

An infinitesimal generator of X_t for $\tau\le t$ is provided as

$$\mathscr{D}_t^n = (K_X^{\mathbb{Q}}(\theta^{\mathbb{Q}}-X_t))'\frac{\partial}{\partial X_t}+\frac{1}{2}\mathrm{Tr}\left(\Sigma_X\Sigma_X'\frac{\partial^2}{\partial X_t\partial X_t'}\right). \tag{8}$$

Applying the Feynman–Kac theorem to the zero coupon bond price $P_{t,u}^n$ with maturity date $T=t+u$ leads to the following partial differential equation (PDE):

$$\left[\frac{\partial}{\partial t}+\mathscr{D}_t^n\right]P_{t,u}^n = r_tP_{t,u}^n,\quad P_{t,0}^n=1. \tag{9}$$

We guess the solution form of Eq.(9) as follows:

$$P_{t,u}^n = \exp(X_t'A_u^nX_t+(b_u^n)'X_t+c_u^n). \tag{10}$$

Substituting Eq.(10) into Eq.(9), we obtain the following system of ordinary differential equations (ODEs) for A_u^n, b_u^n, and c_u^n.

$$
\begin{aligned}
\dot{A}_u^n &= -2K_X^{\mathbb{Q}\prime}A_u^n + 2A_u^n\Sigma_X\Sigma_X' A_u^n - \Psi, \\
(\dot{b}_u^n)' &= 2(K_X^{\mathbb{Q}}\theta^{\mathbb{Q}})'A_u^n - b_u^{n\prime}K_X^{\mathbb{Q}} + 2b_u^{n\prime}\Sigma_X\Sigma_X'A_u^n, \\
\dot{c}_u^n &= (K_X^{\mathbb{Q}}\theta^{\mathbb{Q}})'b_u^n + \mathrm{Trace}\left(\Sigma_X\Sigma_X'\left(A_u^n + \frac{1}{2}b_u^n(b_u^n)'\right)\right),
\end{aligned}
\tag{11}
$$

where $\dot{A}_u^n$, $\dot{b}_u^n$, and $\dot{c}_u^n$ represent the derivatives of A_u^n, b_u^n, and c_u^n with respect to the variable u and the boundary conditions are $A_0^n = 0$, $b_0^n = 0$, and $c_0^n = 0$.

3.2. **Bond pricing under a negative interest rate policy.** In this subsection, we derive a zero coupon bond pricing formula in the case where $t < \tau$. This corresponds to the period when a central bank is conducting NIRP.

First, we deal with the bond price with the maturity date T which arrives before the end date of NIRP τ. Let us denote the zero coupon bond price by $P_{t,u,w}^{unc,1}$ where $u = T - t$ and $w = \tau - T$. The price $P_{t,u,w}^{unc,1}$ is provided as follows:

$$
\begin{aligned}
P_{t,u,w}^{unc,1} &= E^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right)|\mathcal{F}_t\right] \\
&= E^{\mathbb{Q}}\left[\exp\left(-\int_t^T (X_s'\Psi X_s + y_s^\tau)ds\right)|\mathcal{F}_t\right] \\
&= E^{\mathbb{Q}}\left[\exp\left(-\int_t^T X_s'\Psi X_s ds\right)|\mathcal{F}_t\right]E^{\mathbb{Q}}\left[\exp\left(-\int_t^T y_s^\tau ds\right)|\mathcal{F}_t\right] \\
&= P_{t,u}^n E^{\mathbb{Q}}\left[\exp\left(-\int_t^T y_s^\tau ds\right)|\mathcal{F}_t\right],
\end{aligned}
\tag{12}
$$

where $E^{\mathbb{Q}}[\ \ ]$ is the expectation operator under $\mathbb{Q}$. The third equality in Eq.(12) holds true by the independence between X_t and y_t^τ.

Since $P_{t,u}^n$ in the right-hand side of Eq.(12) is obtained from Eqs.(10) and (11), calculating the left-hand side of Eq.(12) reduces to the calculation of $P_{t,u,w}^y$ defined below

$$
P_{t,u,w}^y = E^{\mathbb{Q}}\left[\exp\left(-\int_t^T y_s^\tau ds\right)|\mathcal{F}_t\right]. \tag{13}
$$

An infinitesimal generator of y_t^τ over $t < \tau$ for Eq.(7) is provided as

$$
\mathscr{D}_t^{unc} = -\frac{y_t^\tau}{\tau - t}\frac{\partial}{\partial y_t^\tau} + \frac{1}{2}\sigma_y^2\frac{\partial^2}{\partial y_t^2} = -\frac{y_t^\tau}{u + w}\frac{\partial}{\partial y_t^\tau} + \frac{1}{2}\sigma_y^2\frac{\partial^2}{\partial y_t^2}. \tag{14}
$$

Applying the Feynman–Kac theorem to $P_{t,u,w}^y$ in Eq.(13), we obtain the following PDE:

$$
\left[\frac{\partial}{\partial t} + \mathscr{D}_t^{unc}\right]P_{t,u,w}^y = y_t^\tau P_{t,u,w}^y, \quad P_{t,0,w}^y = 1. \tag{15}
$$

We guess the solution of Eq.(15) as being in the following form:

$$
P_{t,u,w}^y = \exp(d_{u,w}^{unc,1}y_t^\tau + f_{u,w}^{unc,1}). \tag{16}
$$

Substituting Eq.(16) into Eq.(15), we obtain the following ODEs.

$$
\begin{aligned}
\dot{d}_{u,w}^{unc,1} + \frac{d_{u,w}^{unc,1}}{u + w} + 1 &= 0, \\
\dot{f}_{u,w}^{unc,1} &= \frac{1}{2}\sigma_y^2(d_{u,w}^{unc,1})^2,
\end{aligned}
\tag{17}
$$

where the boundary conditions are $d^{unc,1}_{0,w} = 0$ and $f^{unc,1}_{0,w} = 0$, and $\dot{d}^{unc,1}_{u,w}$ and $\dot{f}^{unc,1}_{u,w}$ represent the derivatives of $d^{unc,1}_{u,w}$ and $f^{unc,1}_{u,w}$ with respect to the variable u, respectively. The first equation in Eq.(17) is known as d'Alembert's equation, and its solution is given as follows:

$$d^{unc,1}_{u,w} = -\frac{u(u+2w)}{2(u+w)}. \tag{18}$$

Eq.(18) and the second equation in Eq.(17) lead to the solution of $f^{unc,1}_{u,w}$:

$$\begin{aligned} f^{unc,1}_{u,w} &= \int_0^u \frac{1}{2}\sigma_y^2 (d^{unc,1}_{v,w})^2 dv = \frac{\sigma_y^2}{2}\int_0^u \frac{v^2(v+2w)^2}{4(v+w)^2} dv \\ &= \frac{\sigma_y^2}{24}\left((u+w)^3 - 6w^2u + 2w^3 - \frac{3w^4}{u+w}\right). \end{aligned} \tag{19}$$

