

Analytic Approach for the Walrasian equilibrium Existence Theorem

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Abstract

We will provide the Walrasian equilibrium existence theorem by the methods of elementary analysis or differential geometry rather than topological methods.

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1 Introduction

In this paper, we will investigate some kind of classical existence theorem of Walrasian equilibrium for a perfectly competitive complete market economy. In the field of general equilibrium theory, existence issue of Walrasian equilibria and research of their welfare properties have been considered as crucial and central problem. For the detailed developments of the area, see, for example, [2], [6], [8], [9].

Economic agents, consumers and producers, trade their privately owned commodities by using price or market mechanism embedded in the market economy in order to maximize their economic benefits. Then, price or market mechanism adjusts their behaviors, interactions between economic agents. In the Walrasian equilibrium prices, aggregate excess demand is equal to zero, and hence each economic agent maximizes their benefits under the feasible allocation. The adjustment by price system does not necessarily guarantee the realization of Walrasian equilibrium [4], however, the equilibrium concepts are understood to be important as cornerstones and benchmarks for the interpretations of economic activities. The existence issue of Walrasian equilibrium is concerned with the consistency of price or market mechanism. It makes the consistency condition in the economic analysis clear.

Topological methods in the existence issue of Walrasian equilibrium are main streams in the research areas of general equilibrium theory. On the other hand, analytical methods or methods using differential geometry have been cultivated by many researchers^{*1}. See, for example, [14]. Our approach to the existence issue is heavily dependent on one of the methods of analysis or differential geometry, which have been cultivated by, for example, [11], [12]. As clearly indicated by [16], the existence of Walrasian equilibrium is equivalent to Brouwer's fixed point Theorem. Therefore, the central problem is reduced to the some kinds of fixed point theorem, in particular, Brouwer's fixed point Theorem. There are many methods of proof in (Brouwer's) fixed point theorem, those of which are deeply connected in geometrical natures. As stated in the above, our choice of method is one in differential geometry. The choices of various kind of methodologies are crucial from the viewpoints of comparison, and those plays crucial roles in theoretical development. Our result seems to be transitional in the sense that it is heavily analytical, and there is room to examine algebraic constructions using differential forms or homotopic considerations. Also, several parts of our assumptions are still strong, particularly, Assumption 3. We would like to make remaining issues weakened, and to extend the result to incomplete sequential economies, in the future research.

Our paper is constructed as the followings. In Section 2, assumptions of the economic primitives and main theorem are stated. In Section 3, series of propositions and lemmas are described, and thus, main result is completed in the proofs.

^{*1} It is impossible to overemphasize the importance of the methods of differential topology in these areas. Differential topology have occupied the intermediate and vast position among the topological methods and differential geometry. See, again, [8].

2 The Walrasian Model and Equilibrium Existence Theorem

In the perfectly competitive, complete markets of our Walrasian economy, n kinds of perfectly divisible goods are traded, *i.e.* the commodity space is assumed to be \mathbb{R}^n . Then, our fundamental economic primitive for the perfectly competitive, complete markets is given by the C^1 -excess demand function $x : \mathbb{R}_{++}^n \rightarrow \mathbb{R}^n$, where \mathbb{R}_{++}^n is strictly positive orthant of \mathbb{R}^n , $\{p \in \mathbb{R}^n : p^i > 0 \text{ for any } p = (p^1, \dots, p^n)\}$. The domain \mathbb{R}_{++}^n of x is interpreted as the price space. We assume that excess demand function x satisfies the following Assumption 1.

- Assumption 1.** ^{*2} (i) $x(tp) = x(p)$ for any $t \in \mathbb{R}_{++}$ and $p \in \mathbb{R}_{++}^n$. (homogeneity of degree 0)
(ii) $\langle p|x(p) \rangle = 0$ for any $p \in \mathbb{R}_{++}^n$. ^{*3} (Walras's law)
(iii) $\lim_{p^k \rightarrow p} \|x(p^k)\| = +\infty$ whenever $\lim_{k \rightarrow +\infty} p^k = p \in \partial\mathbb{R}_{++}^n$, where $\partial\mathbb{R}_{++}^n$ is the boundary of \mathbb{R}_{++}^n . (boundary condition, or, the desirability of commodities)
(iv) There exists $l \in \mathbb{R}^n$ such that $x(p) \gg l$. ^{*4} (lower boundedness)