Next, we calculate the price of a zero coupon bond with a maturity date on or after the end date of NIRP, i.e., $t < \tau \le T$. In this case, we denote the zero coupon bond price with $P^{unc,2}_{t,u,w}$, where $u = T - t$ and $w = \tau - T$. Then, $P^{unc,2}_{t,u,w}$ is given by:

$$\begin{aligned} P^{unc,2}_{t,u,w} &= E^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right)|\mathcal{F}_t\right] \\ &= E^{\mathbb{Q}}\left[\exp\left(-\int_t^T (X_s'\Psi X_s + y_s^\tau) ds\right)|\mathcal{F}_t\right] \\ &= E^{\mathbb{Q}}\left[\exp\left(-\int_t^T X_s'\Psi X_s ds\right)|\mathcal{F}_t\right] E^{\mathbb{Q}}\left[\exp\left(-\int_t^T y_s^\tau ds\right)|\mathcal{F}_t\right] \\ &= E^{\mathbb{Q}}\left[\exp\left(-\int_t^T X_s'\Psi X_s ds\right)|\mathcal{F}_t\right] E^{\mathbb{Q}}\left[\exp\left(-\int_t^\tau y_s^\tau ds\right)|\mathcal{F}_t\right] \\ &= P^n_{t,u} P^y_{t,u+w,0}. \end{aligned} \tag{20}$$

Here, one should note that $P^y_{t,u,w} = P^y_{t,u+w,0}$ when $w \le 0$.

$P^y_{t,u+w,0}$ in Eq.(20) is calculated from Eqs.(16), (18), and (19) as follows:

$$P^y_{t,u+w,0} = \exp(d^{unc,1}_{u+w,0} y_t^\tau + f^{unc,1}_{u+w,0}) = \exp\left(-\frac{u+w}{2} y_t^\tau + \frac{\sigma_y^2}{24}(u+w)^3\right). \tag{21}$$

4. Bond pricing in the case where τ is random

In this section, we derive a zero coupon bond pricing formula in the case where the end date of NIRP τ is random. Instead of Eq.(2), we define the risk-free short rate r_t as $r_t = X_t'\Psi X_t + y_t$. By this definition, y_t becomes the lower bound of interest rates. In this section, we model the lower bound of interest rates y_t as the Brownian bridge with a random time interval τ, which is studied in Bedini *et al.* (2017).

To price the zero coupon bonds, we need the probability distribution of τ under $\mathbb{Q}$. Thus, we focus on τ under $\mathbb{Q}$ rather than $\mathbb{P}$ in this section. Let $\tau : \Omega \to (0, +\infty)$ be a strictly positive random variable whose distribution function is denoted with $F(t) = \mathbb{Q}(\tau \le t)$. We assume that τ is independent of $W^{\mathbb{Q}}_{t,x}$ and $W^{\mathbb{Q}}_{t,y}$. $\mathcal{F}^y_t$ denotes the completed natural filtration generated by y_t; that is, $\mathcal{F}^y_t = \sigma(y_s; 0 \le s \le t) \vee \mathcal{N}$ where $\mathcal{N}$ denotes the collection of $\mathbb{Q}$-null sets. When we denote $(C, \mathbf{C})$ as the space of continuous real-valued functions on $\mathbb{R}_+$ endowed with the σ-algebra generated by the canonical process, we define a Brownian bridge with a random time interval τ as the map from $(\Omega, \mathcal{F})$ to $(C, \mathbf{C})$ as follows:

Definition 1. The process $y_t(\omega)$ given by

$$y_t(\omega) = y_t^{\tau(\omega)}(\omega),$$

is the Brownian bridge with a random interval τ, where y_t^r is the Brownian bridge with a deterministic time interval r as defined in Eq.(3).

Bedini *et al.* (2017) prove that the mapping $y : (\Omega, \mathcal{F}) \to (C, \mathbf{C})$ is measurable, $\{y_t = 0\} = \{\tau \le t\}$ for any $t > 0$, $\mathbb{Q}$-a.s., and the process y is a Markov process with respect to the natural filtration generated by y. We present their lemmas which are useful for deriving the zero coupon bond pricing formula in our setting below.

Lemma 1. *Let $\sigma(\tau)$ denote the σ-algebra generated by τ and $\mathcal{B}(A)$ denote the Borel set of A.*

If $h : ((0, +\infty) \times C, \mathcal{B}((0, +\infty)) \otimes \mathbf{C}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a measurable function such that $E[|h(\tau, y)|] < +\infty$, then $E[h(\tau, y)|\sigma(\tau)](\omega) = E[h(r, y^r)]|_{r=\tau(\omega)}$, $\mathbb{Q}$-a.s.

Lemma 2. *Let $0 \le t \le u$ and $g(\tau, y_u)$ be a $\sigma(y_s; s \ge t)$ measurable nonnegative function under $\mathbb{Q}$ where $\sigma(y_s; s \ge t)$ is a sigma algebra generated by the future evolution of the process y. Then,*

$$E^{\mathbb{Q}}[g(\tau, y_u)|\mathcal{F}_t^y] = E^{\mathbb{Q}}[g(\tau, y_u)|y_t], \ \mathbb{Q}\text{-a.s.}$$

Let $f^{\mathbb{Q}}(x)$ be the prior density function of τ under $\mathbb{Q}$. We define $G^{\mathbb{Q}}(t, y_t)$ as follows:

$$G^{\mathbb{Q}}(t, y_t) = \int_t^{\infty} \varphi_t^{\mathbb{Q}}(v, y_t) f^{\mathbb{Q}}(v) dv, \tag{22}$$

where $\varphi_t^{\mathbb{Q}}(r, y)$ represents the density of y_t^r as provided in Eq.(7). In the appendix, we prove that $\varphi_t^{\mathbb{Q}}(r, y)$ is calculated as follows:

$$\varphi_t^{\mathbb{Q}}(r, y) = \sqrt{\frac{r}{2\pi t(r-t)\sigma_y^2}} \exp\left(-\frac{y^2}{2t(r-t)\sigma_y^2}\right). \tag{23}$$

We present another lemma that was proved in Bedini *et al.* (2017) to use for the derivation of the zero coupon bond price representation:

Lemma 3. *Let $t > 0$ and $g(\tau, y_t)$ be a measurable function such that $g(\tau, y_t)$ is integrable. Then, $\mathbb{Q}$-a.s.*

$$E^{\mathbb{Q}}[g(\tau, y_t)|\mathcal{F}_t^y] = g(\tau, 0) 1_{\{\tau \le t\}} + \int_t^{\infty} g(r, y_t) \frac{\varphi_t^{\mathbb{Q}}(r, y) f^{\mathbb{Q}}(r)}{G^{\mathbb{Q}}(t, y_t)} dr 1_{\{t < \tau\}}. \tag{24}$$

Note that Bayes' theorem implies that the expression $\frac{\varphi_t^{\mathbb{Q}}(r, y_t) f^{\mathbb{Q}}(r)}{G^{\mathbb{Q}}(t, y_t)}$ in Eq. (24) can be interpreted as the posterior density of τ conditioned on y_t, while $f^{\mathbb{Q}}(r)$ represents its prior density.