Let S^{n-1} be a unit sphere $\{p \in \mathbb{R}^n : \|p\| = 1\}$ of \mathbb{R}^n , and S_{++}^{n-1} be a strictly positive orthant $\{p = (p^1, \dots, p^n) \in S^{n-1} : p^i > 0 \text{ for any } i\}$ of S^{n-1} . By Assumption 1-(i), we can normalize price set \mathbb{R}_{++}^n to S_{++}^{n-1} . Therefore, by Assumption 1-(ii), we can regard x as a tangent vector field $x : S_{++}^{n-1} \rightarrow \mathbb{R}^n$, which is defined as $p \in S_{++}^{n-1} \mapsto x_p = x(p)$, where $x_p \in T_p S^{n-1}$ is a tangent vector at $p \in S^{n-1}$. ^{*5}

Thus, we can restate Assumption 1 as Assumption 2.

Assumption 2. an excess demand for the Walrasian perfectly competitive economy is a C^1 -tangent vector field $x : S_{++}^{n-1} \rightarrow \mathbb{R}^n$ satisfying the following assumptions

- (i) $\langle p|x_p \rangle = 0$ for any $p \in S_{++}^{n-1}$. (Walras's law)
(ii) $\lim_{p^k \rightarrow p} \|x_{p^k}\| = +\infty$ whenever $\lim_{k \rightarrow +\infty} p^k = p \in \partial S_{++}^{n-1}$, where ∂S_{++}^{n-1} is the boundary of S_{++}^{n-1} . (boundary condition, or, the desirability of commodities)
(iii) There exists $l \in \mathbb{R}^n$ such that $x_p \gg l$. (lower boundedness)

Note that x_p is inward pointing near ∂S_{++}^{n-1} by Assumption 2-(ii). Moreover, we add a technical assumption.

Assumption 3. As function, $x : S_{++}^{n-1} \rightarrow \mathbb{R}^n$ is uniformly continuous on S_{++}^{n-1} .

Our main result is the following theorem.

Theorem 1. There exists $p \in S_{++}^{n-1}$ such that $x_p = 0$.

^{*2} Adding the another regular assumptions, it is wellknown that C^2 -smooth preferences [3] on the common consumption set \mathbb{R}_{++}^n in pure exchange economy generates the excess demand function which satisfies Assumption 1. Production economy under the regular assumptions generates the the excess demand function which satisfies Assumption 1, too [8].

^{*3} $\langle \cdot | \cdot \rangle$ is a usual inner product on $\mathbb{R}^n \times \mathbb{R}^n$.

^{*4} For $a = (a^1, \dots, a^n), b = (b^1, \dots, b^n) \in \mathbb{R}^n$, $a \gg b$ means that $a^i > b^i$ for any i .

^{*5} $T_p S^{n-1}$ is a tangent vector space of S^{n-1} at $p \in S_{++}^{n-1}$.

3 Series of Lemmas and Propositions

To lead a contradiction, suppose that $x_p \neq 0$ for any $p \in S_{++}^{n-1}$, and define a non-zero unit C^1 -tangent vector field $X : S_{++}^{n-1} \rightarrow \mathbb{R}^n$ on S_{++}^{n-1} as $p \in S_{++}^{n-1} \mapsto X_p \equiv \frac{x_p}{\|p\|}$. By Assumption 2, X satisfies the following facts.

Fact 1. (i) $\langle p | X_p \rangle = 0$ for any $p \in S_{++}^{n-1}$.

(ii) $\|X_p\| = 1$ for any $p \in S_{++}^{n-1}$

Note that X_p is outward pointing at ∂S_{++}^{n-1} by Fact 1-(i). Let S_+^{n-1} be a positive orthant $\{p = (p^1, \dots, p^n) \in S^{n-1} : p^i \geq 0 \text{ for any } i\}$ of S^{n-1} .

Proposition 1. *There exists a non-zero unit C^0 -tangent vector field $\tilde{X} : S_+^{n-1} \rightarrow \mathbb{R}^n$ on S_+^{n-1} such that $\tilde{X}_p = X_p$ for any $p \in S_{++}^{n-1}$.*