We derive the pricing formula for the zero-coupon bond price $P_{t,T-t}$ with a maturity date of T at time t during a NIRP period.

Proposition 4. *The following equation holds $\mathbb{Q}$-a.s.:*

$$\begin{aligned}
1_{\{t<\tau\}} P_{t,T-t} &= P_{t,T-t}^n E^{\mathbb{Q}} \left[\exp\left(-\int_t^T y_s ds \right) 1_{\{t<\tau\}} | \mathcal{F}_t^y \right] \\
&= \frac{1_{\{t<\tau\}}}{G^{\mathbb{Q}}(t, y_t)} \left(\int_T^{+\infty} P_{t,T-t,v-T}^{unc,1} \varphi_t^{\mathbb{Q}}(v, y_t) f^{\mathbb{Q}}(v) dv + \int_t^T P_{t,T-t,v-T}^{unc,2} \varphi_t^{\mathbb{Q}}(v, y_t) f^{\mathbb{Q}}(v) dv \right).
\end{aligned}$$

Proof. The first equality holds true due to the independence between X_t and y_t. The term excluing $P^n_{t,T-t}$ on the right-hand side of the first equality is calculated $\mathbb{Q}$-a.s. as follows:

$$\begin{aligned}
&E^{\mathbb{Q}}\left[\exp\left(-\int_t^T y_s ds\right)1_{\{t<\tau\}}|\mathcal{F}^y_t\right] = E^{\mathbb{Q}}\left[\exp\left(-\int_t^T y_s ds\right)|y_t\right]1_{\{t<\tau\}}\\
&= E^{\mathbb{Q}}\left[E^{\mathbb{Q}}\left[\exp\left(-\int_t^T y^r_s ds\right)|y^r_t\right]_{r=\tau}|y_t\right]1_{\{t<\tau\}}\\
&= E^{\mathbb{Q}}\left[P^y_{t,T-t,\tau-T}(y^r_t)|y_t\right]1_{\{t<\tau\}}.
\end{aligned} \tag{25}$$

Since $\exp\left(-\int_t^T y_s ds\right)1_{\{t<\tau\}}$ is a $\sigma(y_s; s\geq t)$ measurable nonnegative function, we obtain the first equality by applying Lemma 2. The second equality in Eq.(25) holds true due to Lemma 1. Third equality is given by Eqs.(13), (16), (20), and (21). Lemma 3 introduces the right-hand side of the final equality in the above equation into the following representation:

$$\begin{aligned}
&E^{\mathbb{Q}}\left[P^y_{t,T-t,\tau-T}(y_t)|y_t\right]1_{\{t<\tau\}} =\\
&\frac{1}{G^{\mathbb{Q}}(t,y_t)}\left(\int_T^{+\infty} P^y_{t,T-t,v-T}\varphi^{\mathbb{Q}}_t(v,y_t)f^{\mathbb{Q}}(v)dv + \int_t^T P^y_{t,T-t,v-T}\varphi^{\mathbb{Q}}_t(v,y_t)f^{\mathbb{Q}}(v)dv\right)1_{\{t<\tau\}}.
\end{aligned}$$

Thus, Eqs.(12), (20), and (25) lead to the conclusion of this proposition. □

We obtain the following pricing formula for $P_{t,T-t}$ by Proposition 4.

Theorem 5. *The following equation holds $\mathbb{Q}$-a.s.:*

$$\begin{aligned}
&P_{t,T-t} = P^n_{t,T-t}1_{\{\tau\leq t\}}\\
&+\frac{1_{\{t<\tau\}}}{G^{\mathbb{Q}}(t,y_t)}\left(\int_T^{+\infty} P^{unc,1}_{t,T-t,v-T}\varphi^{\mathbb{Q}}_t(v,y_t)f^{\mathbb{Q}}(v)dv + \int_t^T P^{unc,2}_{t,T-t,v-T}\varphi^{\mathbb{Q}}_t(v,y_t)f^{\mathbb{Q}}(v)dv\right).
\end{aligned} \tag{26}$$

The first integrand of the right-hand side of Eq.(26) can be computed by applying the Gauss-Laguerre quadrature rule:

$$\begin{aligned}
&\int_T^{+\infty} P^{unc,1}_{t,T-t,v-T}\varphi^{\mathbb{Q}}_t(v,y_t)f^{\mathbb{Q}}(v)dv\\
&= \sum_{i=1}^n w^{GLa}_i P^{unc,1}_{t,T-t,v^{GLa}_i}\varphi^{\mathbb{Q}}_t(v^{GLa}_i+T,y_t)f^{\mathbb{Q}}(v^{GLa}_i+T)e^{v^{GLa}_i},
\end{aligned} \tag{27}$$

where v^{GLa}_i and w^{GLa}_i are nodes ad weights of the Gauss-Laguerre quadrature, respectively.

The Gauss-Legendre quadrature rule is applied in the second integrand on the right-hand side of Eq.(26) as follows:

$$\begin{aligned}
&\int_t^T P^{unc,2}_{t,T-t,v-T}\varphi^{\mathbb{Q}}_t(v,y_t)f^{\mathbb{Q}}(v)dv\\
&= \frac{T-t}{2}\sum_{i=1}^n w^{GLe}_i P^{unc,2}_{t,T-t,v^{GLe}_i-T}\varphi^{\mathbb{Q}}_t(v^{GLe}_i,y_t)f^{\mathbb{Q}}(v^{GLe}_i),
\end{aligned} \tag{28}$$

where v^{GLe}_i and w^{GLe}_i are nodes ad weights of the Gauss-Legendre quadrature, respectively.

Eqs.(27) and (28) contribute to efficient computation of bond prices.

5. Estimation methodology

In this section, we first present a state space representation of our proposed model. Next, we explain the assumptions made for model parameters and the Japanese yield curve historical data used in estimating the model. Additionally, we formulate the posterior distributions of the NIRP duration, which are plotted using the estimated state variables in the next section.

5.1. **State space model representation.** To esimate latent factors X_t and y_t of our model, we apply a filtering method to a state space representation of the model. In this subsection, we formulate our model as a state space model.

The invariant transforms of Dai and Singleton (2000), Ahn *et al.* (2002), and Leippold and Wu (2002) are applicable to our model. This allows us to have the assumption that $\theta^{\mathbb{P}}$ is a zero vector, Σ_X is the identity matrix, and $K_X^{\mathbb{P}}$ is the lower triangular matrix with positive diagonal elements. After applying the invariant transformation, we estimate the model.

The state equation of X_t describes the dynamics of X_t under the physical measure $\mathbb{P}$, as shown in Eq.(1). Replacing Eq.(1) with the discrete time representation under the time step Δt, we obtain the following equation:

$$X_{t+\Delta t} = \exp(-K_X^{\mathbb{P}}\Delta t)X_t + w_{X,t+\Delta t}, \tag{29}$$

where $w_{X,t+\Delta t} \sim N(0, V)$ and V is provided as follows:

$$(K_X^{\mathbb{P}} + (K_X^{\mathbb{P}})')^{-1}(I - \exp(-(K_X^{\mathbb{P}} + (K_X^{\mathbb{P}})')\Delta t)).$$

According to a theorem shown in Bedini *et al.* (2017), y_t for $t < \tau$ satisfies the following equation:

$$\begin{aligned} y_t &= y_0 + \int_0^t E^{\mathbb{P}}\left[\frac{y_s 1_{\{s<\tau\}}}{\tau - s}\,\middle|\, y_s\right] ds + \int_0^t \sigma_y dW_{s,y}^{\mathbb{P}} \\ &= y_0 - \int_0^t ds\; y_s \int_s^\infty dr \frac{\varphi_s^{\mathbb{P}}(r, y_s)}{(r-s)G^{\mathbb{P}}(t, y_t)} f^{\mathbb{P}}(r) 1_{\{s<\tau\}} + \int_0^t \sigma_y dW_{s,y}^{\mathbb{P}}, \end{aligned} \tag{30}$$

where $f^{\mathbb{P}}(v)$ is the prior density function of the end date τ of NIRP under $\mathbb{P}$. $\varphi_t^{\mathbb{P}}(v, y_t)$ is the density function of y_t^τ under $\mathbb{P}$. We now assume that the market price of risk for y is zero; hence, $\varphi_t^{\mathbb{P}}(v, y_t) = \varphi_t^{\mathbb{Q}}(v, y_t)$, whose representation is given in Eq.(23). $G^{\mathbb{P}}(t, y_t)$ is the distribution function of y_t^τ under $\mathbb{P}$ provided as

$$G^{\mathbb{P}}(t, y_t) = \int_t^\infty \varphi_t^{\mathbb{P}}(v, y_t) f^{\mathbb{P}}(v) dv. \tag{31}$$

Let us denote $\int_s^\infty dr \frac{\varphi_s^{\mathbb{P}}(r, y_s)}{(r-s)G^{\mathbb{P}}(t, y_t)} f^{\mathbb{P}}(r) 1_{\{s<\tau\}}$ by $g(s)$. Eq.(30) is rewritten by the following discrete time form:

$$y_{t+\Delta t} = e^{-\int_t^{t+\Delta t} g(s)ds} y_t + w_{y,t+\Delta t}, \tag{32}$$

where $w_{y,t+\Delta t} \sim N(0, V_y)$ and V_y is $\sigma_y^2 \int_t^{t+\Delta t} e^{-2\int_u^{t+\Delta t} g(s)ds} du$.

Let $Yield_t^{u_i}$ be the zero coupon yield with the time to maturity u_i at time t observed in a bond market. The vectors $Y_t^o = (Yield_t^{u_1}, \dots, Yield_t^{u_m})'$ and

$$Y_t(X_t, y_t) = \left(-\frac{1}{u_1}\log P_t^{u_1}(X_t, y_t), \dots, -\frac{1}{u_m}\log P_t^{u_m}(X_t, y_t)\right)'$$

computed from Eq.(26) constitute the observation equation of a state space model:

$$Y_t^o = Y_t(X_t, y_t) + \eta_t,\ \eta_t \sim N(0, \eta^2 I_m), \tag{33}$$

where I_m is the identity matrix of size m. Errors in the observation equation are assumed to follow a normal distribution with a zero mean vector and a diagonal covariance matrix $\eta^2 I_m$, and are assumed to be independent of other random variables.

Our state space model has a nonlinear observation equation; thus, we estimate model parameters and latent factors X_t and y_t using the unscented Kalman filter (UKF) proposed by Julier and Uhlmann (1997). The extended Kalman filter (EKF) is a well-known filter that relies on the Taylor expansion of the nonlinear function. While the EKF is a derivative-based method, the UKF is a derivative-free method. Therefore, in cases where the nonlinear function is challenging to differentiate analytically, the UKF has an advantage over the EKF. In our case, taking derivatives of the observation equations is challenging; thus, we use the UKF to estimate model parameters, while simultaneously using the quasi-maximum likelihood method with latent factor estimates.[5]

5.2. **Parameter setting.** We estimate the model using interest rates in the Japanese government bond (JGB) market. In this subsection, we present the parameter setup for this purpose.

Zero coupon interest rates with maturities of less than one year, estimated from JGB price data, frequently became negative starting in October 2015. With this in mind, we set the initial point of the model to September 30, 2015.

The time it takes for an event to occur is often modeled using an exponential distribution. For example, in Ajevskis and Vitola (2010), the time it takes for a country to join the EMU is assumed to follow an exponential distribution. One characteristic of the exponential distribution is that the random variable obtained from the distribution has the highest probability of being zero, as shown by the solid blue line in Fig. 1. However, this feature is inconsistent with our study since many market participants believed that the Bank of Japan's (BOJ's) unconventional monetary policy would not end soon when negative interest rates on JGBs began to be observed.

Fig. 1 shows the density function of a gamma distribution with the shape parameter $\alpha = 3$ and the scale parameter $\beta = 3$ in the solid red line. The density peaks at a point away from zero, and its expectation is $\alpha\beta$.

For this reason, we prefer a gamma distribution to an exponential distribution as the prior distribution of τ, denoted as $f^{\mathbb{P}}(\tau)$. We assume that the prior distribution of τ follows a gamma distribution with the shape parameter $\alpha = 3$ and the scale parameter $\beta = 3$, reflecting the expectations that the Japanese unconventional monetary policy will continue for prolonged periods. Its density function is provided as follows:

$$f^{\mathbb{P}}(\tau) = \frac{\tau^{\alpha-1} e^{-\frac{1}{\beta}\tau}}{\beta^\alpha \Gamma(\alpha)} = \frac{\tau^2 e^{-\frac{1}{3}\tau}}{27\Gamma(3)}. \tag{34}$$

To maintain simplicity in estimation, we assume that $f^{\mathbb{Q}}(\tau) = f^{\mathbb{P}}(\tau)$ and that $K_X^{\mathbb{Q}}$ is the lower triangular matrix with positive diagonal elements as with $K_X^{\mathbb{P}}$.

We also assume that X_t is a three-dimensional latent state variable.

[5]The UKF is used to estimate latent factors in many finance literatures (e.g., Leippold and Wu (2007), Christoffersen *et al.* (2014), Filipović *et al.* (2016), Branger *et al.* (2021)).

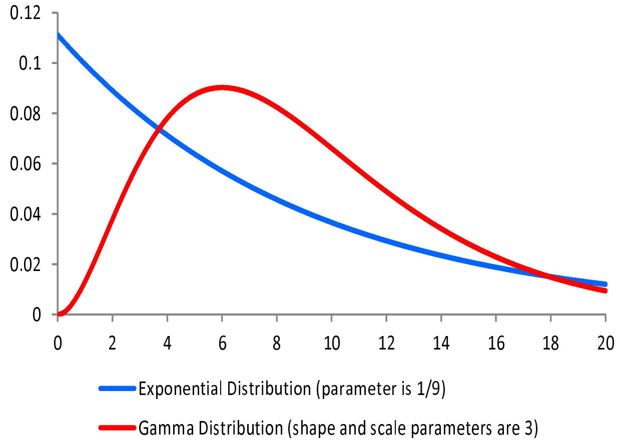

FIGURE 1. Probability density functions of exponential distribution and gamma distribution

5.3. **Data.** In estimating the model, we utilize market data for zero coupon yields of Japanese government bonds with maturities of 6 months and 1, 2, 3, 5, 7, 10, and 20 years. The data covers the period from October 1st, 2015, to June 8th, 2022, with a frequency of every five business days from October 1st, 2015. These yields are estimated based on B-spline regression, as outlined in Steeley (1991) and Kikuchi and Shintani (2012), using Japanese government bond prices from the Japan Securities Dealers Association.