Proof. Fix $\bar{p} \in \partial S_+^{n-1}$. Then, there exists a sequence $\{q^k\}$ such that $q^k \in S_{++}^{n-1}$ and $\lim_{k \rightarrow +\infty} q^k = \bar{p}$, and a convergent subsequence of $\{X_{q^k}\}$, putting the limit as $X_{\bar{p}} \in S_+^{n-1}$. $X_{\bar{p}}$ satisfies that $\langle \bar{p} | X_{\bar{p}} \rangle = 0$, i.e., $X_{\bar{p}} \in T_{\bar{p}} \partial S_+^{n-1}$ and $\|X_{\bar{p}}\| = 1$ by Fact 1-(ii). Vector field X is uniformly continuous on S_{++}^{n-1} by Assumption 3. Hence, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$\|p^k - \bar{p}\| < \delta$ implies $\|X_{p^k} - X_{\bar{p}}\| \leq \|X_{p^k} - X_{q^l}\| + \|X_{q^l} - X_{\bar{p}}\| < \epsilon$. (If necessary, take l large enough.)

Thus, unique tangent vector $X_{\bar{p}} \in T_{\bar{p}} \partial S_+^{n-1}$ can be taken, and we can define a non-zero unit C^0 -tangent vector field $\tilde{X} : S_+^{n-1} \rightarrow \mathbb{R}^n$ on S_+^{n-1} as

$$p \in S_+^{n-1} \mapsto \tilde{X}_p = \begin{cases} X_p & : p \in S_{++}^{n-1} \\ X_{\bar{p}} & : p = \bar{p} \in \partial S_+^{n-1}. \end{cases}$$

□

Applying the Hopf's extension argument [1], [5], [11], there exists an open set U such that $S_+^{n-1} \subset U$ and C^1 -vector field $\tilde{X}^{ext} : U \rightarrow \mathbb{R}^n$ such that $\tilde{X}_p^{ext} = \tilde{X}_p$ for any $p \in S_+^{n-1}$. Abusing the notation, we will merely denote \tilde{X}^{ext} as \tilde{X} in the following argument.

Lemma 1. *There exist a Lipschitz constant $c > 0$ such that $\|\tilde{X}_p - \tilde{X}_q\| \leq c\|p - q\|$ for any $p, q \in S_+^{n-1}$.*

Proof. Let A be a cube, say, $A = \prod_{i=1}^j [a_i, b_i]$, $a_i, b_i \in \mathbb{R}$, $a_i < b_i$, $1 \leq j \leq n$. We can identify \tilde{X} with C^1 -function $\tilde{x} : U \rightarrow \mathbb{R}^n$ since \tilde{X} is C^1 -tangent vector field on U . Applying mean value theorem into any coordinate function $\tilde{x}^i : U \rightarrow \mathbb{R}$,

$$\begin{aligned} \|\tilde{X}_p - \tilde{X}_q\| &= \|\tilde{x}(p) - \tilde{x}(q)\| \leq \sum_{i=1}^n |\tilde{x}^i(p) - \tilde{x}^i(q)| = \sum_{i=1}^n |\langle \partial_{\xi^i} \tilde{x}^i | p - q \rangle| \leq \sum_{i=1}^n \|\partial_{\xi^i} \tilde{x}^i\| \|p - q\|, \\ \xi^i &= \theta_i p + (1 - \theta_i) q, \theta_i \in [0, 1]. \end{aligned}$$

We can take $c^i = \max_{\xi \in A} \|\partial_\xi \tilde{x}^i\|$ since A is compact and \tilde{x} is C^1 . Then, $\|\tilde{X}_p - \tilde{X}_q\| \leq \sum_{i=1}^n c^i \|p - q\|$. Putting $c = \sum_{i=1}^n c^i$, $\|\tilde{X}_p - \tilde{X}_q\| \leq c\|p - q\|$ uniformly.

Let A_α be an open cube such that $S_+^{n-1} \subset \bigcup_\alpha A_\alpha \subset U$. We can assume $\{A_\alpha\}$ is a finite collection since S_+^{n-1} is compact.

(i) If $p, q \in A_\alpha$ for some α , there exists c^α such that $\|\tilde{X}_p - \tilde{X}_q\| \leq c^\alpha \|p - q\|$ uniformly.

(ii) If $(p, q) \in (A \times A) \setminus \bigcup_\alpha (A_\alpha \times A_\alpha)$, $0 < \|p - q\|$ uniformly. Since $(A \times A) \setminus \bigcup_\alpha (A_\alpha \times A_\alpha)$ is compact, a function $F : (A_\alpha \times A_\alpha) \setminus \bigcup_\alpha (A_\alpha \times A_\alpha) \rightarrow \mathbb{R}$, defined by $(p, q) \mapsto F(p, q) = \|\tilde{X}_p - \tilde{X}_q\|$ is uniformly continuous. Therefore, $\|\tilde{X}_p - \tilde{X}_q\| \leq \|p - q\|$ uniformly on $(A \times A) \setminus \bigcup_\alpha (A_\alpha \times A_\alpha)$.