Table 1 displays the summary statistics of the Japanese government bond yields used for our estimation. According to the table, the mean term structure is upward sloping, the yields are negatively skewed except for the 20-year rate, and the medium and long-term rates exhibit thicker tails than the normal distribution.

Maturity	Mean	Std. Dev.	Skew	Kurt	Auto. Correl.
0.5	-0.166	0.0852	-0.697	-0.139	0.962
1	-0.159	0.0729	-0.640	0.114	0.953
2	-0.147	0.0652	-0.207	0.898	0.931
3	-0.138	0.0685	-0.254	1.230	0.926
5	-0.126	0.0876	-0.608	0.905	0.939
7	-0.0864	0.106	-0.420	0.751	0.949
10	0.0423	0.119	-0.0867	0.932	0.964
20	0.532	0.213	0.901	1.841	0.982

TABLE 1. Summary statistics of bond yields: The data are JGB yields expressed as annual percentages. Maturity is indicated in year. Mean is the sample mean, Std. Dev. is the standard deviation, Skew is the skewness, Kurt is the excess kurtosis, and Auto. Correl. is the first order autocorrelation.

5.4. **Expected excess returns and posterior distribution of the NIRP duration.** Once all parameters and state variables are estimated using the UKF and quasi-maximum likelihood methods, various useful measures of bonds can be computed. In particular, the next section presents estimates of bonds' expected excess returns and the posterior distributions of the duration of NIRP (or time until exit from NIRP).

Therefore, this subsection provides mathematical expressions for the bonds' expected excess returns and the posterior distribution of the NIRP duration.

First, we provide the formula for the excess rate of return on the bond. The volatility matrix of a zero-coupon bond with a maturity date of T at time t for the Brownian motions $W_t^{\mathbb{P}}$ and $W_t^{\mathbb{Q}}$ is given by:

$$\frac{1}{P^n_{t,T-t}}\frac{\partial P^n_{t,T-t}}{\partial X_t'}\Sigma_X = \left((A^n_{T-t}+(A^n_{T-t})')X_t + b^n_{T-t}\right)'\Sigma_X.$$

Given the above equation and Eq.(6), we obtain the representation of the excess return on the bond as follows:

$$\left((A^n_{T-t}+(A^n_{T-t})')X_t + b^n_{T-t}\right)'\Sigma_X\Lambda(X_t), \tag{35}$$

where $\Lambda(X_t)$ is defined in Eq.(6).

Essentially, the expected excess rate of return is obtained by taking the expected value under the physical probability measure in Eq.(35). However, for the sake of computational simplicity, we prioritize an approximation where we substitute $X_{t|t}$, the filtered value of X, for X in Eq.(35). In other words, in the next section, we will compute the expected excess returns of bonds as follows:

$$\left((A^n_{T-t}+(A^n_{T-t})')X_{t|t} + b^n_{T-t}\right)'\Sigma_X\Lambda(X_{t|t}). \tag{36}$$

The density functions of the posterior distribution of τ under $\mathbb{P}$ and $\mathbb{Q}$ are provided as

$$\frac{\varphi_t^{\mathbb{P}}(\tau,y_t)f^{\mathbb{P}}(\tau)}{\int_t^\infty \varphi_t^{\mathbb{P}}(v,y_t)f^{\mathbb{P}}(v)dv} = \frac{\varphi_t^{\mathbb{Q}}(\tau,y_t)f^{\mathbb{Q}}(\tau)}{\int_t^\infty \varphi_t^{\mathbb{Q}}(v,y_t)f^{\mathbb{Q}}(v)dv}, \tag{37}$$

where $\varphi_t^{\mathbb{P}}(\tau,y_t)$ is equal to $\varphi_t^{\mathbb{Q}}(\tau,y_t)$ and is provided as Eq.(23). In addition, we assume that $f^{\mathbb{P}}(\tau) = f^{\mathbb{Q}}$ and they are Eq.(34) as we explained in Sect.5.2.

Thus, when $t < \tau$, Eq.(37) leads to the following density of the posterior distribution of the duration $s = \tau - t$ of NIRP,

$$\frac{\varphi_t^{\mathbb{P}}(t+s,y_t)f^{\mathbb{P}}(t+s)}{\int_0^\infty \varphi_t^{\mathbb{P}}(t+s',y_t)f^{\mathbb{P}}(t+s')ds'}. \tag{38}$$

By denoting estimates of y_t by $\tilde{y}_t$, we can compute the posterior densities of the time to exit from NIRP under $\mathbb{P}$ (the same as under $\mathbb{Q}$) for each of the dates, using the estimates of model parameters and $\tilde{y}_t$. From Eq.(38), these densities at time t are provided as:

$$\frac{\varphi_t^{\mathbb{P}}(t+\tilde{\tau},\tilde{y}_t)f^{\mathbb{P}}(t+\tilde{\tau})}{\int_0^\infty \varphi_t^{\mathbb{P}}(t+\tilde{\tau},\tilde{y}_t)f^{\mathbb{P}}(t+\tilde{\tau})d\tilde{\tau}}. \tag{39}$$

6. Estimation results

This section presents the results estimated using the methodology described in the previous section. First, we present the estimates of the model parameters, followed by the estimates of the latent variable X_t and the lower bound of the interest rate y_t. The estimated value of y can provide an idea of how the market perceived the change in the BOJ's stance toward unconventional monetary policy.

Furthermore, we assess the fitting accuracy of the estimates through the root mean square errors (RMSE) and the evolution of the observed and estimated values. Finally, we present the estimates of the posterior probability distributions for the excess return on JGBs and the duration of NIRP based on the estimated values of X_t and y_t.

6.1. **Parameter and latent factor estimates.** The estimates of the model parameters are presented in Table 2.