Putting $c = \max\{1, (c^\alpha)_\alpha\}$, $\|\tilde{X}_p - \tilde{X}_q\| \leq c\|p - q\|$ uniformly. □

For any $t \in \mathbb{R}$, define a function $f_t : A \rightarrow \mathbb{R}^n$ as $p \in A \mapsto f_t(p) = p + t\tilde{X}_p$.

Lemma 2. f_t is injective for $|t|$ small enough.

Proof. Choose t so that $|t| < c^{-1}$, and let $f_t(p) = f_t(q)$ to indicate $p = q$. Suppose $p \neq q$. Then, since $p - q = t(\tilde{X}_q - \tilde{X}_p)$,

$$\|p - q\| = |t| \|\tilde{X}_q - \tilde{X}_p\| \leq c|t| \|p - q\| < \|p - q\|, \text{ which leads to a contradiction.}$$

□

Proposition 2. $\text{vol}(f_t(S_+^{n-1}))$ is a polynomial function of t .

Proof. Differentiating f_t at $p \in S_+^{n-1}$,

$$\partial_p f_t = I_n + t \left[\frac{\partial \tilde{x}^i}{\partial p^j} \right],$$

where I_n is the identity matrix of n -th order, and, $\left[\frac{\partial \tilde{x}^i}{\partial p^j} \right]$ is the Jacobian matrix of \tilde{x} .

Therefore, we can put $\det \partial_p f_t = 1 + \sum_{i=1}^n \pi_i(p)t^i$, which is a polynomial function of t . Note that any $\pi_i(p)$ is continuous in U by the definition of the determinant. Since S_+^{n-1} is compact, any $\pi_i(p)$ is uniformly bounded. Hence, $\det \partial_p f_t > 0$ uniformly over S_+^{n-1} for $|t|$ small enough. By the change of variable formula ^{*6},

$$\begin{aligned} \text{vol}(f_t(S_+^{n-1})) &= \int_{S_+^{n-1}} |\det \partial_p f_t| dp = \int_{S_+^{n-1}} \det \partial_p f_t dp = \int_{S_+^{n-1}} \left(1 + \sum_{i=1}^n \pi_i(p)t^i\right) dp \\ &= \int_{S_+^{n-1}} 1 dp + \sum_{i=1}^n \left(\int_{S_+^{n-1}} \pi_i(p) dp \right) t^i. \end{aligned}$$

□

^{*6} See, for example, [13].

Let $A \subset \mathbb{R}^n$ be a compact region whose indicator function is integrable, $\mathcal{N}A$ be an open set containing A , and $Y : \mathcal{N}A \rightarrow \mathbb{R}^n$ be a C^1 -vector field.

Lemma 3. *There exist a Lipschitz constant $c > 0$ such that $\|Y_p - Y_q\| \leq c\|p - q\|$ for any $p, q \in A$.*

Proof. The proof can be done as similarly as the proof of Lemma 1. □

Take $\alpha > 0$, and let

$$S_{\alpha,+}^{n-1} = \{p \in \mathbb{R}_+^n : \|p\| = \alpha\}, S_{\alpha,t,+}^{n-1} = \left\{p \in \mathbb{R}_+^n : \|p\| = \alpha\sqrt{1+t^2}\right\}.$$

Define a C^1 -function $f_{\alpha,t} : S_{\alpha,+}^{n-1} \rightarrow \mathbb{R}^n$ as $p \in S_{\alpha,+}^{n-1} \mapsto f_{\alpha,t}(p) = p + t\|p\|\tilde{X}_{\frac{p}{\|p\|}}$.