Ψ is a parameter that affects the nonlinearity of the interest rates through X. $\Psi_{2,3}$ and $\Psi_{3,3}$ are significant at the two-sided 95% confidence interval. This finding implies that modeling based on a quadratic Gaussian rather than an affine model has successfully captured the nonlinearity of interest rates in Japan. Moreover, several elements of $K_X^{\mathbb{Q}}$ and $\theta^{\mathbb{Q}}$ become statistically significant, indicating that the market price of X is time-varying. The volatility term σ_y of the stochastic lower bound y_t of interest rates is also statistically significant. This result confirms the validity of introducing a stochastically varying factor rather than a constant in the QTSM.

$$
\begin{aligned}
\Psi &= \begin{pmatrix} 6.77\times10^{-9}\ (0.369) & -4.98\times10^{-10}\ (0.0355) & -5.34\times10^{-9}\ (0.421) \\ -4.98\times10^{-10}\ (0.0355) & 3.41\times10^{-5}\ (1.77) & 4.95\times10^{-5}\ (2.82) \\ -5.34\times10^{-9}\ (0.421) & 4.95\times10^{-5}\ (2.82) & 5.03\times10^{-4}\ (2.03) \end{pmatrix}, \\
K_X^{\mathbb{P}} &= \begin{pmatrix} 0.0227\ (0.126) & 0 & 0 \\ 0.265\ (1.83) & 2.06\times10^{-5}\ (0.0972) & 0 \\ 0.000766\ (0.0349) & -0.0591\ (0.0593) & 4.17\times10^{-6}\ (0.196) \end{pmatrix}, \\
K_X^{\mathbb{Q}} &= \begin{pmatrix} 1.71\times10^{-9}\ (0.0837) & 0 & 0 \\ -0.191\ (1.91) & 0.204\ (6.54) & 0 \\ 0.0362\ (0.179) & -0.266\ (2.84) & 0.289\ (7.87) \end{pmatrix}, \theta^{\mathbb{Q}} = \begin{pmatrix} 9.38\ (1.70) \\ -1.55\ (2.84) \\ 0.860\ (0.194) \end{pmatrix}, \\
\sigma_y &= 0.00174\ (2.22), \quad \eta = 0.01258\%, \quad \mathcal{L} = 18361.2.
\end{aligned}
$$

TABLE 2. Estimates of the model parameters and optimal value of log-likelihood: The absolute magnitude of the t-statistics computed using the BHHH estimator are indicated in parentheses. $\mathcal{L}$ is the optimal log-likelihood. η is defined as Eq.(33).

Fig.2 shows the filtered value of X, or $E_t^{\mathbb{P}}[X_t|Y_t^o]$. In a later subsection, we plot the evolution of the expected excess rate of return on bonds with various maturities using this filtered value, $E_t^{\mathbb{P}}[X_t|Y_t^o]$.

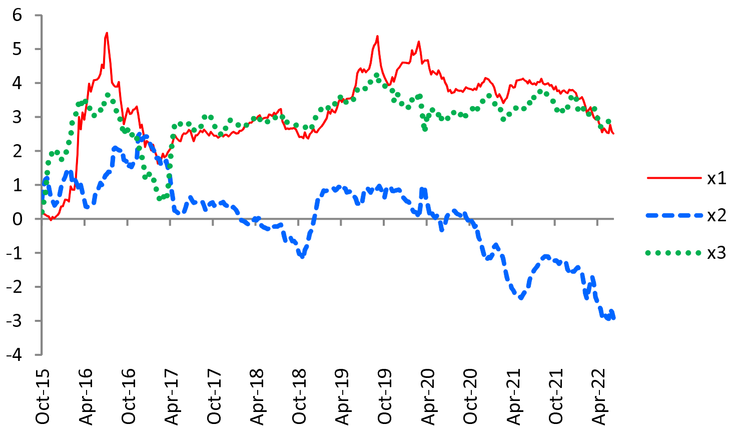

FIGURE 2. Estimates of X, or $E_t^{\mathbb{P}}[X_t|Y_t^o]$, with the first, second, and third elements of X denoted as $x1$", $x2$", and "$x3$", respectively.

Fig.3 displays the estimated values of the stochastic lower bound, y. As shown in Fig.3, y shows a significant decline during the summers of 2016 and 2019. The decline in the level during the summer of 2016 is attributed to the growing belief that the BOJ would deepen its NIRP during this period. Similarly, the decline in the level during the summer of 2019 is due to the BOJ's forward guidance for policy rates at the Monetary Policy Meeting held in April 2019. The Bank's decision to clarify its forward guidance for policy rates and implement several policy actions to continue its strong monetary easing, as discussed in Bank of Japan (2020), likely drove the decline in the level during the summer of 2019.

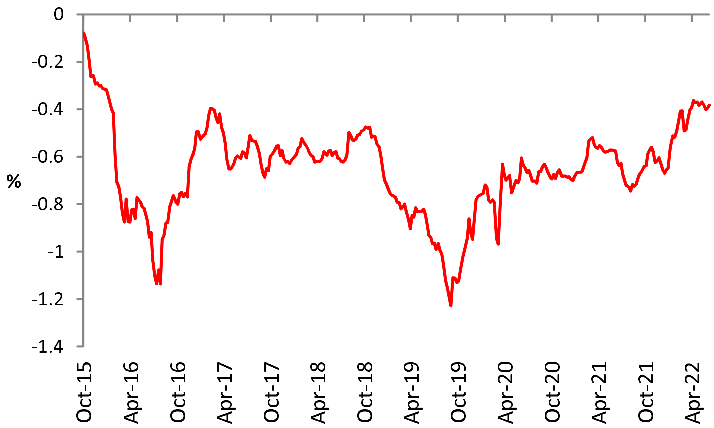

FIGURE 3. Estimates of the stochastic lower bound y, $E_t^{\mathbb{P}}[y_t|Y_t^o]$.

6.2. **Fitting results.** Table 3 shows the root mean squared errors for the estimated yields and demonstrates good in-sample performance. For all time-to-maturities, the root mean squared errors are within two basis points.

Maturity	0.5	1	2	3	5	7	10	20
RMSE	0.0115	0.00726	0.00857	0.00954	0.0106	0.0154	0.0141	0.00860

TABLE 3. The root mean squared errors (RMSE) for estimated yields are reported as annual percentages, with maturity indicated in years.

Table 4 shows the descriptive statistics of the estimation errors. The absolute values of the fitting errors are confirmed to be at most 10 basis points.

Maturity	0.5	1	2	3	5	7	10	20
Mean	0.00316	−0.00243	−0.00104	0.00436	0.00124	−0.00501	0.00249	−0.000182
Std	0.0111	0.00684	0.00851	0.00848	0.0105	0.0145	0.0139	0.00860
Min	−0.0540	−0.0459	−0.0308	−0.0197	−0.0511	−0.0638	−0.0462	−0.0614
25%	−0.00193	−0.00409	−0.00746	−0.00197	−0.00558	−0.0143	−0.00879	−0.00309
50%	0.00312	−0.00157	−0.00128	0.00440	0.00117	−0.00505	0.00302	0.000768
75%	0.00840	0.000400	0.00531	0.00987	0.00901	0.00467	0.0110	0.00380
Max	0.0500	0.0212	0.0367	0.0382	0.0300	0.0347	0.0714	0.0338

TABLE 4. Fitting errors are reported as annual percentages, with maturity indicated in years. The mean represents the sample average of fitting errors, the standard deviation is denoted by Std, the minimum and maximum values are denoted by Min and Max, respectively. The fitting errors' quartiles are presented as 25%, 50%, and 75%.