Proposition 3. *$f_{\alpha,t} : S_{\alpha,+}^{n-1} \rightarrow \mathbb{R}^n$ is a bijective mapping onto $S_{\alpha,t,+}^{n-1}$ for $|t|$ small enough.*

Proof. Applying Lemma 3, we can take $c > 0$ such that $\|\tilde{X}_{\frac{q}{\|q\|}} - \tilde{X}_{\frac{p}{\|p\|}}\| \leq c\|p - q\|$ for any $p, q \in S_{\alpha,+}^{n-1}$. Then, choose t so that $|t| < c^{-1}\alpha^{-1}$, and let $f_{\alpha,t}(p) = f_{\alpha,t}(q)$ to indicate $p = q$. Suppose $p \neq q$. Then, since $p - q = t \cdot \alpha(\tilde{X}_{\frac{q}{\|q\|}} - \tilde{X}_{\frac{p}{\|p\|}})$,

$$\|p - q\| = \alpha|t|\|\tilde{X}_{\frac{q}{\|q\|}} - \tilde{X}_{\frac{p}{\|p\|}}\| \leq c\alpha|t|\|p - q\| < \|p - q\|, \text{ which leads to a contradiction.}$$

Hence, $f_{\alpha,t}$ is injective.

Let $c_A > 0$ be the Lipschitz constant of a vector field $\|p\|\tilde{X}_{\frac{p}{\|p\|}}$ for $A = \left\{p \in \mathbb{R}_+^n : \frac{\alpha}{2} \leq \|p\| \leq \frac{3\alpha}{2}\right\}$ in Lemma 3, let $t \in \mathbb{R}$ be such that $|t| < c_A^{-1}$ and $|t| < \frac{1}{3}$, and fix $P_0 \in S_{\alpha,t,+}^{n-1}$. Define a function $g_t : A \rightarrow \mathbb{R}^n$ as $p \in A \mapsto g_t(p) = P_0 - t\|p\|\tilde{X}_{\frac{p}{\|p\|}}$. Then,

$$\frac{\alpha}{2} \leq \|P_0\| - |t|\|p\| \leq \|g_t(p)\| \leq \|P_0\| + |t|\|p\| \leq \frac{3\alpha}{2}$$

Therefore, $g_t(p) \in A$. For any $p, q \in A$,

$$\|g_t(p) - g_t(q)\| = |t| \left| \left\| \|p\|\tilde{X}_{\frac{p}{\|p\|}} - \|q\|\tilde{X}_{\frac{q}{\|q\|}} \right\| \right| \leq |t|c_A\|p - q\|, \text{ and } 0 < |t|c_A < 1.$$

Consequently, function g_t has a fixed point $p^* \in A$ by Banach's shrinking mapping theorem since A is a compact, hence, a Banach space. Therefore,

$$p^* = g_t(p^*), \text{ i.e. } P_0 = p^* + t\|p^*\|\tilde{X}_{\frac{p^*}{\|p^*\|}} = f_{\alpha,t}(p^*).$$

On the other hand,

$$\begin{aligned} \|P_0\|^2 &= \|f_{\alpha,t}(p^*)\|^2 = \|p^*\|^2 + t^2\|p^*\|^2 = (1+t^2)\|p^*\|^2 \\ \|p^*\| &= \frac{\alpha}{\sqrt{1+t^2}}. \end{aligned}$$

Hence, we can write

$$\sqrt{1+t^2}P_0 = f_{\alpha,t}\left(\sqrt{1+t^2}p^*\right).$$

This implies that $f_{\alpha,t} : S_{\alpha,+}^{n-1} \rightarrow \mathbb{R}^n$ is surjective onto $S_{\alpha,t,+}^{n-1}$. □

Let $\alpha = 1$. Then,

$$S_{\alpha,+}^{n-1} = S_+^{n-1}, \text{ and } S_{\alpha,t,+}^{n-1} = S_{t,+}^{n-1}, \text{ where } S_{t,+}^{n-1} = \left\{ p \in \mathbb{R}_+^n : \|p\| = \alpha\sqrt{1+t^2} \right\}.$$

Note that a C^1 -function $f_t : S_+^{n-1} \rightarrow \mathbb{R}^n$ as $p \in S_+^{n-1} \mapsto f_t(p) = p + t\|p\|\tilde{X}_{\frac{p}{\|p\|}}$ is bijective onto $S_{t,+}^{n-1}$. Let

$$B_{\alpha,+}^n = \{p \in \mathbb{R}_+^n : \|p\| \leq \alpha\}, B_{\alpha,t,+}^{n-1} = \left\{ p \in \mathbb{R}_+^n : \|p\| \leq \alpha\sqrt{1+t^2} \right\}.$$

Fix $a, b \in \mathbb{R}$ so that $0 < a < b$, and put a compact region A_a^b as $B_{b,t}^n \setminus \text{int}B_{a,t}^n = \{p \in \mathbb{R}_+^n : a \leq \|p\| \leq b\}$, and define a vector field $\tilde{X}_a^b : A_a^b \rightarrow \mathbb{R}^n$ as $p \in A_a^b \mapsto \tilde{X}_p^b = \|p\|X_{\frac{p}{\|p\|}}$, which is a C^1 -vector field.

Proof of Theorem 1. First, let n be an odd number. Define a function $f_{t,a}^b : A_a^b \rightarrow B_{b,+}^n \setminus \text{int}B_{a,+}^n$ as $p \in A_a^b \mapsto p + t\|p\|\tilde{X}_{\frac{p}{\|p\|}}$. Let $|t|$ be small enough so that Proposition 3 holds. Then, $f_{t,a}^b : A_a^b \rightarrow B_{b,+}^n \setminus \text{int}B_{a,+}^n$ is a bijective C^1 -mapping onto $B_{b,+}^n \setminus \text{int}B_{a,+}^n$. Since the volume of n -dim ball is proportional to the n -th power of the radius,

$$\text{vol}(f_t(A)) = \left(\sqrt{1+t^2}\right)^n \cdot \text{vol}A$$

must hold. But, the right hand side of the equation is not a polynomial function since n is an odd number. This is a contradiction to Proposition 2.

Next, let n be an even number, and suppose $x_p \neq 0$ for any $p \in S_+^{n-1}$, which satisfies Assumption 2 and 3. Then, applying Proposition 1 to x_p , we can get an extended C^1 -vector field $\tilde{X} : S_+^{n-1} \rightarrow \mathbb{R}^n$, which satisfies Fact 1, Proposition 1, and uniform continuity. Define a C^1 -vector field $\hat{X} : S_+^n \rightarrow \mathbb{R}^{n+1}$ as $(p, q) \in S_+^n \mapsto (\tilde{X}_{\frac{p}{\|p\|}}, 0)$. Since \hat{X} is uniformly continuous and outward pointing on ∂S_+^n , we can apply the argument in this chapter, which leads to a contradiction since $n+1$ is an odd number. Thus, proofs are completed. □

References

- [1] Acemoglu, D., Ozdaglar, A., Simsek, A. 2007. Generalized Poincare-Hopf Theorem for compact nonsmooth regions. *Mathematics of operations research*. **32(1)** 193–214.
- [2] Debrue, G. 1982. Existence of competitive equilibrium. in: Arrow, K., Hildenbrand, M. eds., *Handbook of mathematical economics*. **2(15)** 697–743. Elsevier..
- [3] Debrue, G. 1972. Smooth preferences. *Econometrica*. **40(4)** 603–615.
- [4] Hahn, F. 1982. Stability. in: Arrow, K., Hildenbrand, M. eds., *Handbook of mathematical economics*, **2(16)** 745–793. Elsevier..
- [5] Krulwiski, C., Walsh, J. 2017. The Hopf degree theorem : homotopy groups and vector fields. *Math 132 Project II*.
- [6] Magil, M., Shafer, W. 1991. Incomplete market. in: Hildenbrand, W., Sonnenschein, H. eds., *Handbook of mathematical economics*. **4(30)** 1523–1614. Elsevier.
- [7] Mas-Colell, A. 2016. *General equilibrium and game theory: ten papers*. Harvard University Press.

- [8] Mas-Colell, A. 1985. *The theory of general economic equilibrium a differentiable approach*. Cambridge University Press.
- [9] McKenzie, L.W. 2005. *Classical general equilibrium theory*. The MIT Press.
- [10] Milnor, J. 1978. Analytic proofs of the "hairy ball theorem" and the Brouwer fixed point theorem. *The American Mathematical Monthly*. **85(7)** 521–524.
- [11] Milnor, J. 1997. *Topology from the differentiable view point*. Princeton University Press.
- [12] Rogers, C.A. 1980. A less strange version of Milnor's proof of Brouwer's fixed point theorem. *The American Mathematical Monthly*. **87(7)** 525–527.
- [13] Rudin, W. 1976. *Principles of mathematical analysis third edition*. McGraw-Hill.
- [14] Smale, S. 1981. Global analysis and economics. in: Arrow, K., Hildenbrand, M. eds., *Handbook of mathematical economics*, **1(8)** 331–370. Elsevier.
- [15] Tu, L.W. 2011. *An introduction to manifold second edition*. Springer.
- [16] Uzawa, H. 1962. Walras's Existence Theorem and Brouwer's Fixed Point Theorem. *Kikan Rironkeizaigaku* (in Japanese). **13(1)** 59–62.