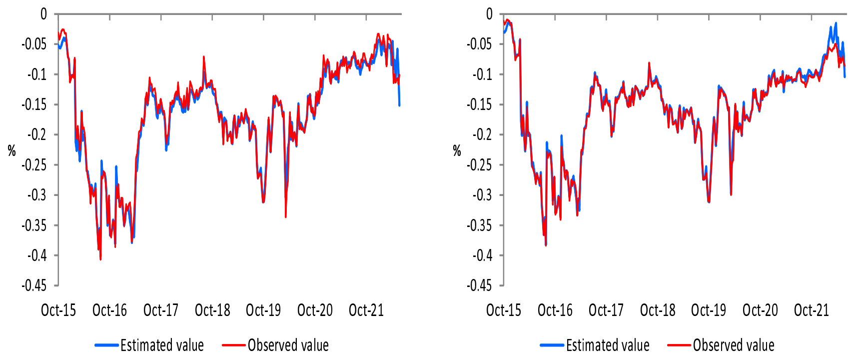

FIGURE 4. Comparison between observed and estimated values of yields with short-term maturities. The figure on the left shows the 6-month maturity, while the one on the right shows the 20-year maturity.

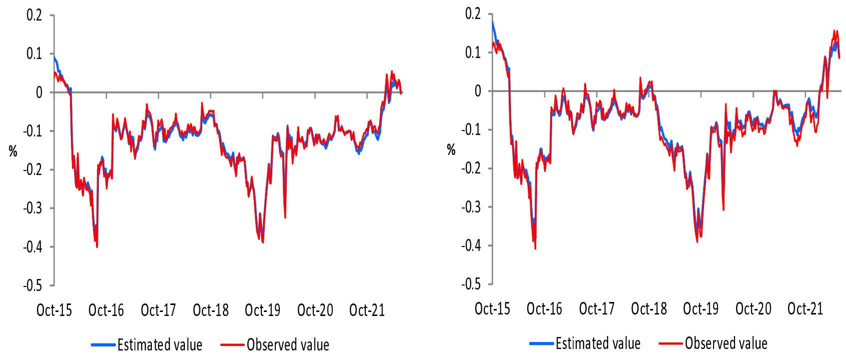

FIGURE 5. Comparison between observed and estimated values of yields with medium-term maturities. The figure on the left shows the 5-year maturity, while the one on the right shows the 7-year maturity.

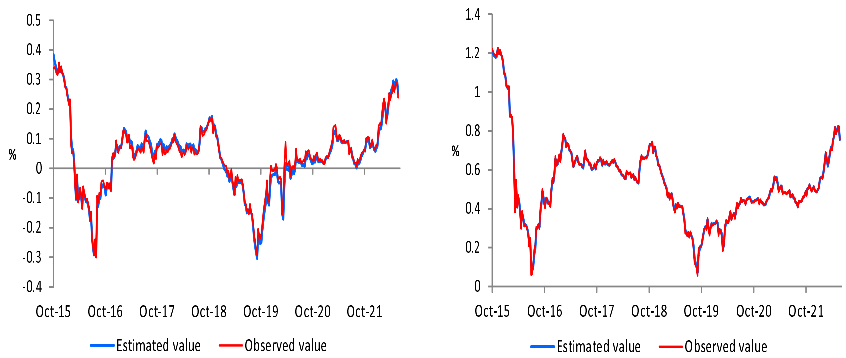

FIGURE 6. Comparison between observed and estimated values of yields with long-term maturities. The figure on the left shows the 10-year maturity, while the one on the right shows the 20-year maturity.

Figs.4, 5, and 6 compare the time series of observed and estimated yields, providing comparisons of short-, medium-, and long-term interest rates. These figures demonstrate the high accuracy of our estimates.

6.3. **Expected excess returns on some bonds.** Once we have estimated the state variables X, we can calculate informative indicators. In this subsection, we estimate the expected excess returns on one-, two-, five-, ten-, and twenty-year bonds.

As shown in Fig.7, the expected excess returns on bonds of all maturities remained consistently low throughout the sample period, with most values being negative. This result can be attributed to the significant monetary easing that has been implemented during this period.

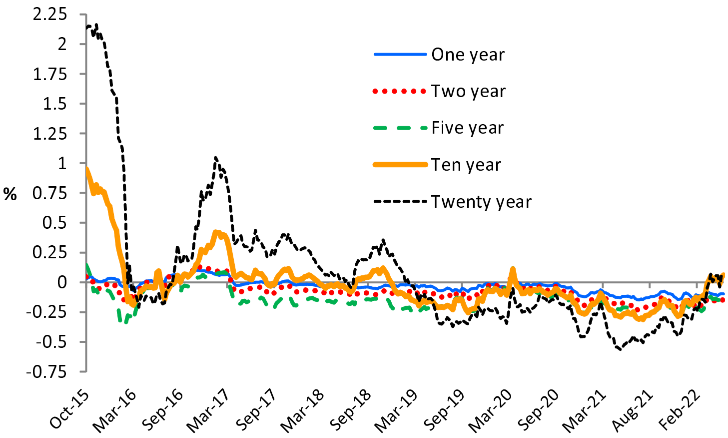

FIGURE 7. Expected excess returns on some bonds

6.4. **Implied posterior distribution of the NIRP duration.** Market participants are very interested in when NIRP will end. To shed light on this issue, we present how market participants' perceptions of the duration of NIRP have changed over time based on the posterior distributions of the duration of NIRP provided in Eq.(39).

Fig.8 shows the changes in the posterior distribution of the duration of NIRP over time. It can be seen that the posterior distribution changes towards shorter NIRP durations at the end of the sample period.

Fig.9 shows the expected values and modes of the NIRP duration that we computed from the distributions in Fig.8. The expected values and modes in Fig.9 show a reversal compared to Fig.3. They have shown a decreasing trend since the fall of 2019. In particular, it can be seen that the decrease is significant and marked since the beginning of 2022.

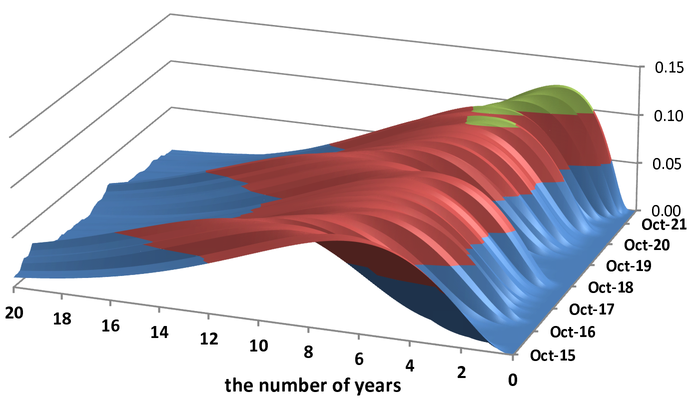

FIGURE 8. Changes in the distribution of NIRP duration over time

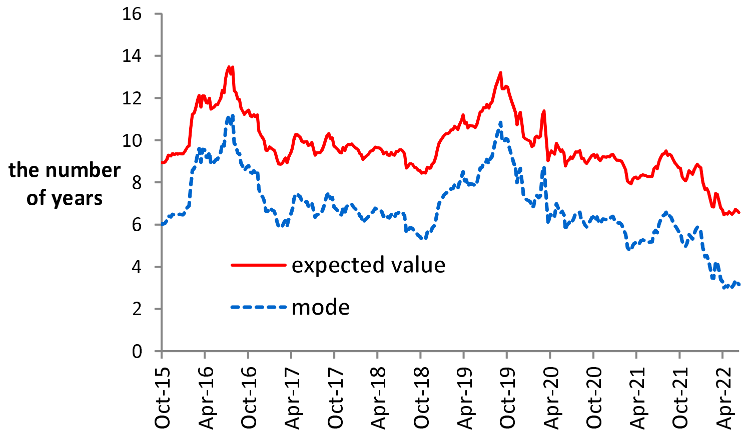

FIGURE 9. Estimated values and modes computed from implied posterior distributions of NIRP duration

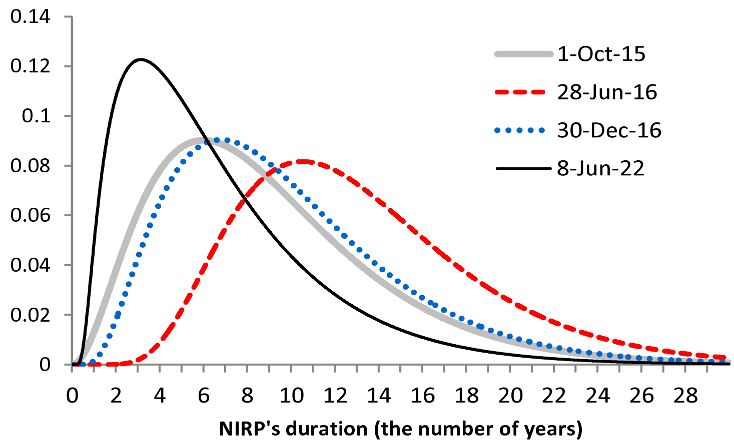

FIGURE 10. Implied posterior distributions of NIRP duration

Fig.10 displays a side-by-side comparison of the implied posterior distributions for the duration of NIRP on four selected days from the sample. The BOJ introduced NIRP in January 2016, and by the summer of that year, market speculation that the policy would be deepened was growing. Therefore, the peak of the distribution in Fig.10 on June 28, 2016, may have shifted to the right relative to that in October 2015. However, the BOJ did not deepen NIRP and instead introduced a yield curve control (YCC) policy in September 2016. Fig.10 suggests that this action by the BOJ led to a shortening of the forecasts for the duration of NIRP. In early 2022, the market became more aware of the possibility of a revision to the YCC, and the implied distribution in June 2022 shifted to the left relative to the rest of the chart. As of 2023, the market's interest in the possibility of a BOJ YCC revision has become even stronger. Therefore, if we extend the sample period to 2023 and estimate the model, the resulting posterior probability distribution may differ from the one obtained in 2022.

7. Conclusion

This study introduces a novel model of the term structure of interest rates to analyze time series data of yield curves, including the period of negative interest rate policy in the sample. The proposed model represents the short rate as the sum of a quadratic function of Gaussian state variables and a stochastic lower bound on interest rates modeled by a Brownian bridge starting and ending at zero. This Brownian bridge is intended to represent a lower bound on interest rates and is characterized by a random time interval corresponding to the duration of the negative interest rate policy period. The introduction of this Brownian bridge lower bound allows the interest rate term structure in the model to flexibly capture the actual yield curves corresponding to both the tightening and loosening of unconventional monetary policy. Under the no-arbitrage condition, we derive a pricing formula for zero coupon bonds.

To demonstrate the effectiveness of the proposed model, we estimate the state variables and parameters using time series data on the Japanese yield curve. The results show that the model fits the data well, with very small root mean squared errors between the estimated and observed yield curves. Moreover, the lower bound of the interest rate, which is the latent variable resulting from the estimation, shows a significant decline in the summer of 2016 and the fall of 2019. This result is consistent with the fact that speculation about the strengthening of the negative interest rate policy spread in the financial market in the summer of 2016, and the observation of a prolonged period of monetary easing strengthened in the fall of 2019.

After estimating the model parameters and state variables, we compute the expected excess returns of bonds with different maturities as financial market indicators of interest. The results show that the expected excess returns on bonds have been consistently low throughout the sample period, with many remaining negative. This can be attributed to the continuation of a large monetary easing policy. Furthermore, we compute posterior probability distributions for the duration of NIRP based on the estimated values of the parameters and state variables to shed light on the formation of financial market participants' expectations of NIRP. Our results show that financial market participants expected a longer duration of NIRP in the summer of 2016 and the fall of 2019, but the duration shortened rapidly from the beginning of 2022.

While we demonstrated the application of our proposed model using time series data of the Japanese yield curve, we believe it could also be applied to analyze the yield curves of European countries that have experienced negative interest rates. Although Europe returned to positive interest rates in 2022, it has experienced negative interest rates in the past. Therefore, investigating the model's performance using time series data that includes both periods of negative and positive interest rates in Europe would be a valuable topic for future research.

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Appendix

A proof of Eq.(23). Eq.(23) represents the density of y_t^r as defined in Eq.(7). Specifically, the following equation holds on $\{t < \tau\}$:

$$\varphi_t^{\mathbb{Q}}(r, y) = \sqrt{\frac{r}{2\pi t(r-t)\sigma_y^2}} \exp\left(-\frac{y^2}{2t(r-t)\sigma_y^2}\right),$$

where we denote the density of y_t^r in Eq.(7) by $\varphi_t^{\mathbb{Q}}(r, y)$.

We provide the proof of the above equation. We first carry out the Ito derivative for $\dfrac{y_t^r}{r-t}$ as follows:

$$d\left(\frac{y_t^r}{r-t}\right) = \frac{dy_t^r}{r-t} + \frac{y_t^r dt}{(r-t)^2} = -\frac{y_t^r}{(r-t)^2}dt + \frac{\sigma_y dW_{t,y}^{\mathbb{Q}}}{r-t} + \frac{y_t^r dt}{(r-t)^2} = \frac{\sigma_y dW_{t,y}^{\mathbb{Q}}}{r-t}.$$

The second equality holds due to Eq.(7).

Integrating both sides of the above equation from 0 to t, we obtain the following equation:

$$\int_0^t d\left(\frac{y_s^r}{r-s}\right) = \frac{y_t^r}{r-t} - \frac{y_0^r}{r} = \int_0^t \frac{\sigma_y dW_{s,y}^{\mathbb{Q}}}{r-s}$$

Since $y_0^r = 0$, we have

$$y_t^r = \sigma_y(r-t)\int_0^t \frac{dW_{s,y}^{\mathbb{Q}}}{r-s}.$$

Hence, y_t^r follows a normal distribution with expectation $E[y_t^r] = 0$ and variance

$$Var[y_t^r] = \sigma_y^2(r-t)^2\int_0^t \frac{ds}{(r-s)^2} = \frac{\sigma_y^2 t(r-t)}{r}.$$

This means that Eq.(23) holds.

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(K.Kikuchi) Fuculty of Economics, Shiga University
Email address: kentaro-kikuchi@biwako.shiga-u.ac.jp